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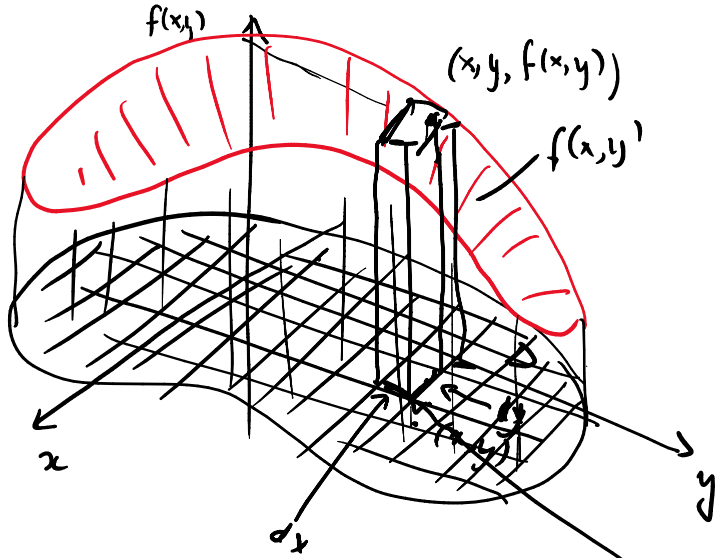
$f = f(x,y) : D \subset \mathbb{R}^2 \longrightarrow [0,+\infty[$



$$\int_D f(x,y)\,dxdy$$

$$= \mathrm{Vol}\,(\mathrm{Trap}\,(f))$$

$$\mathrm{Trap}\, f = \{ (x,y,z) \in \mathbb{R}^3 : (x,y) \in D,\ 0 \leq z \leq f(x,y) \}$$

$$\boxed{\int_D f(x,y)\,dxdy \approx \sum \underline{f(x,y)}\, dxdy}$$

We assume the integral of a funct f be well defined.

Pb: How do we compute $\int_D f(x,y)\,dx\,dy$.

There're two major tools:

1) Reduction Formula: a way to reduce calc. of a multiple integral to the calculus of integrals in 1 single var.

2) Change of variables

Reduction Formula

Suppose we wish to compute

$$\int f(x,y)\,dxdy \qquad \sum \overset{a_{x,y}}{\boxed{f(x,y)\,dxdy}}$$

''

$$\int_D f(x,y)\,dx\,dy = \sum_{(x,y)\in D} \overbrace{f(x,y)\,dx\,dy}^{a_{x,y}}$$

$$= \sum_x \left(\sum_{y:\,(x,y)\in D} f(x,y)\,dx\,dy \right) \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & & a_{2n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & \cdots & & a_{mn} \end{bmatrix}$$

$$\left[a_{x_1,y_1} + a_{x_1 y_2} + a_{x_1 y_3} + a_{x_1 y_4} \cdots \right] +$$

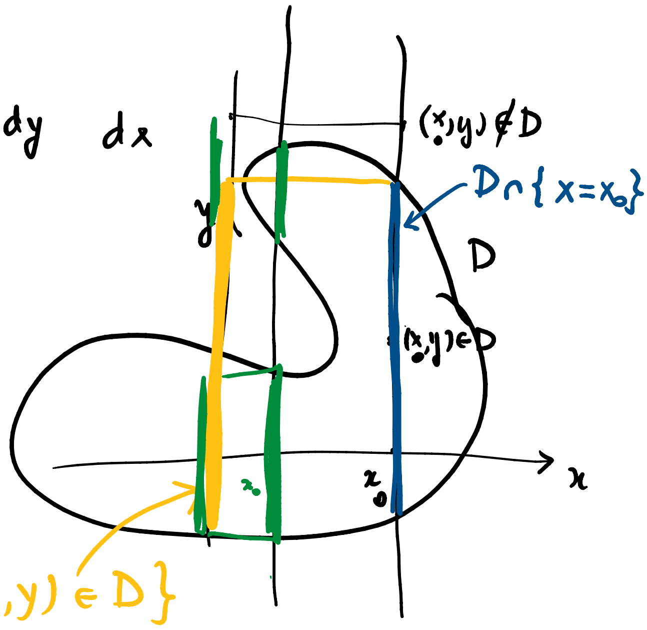
$$\left[a_{x_2,y_1} \quad a_{x_2 y_2} \quad a_{x_2,y_3} \cdots \right] +$$

$$\left[\quad \right]$$

$$= \sum_x \left(\sum_{y:\,(x,y)\in D} f(x,y)\,dy \right) dx$$

$$\sum_x \int_{\underbrace{\{y\,:\,(x,y)\in D\}}_{D_x}} f(x,y)\,dy\,dx$$

$$D_x := \{ y \in \mathbb{R} : (x,y) \in D \}$$



$$= \sum_x \left(\int_{D_x} f(x,y)\,dy \right) dx$$

$$\sum_x \boxed{\text{funct of } x}\, dx$$

$$\angle_{x} \boxed{\text{funct of } x} \, dx$$

$$\sum_{x} g(x)\, dx$$

$$= \int_{-\infty}^{+\infty} \left(\int_{D_x} f(x,y)\, dy \right) dx.$$

<u>Thm:</u> If f is <u>integrable</u> on D $\Rightarrow$

$$\int_{D} f(x,y)\, dx\, dy = \int_{-\infty}^{+\infty} \int_{D_x} f(x,y)\, dy \; dx$$

where $D_x = \{ y \in \mathbb{R} : (x,y) \in D \}$

<u>Rmk:</u> $f \in \mathcal{C}(D)$, D compact are integrable.

