lunedì 9 marzo 2020 08:26

Ex 4.5.5: $M = \{ (x,y,z) \in \mathbb{R}^3 : (x^2+y^2+z^2)^2 - xyz = 1 \}$

i) Show $M \neq \emptyset$ and it is the zero set of a submersion

ii) Is M compact?

iii) Det pts of M at min/max dist to $\vec{0}$.

Sol: $(x,0,0) \in M \iff x^4 = 1 \iff x = \pm 1$

$(\pm 1, 0, 0) \in M \implies M \neq \emptyset$.

$M = \{ g = 0 \}$ where $g = (x^2+y^2+z^2)^2 - xyz - 1$

g is a submersion on $M \iff \nabla g \neq \vec{0}$ on M

Notice that

$$\nabla g = \vec{0} \iff \begin{cases} \partial_x g = 0 \\ \partial_y g = 0 \\ \partial_z g = 0 \end{cases} \iff \begin{cases} \cancel{2}4(x^2+y^2+z^2)\cancel{2}x - yz = 0 \\ \cancel{2}4(x^2+y^2+z^2)\cancel{2}y - xz = 0 \\ \cancel{2}4(x^2+y^2+z^2)\cancel{2}z - xy = 0 \end{cases}$$

$$\iff \begin{cases} 4(x^2+y^2+z^2) = \dfrac{yz}{x} \\ x \neq 0 \\ y\,\dfrac{yz}{x} - x^2 z = 0 \\ z\,\dfrac{yz}{x} - x^2 y = 0 \end{cases} \vee \begin{cases} x = 0 \\ yz = 0 \\ 4y(y^2+z^2) = 0 \\ 4z(y^2+z^2) = 0 \end{cases}$$

$\Uparrow$

$L_z\quad \frac{yz}{\cancel{x}} - x^2 y = 0$

$4z(y+z)=0$

$\Updownarrow$

$$\begin{cases} x \neq 0 \\ 4(x^2+y^2+z^2) = \frac{yz}{x} \\ z(y-x)(y+x) = 0 \\ y(z^2-x^2) = 0 \end{cases}$$

$$\begin{cases} x = 0 \\ y = 0 \\ 0 = 0 \end{cases} \vee \begin{cases} x = 0 \\ z = 0 \\ 4y^3 = 0 \\ 0 = 0 \end{cases}$$

$4z^3 = 0 \Rightarrow (0,0,0)$

$\Rightarrow (0,0,0)$

$\Updownarrow$

$$\begin{cases} x \neq 0 \\ z = 0 \\ 4(x^2+y^2) = 0 \\ 4x^2 = 0 \end{cases} \vee \begin{cases} x \neq 0 \\ y = x \\ 4(2x^2+z^2) = z \\ \cancel{x}(z^2-x^2) = 0 \end{cases} \vee \begin{cases} x \neq 0 \\ y = -x \\ 4(2x^2+z^2) = -z \\ \cancel{x}(z^2-x^2) = 0 \end{cases}$$

$\Downarrow$

impos.

$\Updownarrow$

$$\begin{cases} x \neq 0 \\ y = x \\ z = x \\ 12x^{\cancel{2}} = \cancel{x}1 \end{cases} \vee \begin{cases} x \neq 0 \\ y = x \\ z = -x \\ 12x^{\cancel{2}} = -\cancel{x}1 \end{cases} \qquad \begin{cases} x \neq 0 \\ y = -x \\ z = x \\ 12x^{\cancel{2}} = -\cancel{x}1 \end{cases} \vee \begin{cases} x \neq 0 \\ y = -x \\ z = -x \\ 12x^{\cancel{2}} = \cancel{x}1 \end{cases}$$

$\Updownarrow$

$\left(\frac{1}{12}, \frac{1}{12}, \frac{1}{12}\right), \left(-\frac{1}{12}, -\frac{1}{12}, +\frac{1}{12}\right) \quad \left(-\frac{1}{12}, \frac{1}{12}, -\frac{1}{12}\right)$

$\left(\frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}\right)$

Now clearly $(0,0,0) \notin M$ while for each of prev.

pts:

$$(x^2+y^2+z^2)^2 - xyz = \left(3\frac{1}{12^2}\right)^2 - \frac{1}{12^3}$$

$$= \underbrace{\frac{9}{12^4} - \frac{1}{12^3}}_{<0} \neq 1$$

$\Rightarrow$ so on M, $\nabla g \neq \vec{0}$.

ii) Is M compact (closed & bded).

Clearly M is closed because $M = \{g=0\}$ $g \in C$.

Bded? Using spher coords we see that

$$(x,y,z) = (\qquad\qquad) \in M$$

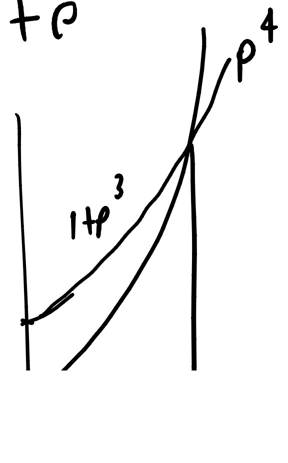
$$\rho \quad \theta \quad \varphi$$

$$\iff (\rho^2)^2 - \rho^3(\sin\varphi)^2\cos\theta\sin\theta\cos\varphi = 1$$

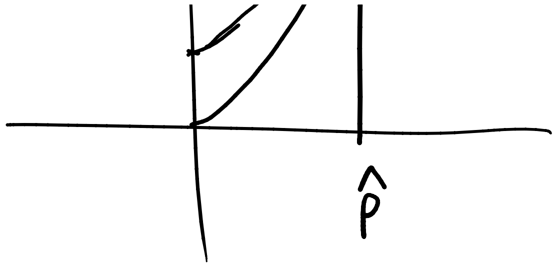
$$\iff \rho^4 - \rho^3 C(\varphi,\theta) = 1$$

$$\iff \rho^4 = 1 + \rho^3 \underbrace{C(\theta,\varphi)}_{\leq 1} \leq 1+\rho^3$$

$$\Rightarrow \boxed{\rho^4 \leq 1+\rho^3}$$



$$\Rightarrow \quad \boxed{\rho \leq 1+\rho}$$



$$\Downarrow$$

$$|x|,|y|,|z| \leq \|(x,y,z)\| = \rho \leq \hat{\rho} \quad \text{const}$$

$\Rightarrow$ $\mathcal{M}$ is bded.

iii) We have to det $\min/\max_{\mathcal{M}} \sqrt{x^2+y^2+z^2}$.

which is the same of $\min/\max_{\mathcal{M}} (\underbrace{x^2+y^2+z^2}_{f(x,y,z)})$.

Because $\mathcal{M}$ is compact and $f \in \mathcal{C} \Rightarrow$ (W thm) $\exists$ min/max f on $\mathcal{M}$.

To det these pts, because we also know that g is submersive on $\mathcal{M}$, we may apply Lagr. mult. thm: at any min/max point (x,y,z) for f on $\mathcal{M}$ we have

$$\nabla f = \lambda \nabla g$$

$$\Leftrightarrow \quad rk \begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} < 2 \Leftrightarrow rk \begin{bmatrix} 2x & 2y & 2z \\ 4x(x^2+y^2+z^2)-yz & \downarrow & \downarrow \end{bmatrix} < 2$$

$$\Longleftrightarrow \begin{cases} \det\begin{bmatrix} 2x & 2y \\ * & * \end{bmatrix} = 0 \\ \det\begin{bmatrix} 2x & 2z \\ * & * \end{bmatrix} = 0 \\ \det\begin{bmatrix} 2y & 2z \\ * & * \end{bmatrix} = 0 \end{cases} \Longleftrightarrow \begin{cases} 2x\left(4y\cancel{(x^2+y^2+z^2)} - xz\right) - 2y\left(4x\cancel{(x^2+y^2+z^2)} - yz\right) = 0 \\ = \end{cases}$$