If D is non compact to check integrability we should prove first that $\int_D |f| < +\infty$

If $f \geq 0 \Rightarrow |f| = f \Rightarrow f$ is int if

$$\int_D f \, dx\, dy = \int_{-\infty}^{+\infty} \left(\int_{D_x} f \, dy \right) dx < +\infty$$

Ex 5.2.4. $f(x,y) = x^3 e^{-yx^2}$ on $D = [0, +\infty[\times [1,2]$.

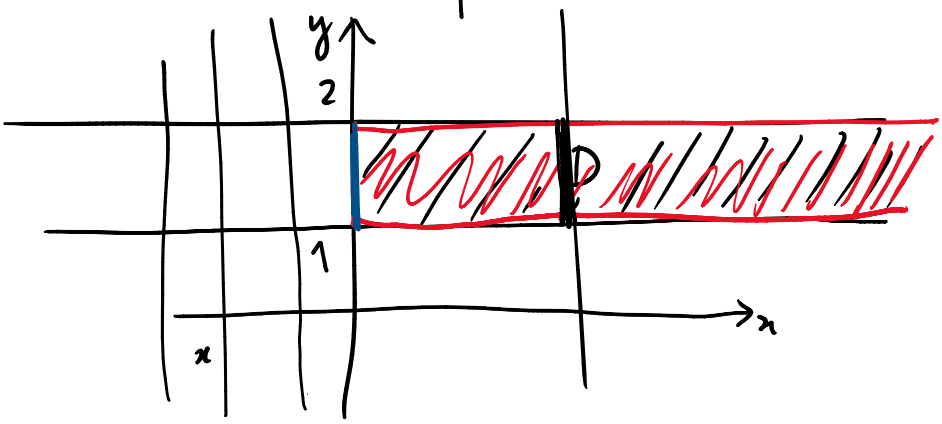
Compute $\int_D f$.

<u>Sol:</u> In this case we've to compute

$$\int_{D} x^3 e^{-yx^2} \, dx\, dy = \int_{-\infty}^{+\infty} \left(\int_{D_x} x^3 e^{-yx^2} dy \right) dx$$

$$\int_D x^3 e^{\cdots}\, dx\,dy = \int_{-\infty} \left(\int_{D_x} \cdots dy \right) dx$$

$$= \int_{-\infty}^{+\infty} \left(x^3 \int_{D_x} e^{-yx^2}\, dy \right) dx$$



$$D_x = \begin{cases} \phi & \text{if} \quad x < 0 \\ [1,2] & \text{if} \quad x \geqslant 0 \end{cases}$$

$$= \int_{-\infty}^{0} x^3 \underbrace{\int_{\phi} e^{-yx^2}\, dy}_{=0}\, dx + \int_{0}^{+\infty} x^3 \underbrace{\int_1^2 e^{-yx^2}\, dy}\, dx$$

$$\int_{-\infty}^{0} 0\, dx = 0$$

$$\int_1^2 e^{-yx^2}\, dy$$

$$\partial_y \left(e^{-yx^2} \right) = e^{-yx^2} (-x^2)$$

$$-\int_1^2 -x^2 e^{-yx^2}\, dy = \left[e^{-yx^2} \right]_{y=1}^{y=2}$$

$$= e^{-2x^2} - e^{-x^2}$$

$e^{\alpha y}$

$$= + \int_0^{+\infty} x \cdot \left(e^{-2x^2} - e^{-x^2} \right) dx$$

$$\frac{1}{4} \cdot \left(-4x\, e^{-2x^2}\right) \; \frac{+}{-2}\!\!{}^{-2}\; x\, e^{-x^2} \qquad \partial_x e^{-x^2} = e^{-x^2}(-2x)$$