$$\Longleftrightarrow \begin{cases} -\cancel{2}x^2 z + \cancel{2}y^2 z = 0 \\ -\cancel{2}x^2 y + \cancel{2}z^2 y = 0 \\ -\cancel{2}y^2 x + \cancel{2}z^2 x = 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} z(y-x)(y+x) = 0 \\ y(z-x)(z+x) = 0 \\ x(z-y)(z+y) = 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} z = 0 \\ yx^2 = 0 \\ \cancel{xy^2 = 0} \end{cases} \vee \begin{cases} y = x \\ x(z-x)(z+x) = 0 \end{cases} \vee \begin{cases} y = -x \\ -x(z-x)(z+x) = 0 \end{cases}$$

$$\Updownarrow \qquad\qquad \Updownarrow \qquad\qquad \Updownarrow$$

$$\begin{cases} z = 0 \\ y = 0 \end{cases} \vee \begin{cases} z = 0 \\ x = 0 \end{cases}$$

$$(x,0,0) \qquad (0,y,0)$$

$$\begin{cases} y = x^{=0} \\ x = 0 \end{cases} \vee \begin{cases} y = x \\ z = x \end{cases} \vee \begin{cases} y = x \\ z = -x \end{cases} \vee \begin{cases} y = -x \\ z = x \end{cases}$$

$$\begin{cases} y = -x \\ x = 0 \end{cases} \quad (0,0,z)$$

$$(0,0,z) \qquad (x,x,x) \qquad (x,x,-x) \qquad (x,-x,x)$$

$\vee \ldots$

$(x,0,0)$ $\quad (0,y,0)$ $\quad (0,0,z)$ $\quad (x,x,x)$ $\quad (x,x,-x)$

$\vee \quad \begin{cases} y=-x \\ z=-x \end{cases}$

$(x,-x,-x)$

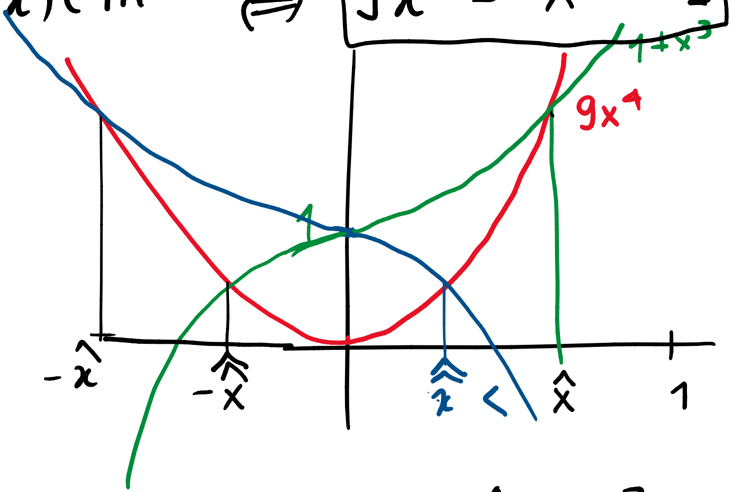
Let's see which of these pts $\in M$:

$(x,0,0)\in M \iff x=\pm 1$ $\qquad 1 \xleftarrow{f} (\pm 1,0,0)$

$(0,y,0)\in M \iff y=\pm 1$ $\qquad 1 \leftarrow (0,\pm 1,0)$

$(0,0,z)\in M \iff z=\pm 1$ $\qquad 1 \leftarrow (0,0,\pm 1)$

$(x,x,x)\in M \iff \boxed{9x^4-x^3=1} \iff 9x^4=1+x^3$



$\exists!\ 0<\hat{x}<1$ sol of

$\exists!\ -\hat{x}<-\hat{\hat{x}}<0$

$(\hat{x},\hat{x},\hat{x}) \xrightarrow{f} 3\hat{x}^2$

$(\hat{\hat{x}},\hat{\hat{x}},\hat{\hat{x}}) \xrightarrow{f} 3\hat{\hat{x}}^2$

$(x,x,-x)\in M \iff 9x^4+x^3=1 \iff 9x^4=1-x^3$

$\Downarrow$

$3\hat{x}^2 \xleftarrow{f} (\hat{x},\hat{x},-\hat{x})$, $\quad (\hat{\hat{x}},\hat{\hat{x}},-\hat{\hat{x}}) \xrightarrow{f} 3\hat{\hat{x}}^2$

$(x,-x,x)\in M \iff 9x^4+x^3=1$

$3\hat{x}^2 \leftarrow (\hat{x},-\hat{x},-\hat{x})$

$(x,-x,-x)\in M \iff 9x^4-x^3=1 \implies$

$3\hat{\hat{x}}^2 \leftarrow (-\hat{\hat{x}},\hat{\hat{x}},\hat{\hat{x}})$

So values attained at these pts are

$$1, \quad 3\hat{x}^2, \quad \underset{\smile}{3\hat{\hat{x}}^2}.$$

We need num. estimation to figure out which are min and which max. ∎

Multiple Integration

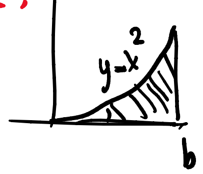
Pb: We want to define integrals as

$$\int_D f(x,y)\,dx\,dy, \qquad \int_D f(x,y,z)\,dx\,dy\,dz.$$

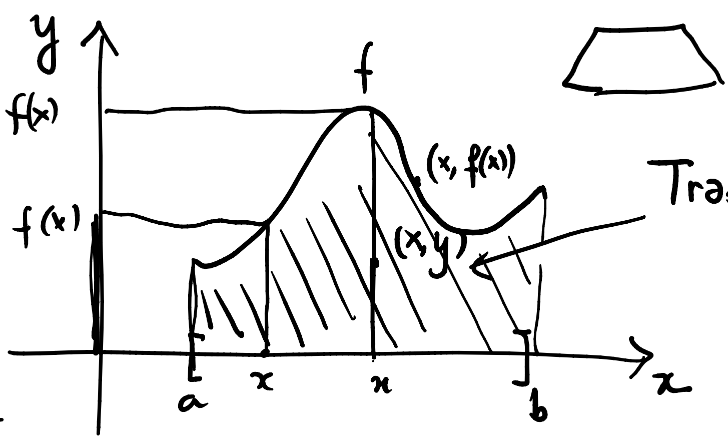
or more in gen.

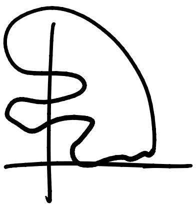
$$\int_D f(\vec{x})\,d\vec{x} \qquad \vec{x} \in \mathbb{R}^d.$$

$$\int_0^b x^2\,dx$$



Recall that, if $f : [a,b] \longrightarrow [0,+\infty[$,

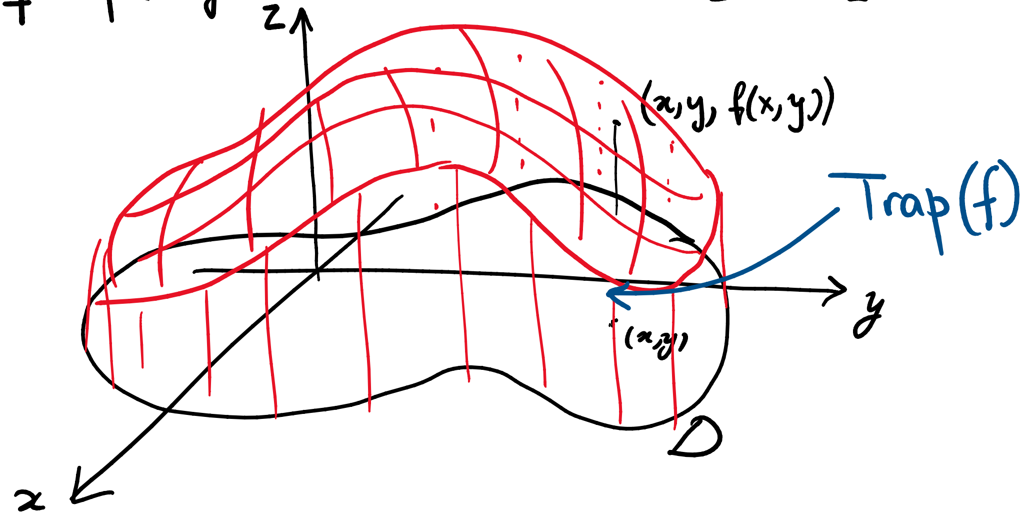




$$\mathrm{Trap}(f) = \{(x,y) \in \mathbb{R}^2 : a \le x \le b,\ 0 \le y \le f(x)\}$$