$$\partial_x e^{-2x^2} = e^{-2x^2}(-4x)$$

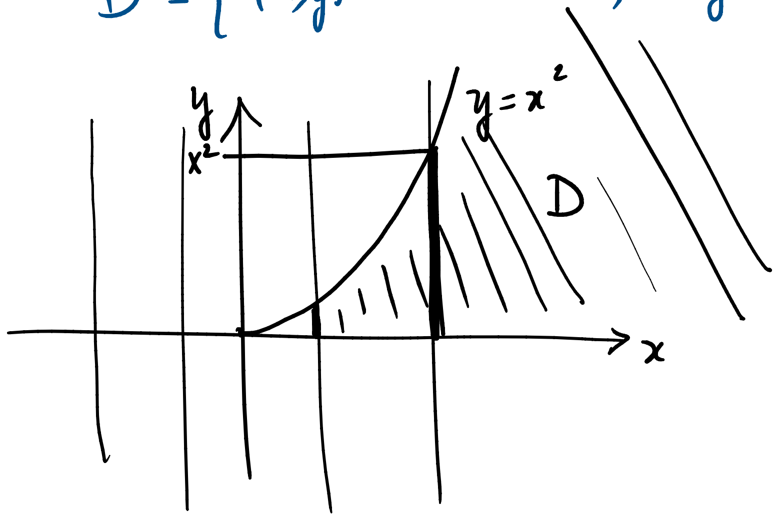
$$= \frac{1}{4} \int_0^{+\infty} \underbrace{\left(-4x\, e^{-2x^2}\right)}_{\partial_x e^{-2x^2}} dx - \frac{1}{2} \int_0^{+\infty} \underbrace{\left(-2x\, e^{-x^2}\right)}_{\partial_x e^{-x^2}} dx$$

$$= \frac{1}{4} \underbrace{\left[e^{-2x^2} \right]_0^{+\infty}}_{0-1} - \frac{1}{2} \underbrace{\left[e^{-x^2} \right]_0^{+\infty}}_{0-1}$$

$$= -1/4 + 1/2 = 1/4 . \qquad \blacksquare$$

Ex 5.2.5 $\quad f(x,y) = e^{-x} \qquad D = \{ (x,y) \in \mathbb{R}^2 : \; x \geq 0, \; 0 \leq y \leq x^2 \}$

Compute $\int_D f$.



Sol:

$$\int_D f(x,y)\, dx\, dy \overset{RF}{=}$$

$$= \int_{-\infty}^{+\infty} \int_{D_x} e^{-x}\, dy\, dx$$

$$= \int_{-\infty}^{+\infty} e^{-x} \int_{D_x} 1\, dy\, dx$$

$$D_x = \begin{cases} \emptyset & x \leq 0 \\ [0, x^2] & x \geqslant 0 \end{cases}$$

$$= \underbrace{\int_{-\infty}^{0} e^{-x} \underbrace{\int_{\emptyset} 1\,dy}_{=0}\,dx}_{=0} + \int_{0}^{+\infty} \Big(e^{-x} \underbrace{\int_{0}^{x^2} 1\,dy}_{=x^2} \Big)\,dx$$

$$= \int_0^{+\infty} x^2 \underbrace{e^{-x}}_{=\partial_x(-e^{-x})}\,dx$$

$$= \underbrace{-e^{-x} x^2 \Big|_0^{+\infty}}_{=0 \quad \frac{x^2}{e^x} \xrightarrow[x\to+\infty]{} 0} + \int_0^{+\infty} 2x\, e^{-x}\,dx$$

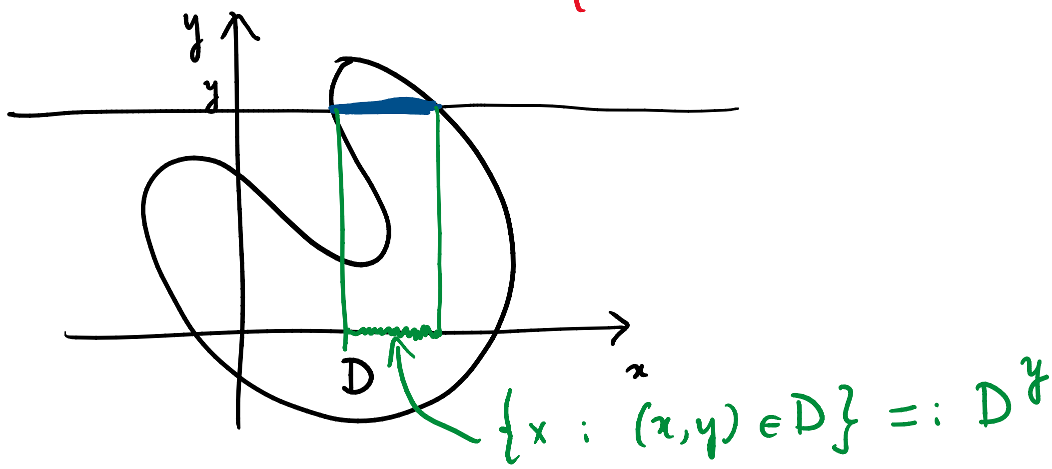
$$= 2 \int_0^{+\infty} x \underbrace{e^{-x}}_{=\partial_x(-e^{-x})}\,dx = 2\Big[\underbrace{-e^{-x} x \Big|_0^{+\infty}}_{=0} + \int_0^{+\infty} 1\, e^{-x}\,dx \Big]$$

$$= 2 \int_0^{+\infty} e^{-x}\,dx = 2\big[-e^{-x}\big]_0^{+\infty} = 2 \quad \blacksquare$$