$f \in \mathscr{C}([a,b])$, then $\int_a^b f(x)\,dx := \text{Area}\,(\text{Trap}(f))$.

If $f = f(x,y) : D \subset \mathbb{R}^2 \longrightarrow [0,+\infty[$



$$\text{Trap}(f) = \{ (x,y,z) \in \mathbb{R}^3 : \quad (x,y) \in D, \quad 0 \leq z \leq f(x,y) \}$$

A natural idea would be then

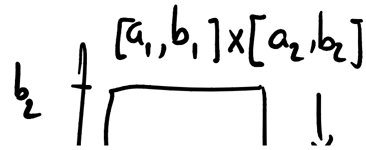
$$\int_D f(x,y)\,dx\,dy = \text{Volume}\,(\text{Trap}(f))$$

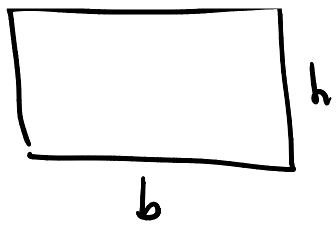
Similarly, for $f = f(x,y,z) : D \subset \mathbb{R}^3 \longrightarrow [0,+\infty[$

$$\text{Trap}(f) = \{ (x,y,z,w) \in \mathbb{R}^4 : \quad (x,y,z) \in D, \quad 0 \leq w \leq f(x,y,z) \}$$

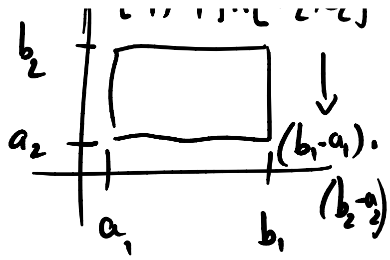
and

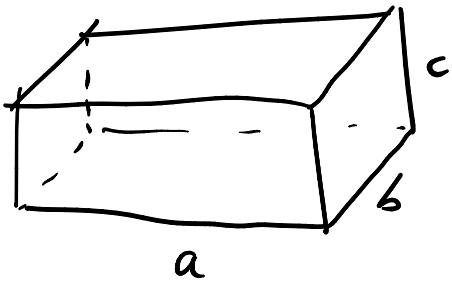
$$\int_D f(x,y,z)\,dx\,dy\,dz \overset{?}{=} 4 \text{ dim Volume}\,(\text{Trap}(f))$$

2dim:  $A = b \cdot h$ 

2dim: 

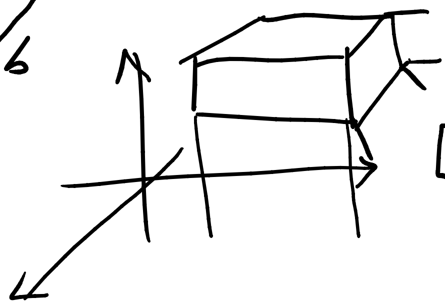
$A = b \cdot h$



3dims 

$V = a \cdot b \cdot c.$ $\quad (b_1-a_1)(b_2-a_2)(b_3-a_3)$

$\uparrow$



$[a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]$

4 dim:

$$[a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]\times[a_4,b_4] \longrightarrow$$

$$(b_1-a_1)(b_2-a_2)(b_3-a_3)(b_4-a_4)$$

By these preliminary considerations we see that, ion general, to define the operation of integral for a funct