Clearly, the RF may be applied by inverting the order of integrations:

$$\int_D f(x,y)\,dx\,dy = \int_{-\infty}^{+\infty} \Big(\int_{\{x:\,(x,y)\in D\}} f(x,y)\,dx \Big)\,dy$$

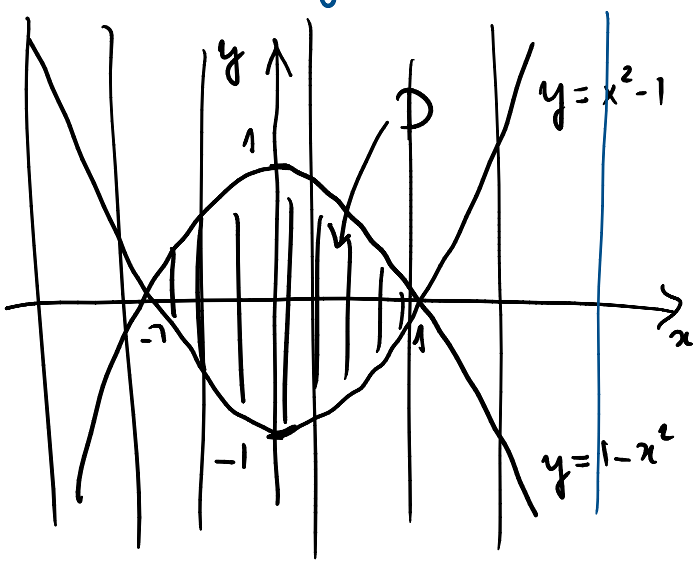
D $\qquad -\infty \qquad \{x : (x,y) \in D\}$



Ex. 5.5.1

Do # 1, 2, 3

#3: $\displaystyle\int_{|y|\leq 1-x^2} \frac{1}{1+y}\, dx\, dy$



$$|y| \leq 1 - x^2$$
$$\Updownarrow$$
$$\boxed{x^2 - 1 \leq y \leq 1 - x^2}$$

$$= \int_{-\infty}^{+\infty} \left(\int_{D_x} \frac{1}{1+y}\, dy \right) dx \overset{(*)}{=}$$

$$= \int_{-\infty}^{+\infty} \left(\int_{D_y} \frac{1}{1+y}\, dx \right) dy$$

$$= \int_{-\infty} \left(\int_{D_y} \frac{1}{1+y} \, dx \right) dy$$

$$\overset{(*)}{=} \int_{-1}^{1} \left(\int_{x^2-1}^{1-x^2} \frac{1}{1+y} \, dy \right) dx$$

$$\left\| \quad \log(1+y) \Big|_{x^2-1}^{1-x^2} \right.$$

$$= \int_{-1}^{1} \log(2-x^2) - \log x^{2} \, dx$$

$$= 2 \underbrace{\int_0^1 \overset{= \partial_x x}{1} \log(2-x^2)} - 2 \cdot 1 \log x \, dx$$

$$= 2 \Bigg[\underbrace{x \log(2-x^2) \Big|_0^1}_{=0} - \int_0^1 x \frac{-2x}{2-x^2} \, dx$$

$$2 \Big[\underbrace{x \log x \Big|_0^1}_{=0} - \int_0^1 x \cdot \frac{1}{x} \, dx \Big] \Bigg]$$

$$= 2 \Big[+-2 \int_0^1 \frac{2-x^2-2}{2-x^2} \, dx - 2 \int_0^1 1 \, dx \Big]$$

$$= 2 \Big[-4 \underbrace{\int_0^1 1 \, dx}_{=1} + 4 \int_0^1 \frac{1}{2-x^2} \, dx \Big]$$

$$\left\| \quad -4 \int_0^1 \frac{1}{x^2-2} = \frac{1}{(x-\sqrt{2})(x+\sqrt{2})} \right.$$

$$\frac{1}{(x-\sqrt{2})(x+\sqrt{2})} = \left(\frac{1}{x-\sqrt{2}} - \frac{1}{x+\sqrt{2}}\right)\frac{1}{2\sqrt{2}}$$

$$= 2\left[-4 - 4\,\frac{1}{2\sqrt{2}}\int_0^1 \frac{1}{x-\sqrt{2}} - \frac{1}{x+\sqrt{2}}\,dx\right]$$

$$\| $$

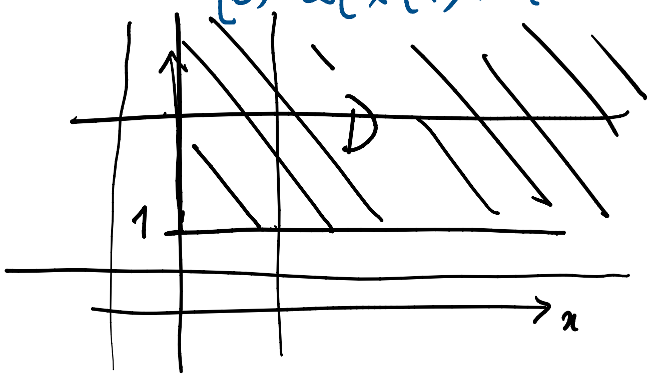
$$\log|x-\sqrt{2}|\Big|_0^1 - \log(x+\sqrt{2})\Big|_0^1$$

$$= 2\left[-4 - \frac{2}{\sqrt{2}}\left(\log(\sqrt{2}-1) - \cancel{\log\sqrt{2}} - \log(1+\sqrt{2}) + \cancel{\log\sqrt{2}}\right)\right]$$

∎

Ex 5.5.2. Do #1, 2, 3, 4.

#3 $\displaystyle\int_{[0,+\infty[\times[1,+\infty[} e^{-xy^4}\,dx\,dy$



$$= \int_{-\infty}^{+\infty}\left(\int_{D_x} e^{-xy^4}\,dy\right)dx$$

$$= \int_0^{+\infty}\left(\int_1^{+\infty} e^{-xy^4}\,dy\right)dx$$

?

$$= \int_{-\infty}^{+\infty}\left(\int_{D_y} f(x,y)\,dx\right)dy$$

$$= \int_1^{+\infty}\left(\int_0^{+\infty} \frac{-y^4 e^{-xy^4}}{(-y^4)}\,dx\right)dy$$

$$= \int_1 \left(\int -y^4 e \quad dx \right) dy$$

$$\partial_x \left(e^{-xy^4} \right) = e^{-xy^4} (-y^4)$$

$$= \int_1^{+\infty} -\frac{1}{y^4} \underbrace{\int_0^{+\infty} \partial_x e^{-xy^4} \, dx}_{\substack{\| \\ \left[e^{-xy^4} \right]_{x=0}^{x=+\infty} \\ 0-1}} \, dy$$

$$= \int_1^{+\infty} \frac{1}{y^4} \, dy = \left[\frac{y^{-3}}{-3} \right]_1^{+\infty} = -\frac{1}{3} \left[\frac{1}{y^3} \right]_{y=1}^{y=+\infty}$$

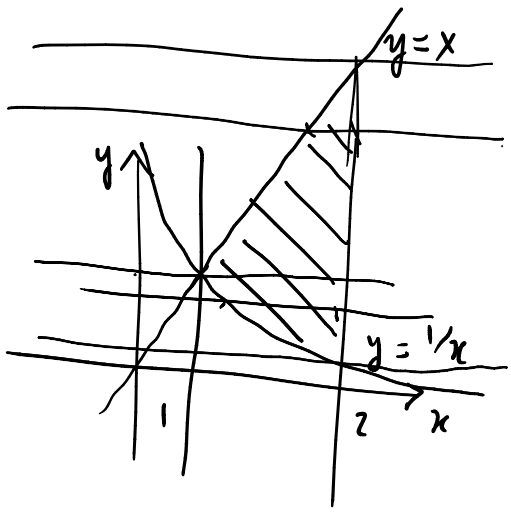
$$0 - 1$$

$$= +\frac{1}{3} \quad \blacksquare$$

4

$$\int_{\boxed{1 \leq x \leq 2 \,,\ \frac{1}{x} \leq y \leq x}} \frac{x}{y} \, dx \, dy$$

$$= \int_1^2 \left(\int_{1/x}^{x} \frac{x}{y} \, dy \right) dx$$



$$= \int_1^2 x \int_{1/x}^{x} \frac{1}{y} \, dy \, dx$$

$$\|$$

$$\log |y| = \log y \Big|_{1/x}^{x} = \log x - \log 1/x$$

$$= 2 \log x$$

$= 2\log x$

$$= \int_1^2 \underset{\substack{\| \\ \partial_x(x^2)}}{x\,2} \cdot \log x \, dx = \left[x^2 \log x \right]_{x=1}^{x=2} - \int_1^2 x^2 \cdot \underset{\substack{\| \\ x}}{\frac{1}{x}} \, dx$$

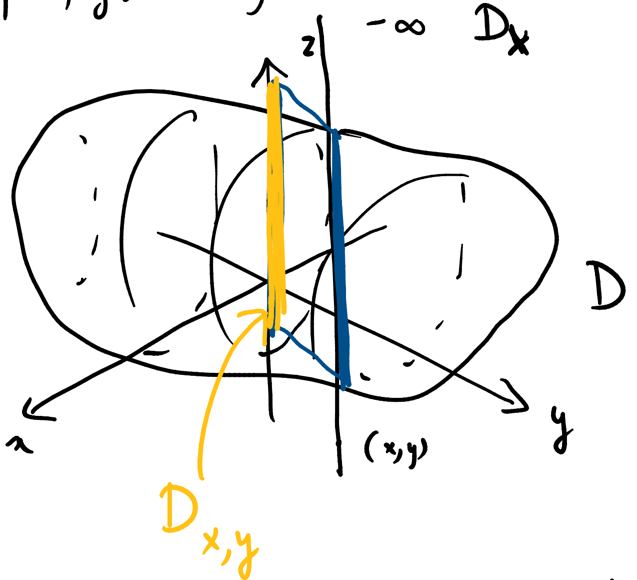
$$= 4\log 2 - \int_1^2 x \, dx = 4 \log 2 - \left[\frac{x^2}{2} \right]_1^2 =$$