$$f: D \subset \mathbb{R}^d \longrightarrow [0,+\infty[$$

by posing

$$\int_D f(\vec{x})\,d\vec{x} = \lambda_{d+1}\,(\mathrm{Trap}(f))$$

d+1 dim measure

where

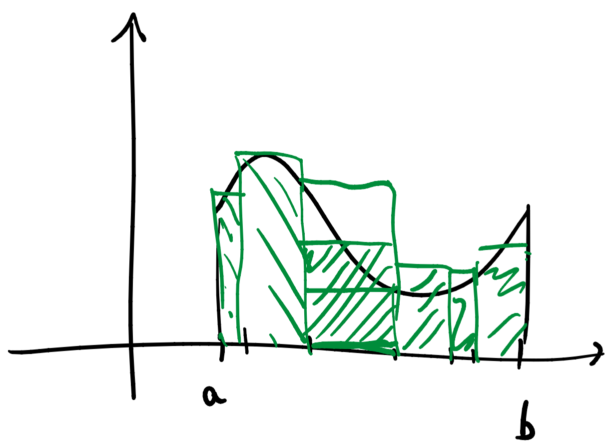
$\mathrm{Trap}(f) = \{(\vec{x}, y) \in \mathbb{R}^{d+1} : \vec{x} \in D, \quad 0 \le y \le f(\vec{x})\}$

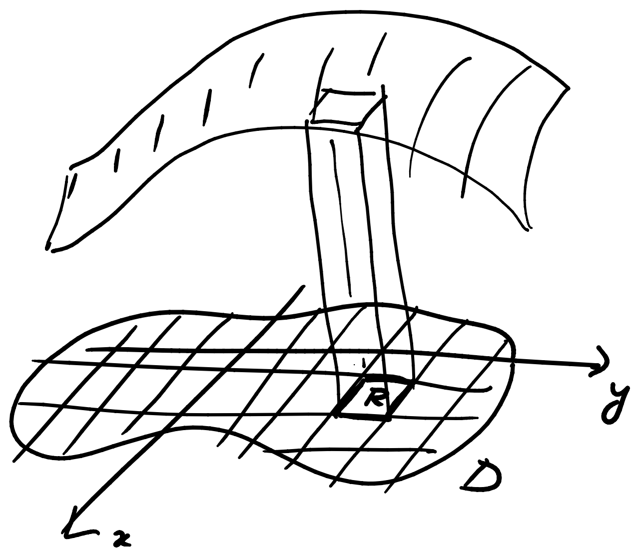
where

$$\operatorname{Trop}(f) = \{ (\vec{x}, y) \in \mathbb{R}^{d+1} : \vec{x} \in D, \ 0 \leq y \leq f(\vec{x}) \}$$

We need a concept of $d+1$ dim meas. on $\mathbb{R}^{d+1}$, or, which is the same, the concept of d-dim meas on $\mathbb{R}^d$.

We have seen this construction in dim 2





The final product of this procedure is a <u>function</u>

$$\underset{\substack{\uparrow \\ \text{set}}}{S} \subset \mathbb{R}^d \longrightarrow \lambda_d(S) \ \text{meas. of } S.$$

$$\lambda_d : \underset{\substack{\downarrow \\ \text{parts of } \mathbb{R}^d}}{\mathcal{M}_d \subset \mathcal{P}(\mathbb{R}^d)} \longrightarrow [0,+\infty]$$

(all the subsets of $\mathbb{R}^d$)

parts of $\mathbb{R}$

(all possible subsets of $\mathbb{R}^d$)

such that a certain number of properties is fulfilled:

<u>Thm:</u> $\exists\ \lambda_d : \mathcal{M}_d \subset \mathcal{P}(\mathbb{R}^d) \longrightarrow [0,+\infty]$ with the following properties:

1. good and natural sets as closed sets and open sets belong to $\mathcal{M}_d$
2. $A \in \mathcal{M}_d\ ,\ B \in \mathcal{M}_n \Rightarrow A \times B \in \mathcal{M}_{d+n}$

   and

$$\lambda_{d+n}(A \times B) = \lambda_d(A)\,\lambda_n(B)$$

with the agreement that here $0 \cdot \infty = 0$.