$$= 4\log 2 - \frac{1}{2}(4-1) = 4 \log 2 - \frac{3}{2}. \quad \blacksquare$$

Reduction formula may be used also in case of triple integrals:

$$\int_D f(x,y,z) \, dx\, dy\, dz = \int_{\mathbb{R}^2} \left(\int_{\underbrace{\{z : (x,y,z) \in D\}}_{D_{x,y}}} f(x,y,z) \, dz \right) dx\, dy$$

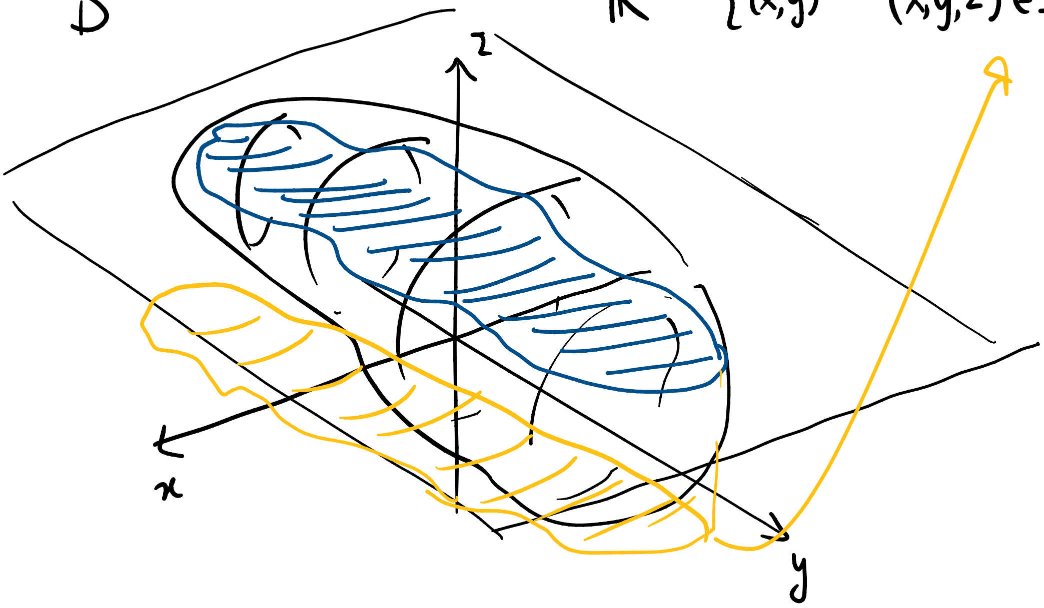
$$\int_D f(x,y) \, dx\, dy = \int_{-\infty}^{+\infty} \int_{D_x} f(x,y) \, dy \quad dx$$



We could also decide to apply RF integrating

We could also decide to apply KT integrating first in (x,y) and then in z

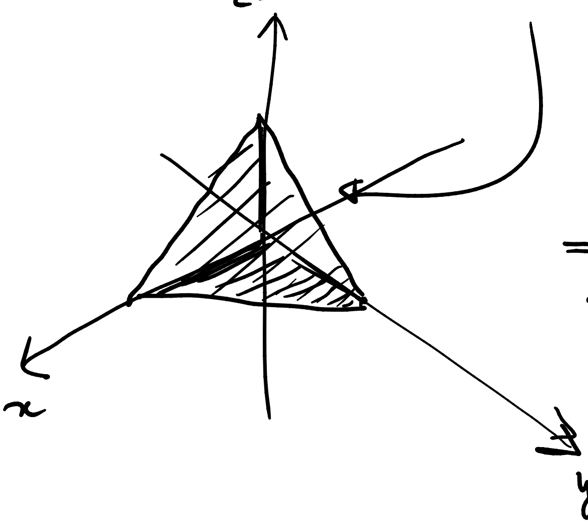
$$\int_D f(x,y,z)\,dx\,dy\,dz = \int_{\mathbb{R}} \left(\int_{\{(x,y)\,:\,(x,y,z)\in D\}} f(x,y,z)\,dx\,dy \right) dz$$



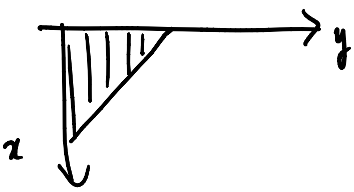
5.5.1.

#5. $\displaystyle\int_{x\geq 0,\ y\geq 0,\ z\geq 0,\ x+y+z\leq 1} xyz\,dx\,dy\,dz$

$0 \leq z \leq 1-(x+y)$



$$= \int_{\substack{x\geq 0,\ y\geq 0\\ x+y\leq 1}} \left(\int_{0\leq z\leq 1-(x+y)} xyz\,dz \right) dx\,dy$$



$$= \int_{\substack{x\geq 0,\ y\geq 0\\ x+y\leq 1}} \left(xy \int_0^{1-(x+y)} z\,dz \right) dx\,dy$$

$$\| $$

$$\left[\frac{z^2}{2}\right]_0^{1-(x+y)}$$

$$= \frac{1}{2}\int_{\substack{x \geq y,\ y \geq 0 \\ x+y \leq 1}} xy\,\big(1-(x+y)\big)^2\,dx\,dy$$

$0 \leq y \leq 1-x$

$$= \frac{1}{2}\int_{\substack{0\leq x \\ 1-x\geq 0}} \int_0^{1-x} xy\big(1+(x+y)^2-2(x+y)\big)\,dy$$

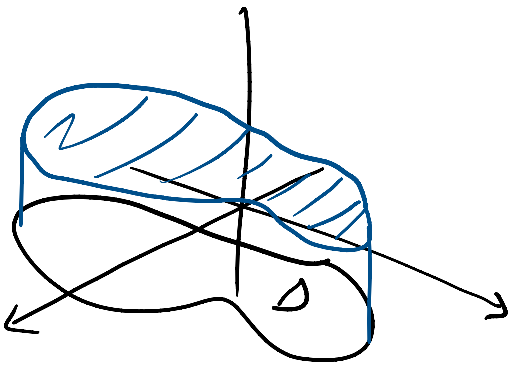
$(1+x^2+y^2+2xy-2x-2y)$

$(xy + x^3y + xy^3 + 2x^2y^2 - 2x^2y - 2xy^2)$

$$= \frac{1}{2}\int_0^1 (x-2x^2+x^3)\,\frac{y^2}{2}\Big|_0^{1-x} + (2x^2-2x)\,\frac{y^3}{3}\Big|_0^{1-x} + x\,\frac{y^4}{4}\Big|_0^{1-x}\,dx$$

$$= \frac{1}{2}\int_0^1 \frac{1}{2}(x-2x^2+x^3)(1-x)^2 + \frac{1}{3}(2x^2-2x)(1-x)^3 + \frac{1}{4}x(1-x)^4\,dx$$

= you finish....



Rmk: $\int_D 1\,dxdy$

$= \lambda_2(D)\cdot 1$

area of D

$\Rightarrow \quad \lambda_2(D) = \int_D 1\,dxdy$

Similarly!

$$\lambda_3(D) = \int_D 1 \, dx\,dy\,dz.$$

volume of D

Example: Volume of a ball of radius r:

$$D = \{ (x,y,z) : x^2 + y^2 + z^2 \leq r^2 \}$$

$$\lambda_3(D) = \int_D 1 \, dx\,dy\,dz = \int_{x^2+y^2+z^2 \leq r^2} dx\,dy\,dz$$

Vol of D

$|z| \leq \sqrt{r^2 - (x^2+y^2)} \Leftrightarrow z^2 \leq r^2 - (x^2+y^2)$ if $x^2+y^2 \leq r^2$

$$= \int_{x^2+y^2 \leq r^2} \left(\int_{-\sqrt{}}^{+\sqrt{}} 1 \, dz \right) dx\,dy$$

$$= \int_{x^2+y^2 \leq r^2} 2\sqrt{r^2 - (x^2+y^2)} \, dx\,dy \qquad ?$$

What if

$$\lambda_3(D) = \int_{-r}^{r} \int_{x^2+y^2 \leq r^2 - z^2} 1 \, dx\,dy \, dz$$

$x^2+y^2+z^2 \leq r^2 \Leftrightarrow x^2+y^2 \leq r^2 - z^2$ if $r^2 - z^2 \geq 0$

$\Updownarrow$

$z^2 \leq r^2$

$\Updownarrow$

$|z| \leq r$

$$\int 1 \, dx\,dy = \lambda_2\left(\{ (x,y) : x^2+y^2 \leq r^2 - z^2 \} \right)$$

$$\int_{x^2+y^2 \le r^2-z^2} 1 \, dx\,dy = \lambda_2\left(\{(x,y) : x^2+y^2 \le r^2-z^2\}\right) \qquad \Downarrow \; |z| \le r$$

disk centered at $(0,0)$
radius $\sqrt{r^2-z^2}$

$$= \pi(r^2-z^2)$$

$$= \int_{-r}^{r} \pi(r^2-z^2)\, dz$$

$$= \pi\Big[\underbrace{r^2\int_{-r}^{r} 1\, dz}_{2r} - \int_{-r}^{r} z^2\, dz\Big]$$

$$\left[\frac{z^3}{3}\right]_{-r}^{r} = \frac{1}{3}\left[r^3-(-r)^3\right] = \frac{2}{3}r^3$$

$$= \pi\left[2r^3 - \frac{2}{3}r^3\right] = \frac{4}{3}\pi r^3. \quad \boxed{}$$

Do 5.5.1 # 4, 6 (long, boring but not difficult)

5.5.2 # 1, # 5 (tricky)

6

5.5.3 # 1,2,3

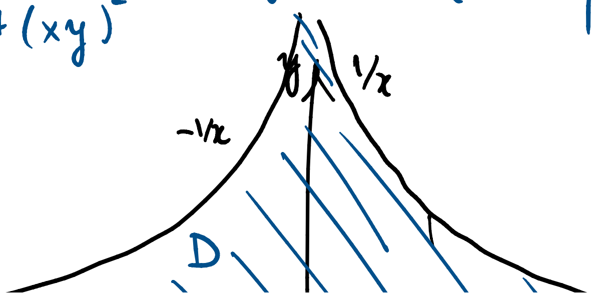
5.5.5

5.5.6 # 4 $\leftarrow$

5.5.3 # 2 $\displaystyle\int_D \frac{x^2 e^{-x^2}}{1+(xy)^2}\, dx\,dy \qquad D = \{(x,y) \in \mathbb{R}^2 : |xy| \le 1\}$

$|xy| \le 1$

$\Updownarrow$



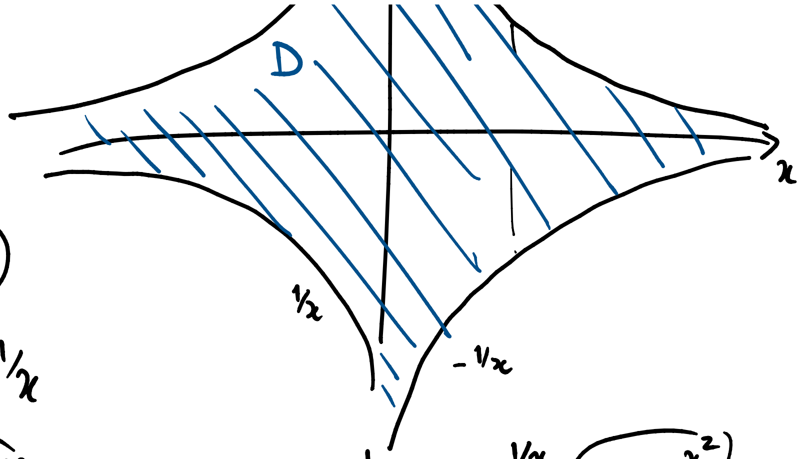
$|xy|$

$\Updownarrow$

$-1 \le xy \le 1$

$\Updownarrow$ $x > 0$ $\qquad$ $\Updownarrow$ $x < 0$

$-\frac{1}{x} \le y \le \frac{1}{x}$ $\qquad$ $\frac{1}{x} \le y \le -\frac{1}{x}$



$$= \int_{-\infty}^{0} \int_{1/x}^{-1/x} \frac{x^2 e^{-x^2}}{1 + (xy)^2}\, dy\, dx + \int_{0}^{+\infty} \left(\int_{-1/x}^{1/x} \frac{x^2 e^{-x^2}}{1+(xy)^2} dy \right) dx$$

$$= \int_{-\infty}^{0} x^{\cancel{2}} e^{-x^2} \int_{1/x}^{-1/x} \frac{x}{1+(xy)^2}\, dy\, dx + \int_{0}^{+\infty} x^{\cancel{2}} e^{-x^2} \left(\int_{-1/x}^{1/x} \frac{x}{1+(xy)^2} dy \right) dx$$

$$\partial_y \operatorname{arctg}(xy) = \frac{1}{1+(xy)^2}\, x$$

$$= \int_{-\infty}^{0} x e^{-x^2} \Big[\operatorname{arctg}(xy) \Big]_{y=1/x}^{y=-1/x} dx +$$

$$+ \int_{0}^{+\infty} x e^{-x^2} \Big[\operatorname{arctg}(xy) \Big]_{y=-1/x}^{y=1/x} dx$$

$$= -\frac{1}{2}\int_{-\infty}^{0} -2x e^{-x^2} \left[-\frac{\pi}{4} - \frac{\pi}{4} \right] dx + \int_{0}^{+\infty} x e^{-x^2} \left[\frac{\pi}{4} - \left(-\frac{\pi}{4} \right) \right] dx$$

$$\partial_x e^{-x^2} = e^{-x^2}(-2x)$$

$$= -\frac{\pi}{2}\left(-\frac{1}{2}\right) \left[e^{-x^2} \right]_{-\infty}^{0} + \frac{\pi}{2}\left(-\frac{1}{2}\right) \left[e^{-x^2} \right]_{0}^{+\infty}$$

$1 - 0$ $\qquad$ $0 - 1$

$$= \frac{\pi}{4} + \frac{\pi}{4} = \pi/2 \quad \blacksquare$$