giovedì 12 marzo 2020 07:34

2) Change of vars.

Recall that

$$\int f(x)\,dx \overset{y=\phi(x),\ x=\phi^{-1}(y),\ dx=(\phi^{-1})'(y)dy}{=} \int f(\phi^{-1}(y))\,(\phi^{-1})'(y)\,dy$$

(here $f=f(x)$, $x\in\mathbb{R}$) (change of var formula for primitives)

By this it follows

$$\int_a^b f(x)\,dx \overset{y=\phi(x)}{=} \int_{\phi(a)}^{\phi(b)} f(\phi^{-1}(y))\,(\phi^{-1})'(y)\,dy$$

$x\in[a,b]$

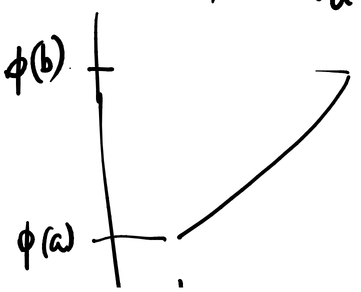
(change of var formula for integrals)

We notice that:

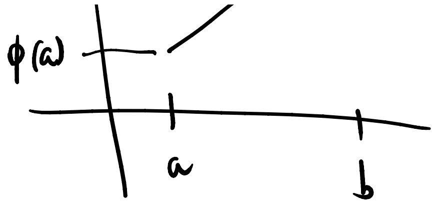
- if $\phi\nearrow$, $\phi(b)>\phi(a)$

$$\phi:[a,b]\longrightarrow[\phi(a),\phi(b)]$$

$$\Rightarrow \int_a^b f(x)\,dx = \int_{[\phi(a),\phi(b)]} f(\phi^{-1}(y))\,(\phi^{-1})'(y)\,dy$$



$$= \int_{\phi([a,b])} f(\phi^{-1}(y))\,\underbrace{(\phi^{-1})'(y)}_{\geq 0}\,dy$$



$$= \int_{\phi([a,b])} \qquad \phi^{-1} \nearrow \quad \geq 0$$

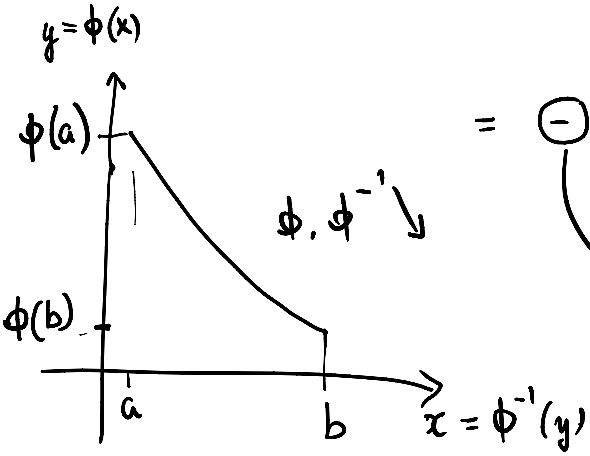
$$= \int_{\phi([a,b])} f(\phi^{-1}(y))\, |(\phi^{-1})'(y)|\, dy.$$

. if $\phi \searrow$, $\phi(b) < \phi(a)$

$$\phi : [a,b] \longrightarrow [\phi(b), \phi(a)] = \phi([a,b])$$

and
$$\int_a^b f(x)\,dx = \int_{\phi(a)}^{\phi(b)} f(\phi^{-1}(y))\, (\phi^{-1})'(y)\,dy$$

$$= - \int_{\phi(b)}^{\phi(a)} f(\phi^{-1}(y))\, (\phi^{-1})'(y)\, dy$$



$\phi, \phi^{-1} \searrow$

$$= - \int_{\phi([a,b])} f(\phi^{-1}(y))\, \underbrace{(\phi^{-1})'(y)}_{\leq 0}\, dy$$

$$= \int_{\phi([a,b])} f(\phi^{-1}(y))\, |(\phi^{-1})'(y)|\, dy$$

We obtained

$$\boxed{y = \phi(x),\ x = \phi^{-1}(y) \qquad \int_{[a,b]} f(x)\,dx = \int_{\phi([a,b])} f(\phi^{-1}(y))\, |(\phi^{-1})'(y)|\, dy}$$

Let's consider $f = f(\vec{x})$, $\vec{x} \in \mathbb{R}^d$ and we have to compute

$$\int_D f(\vec{x})\, d\vec{x}$$

We wish to change var $\vec{x} = \phi(\vec{y})$ where ϕ is invertible $\vec{y} = \phi^{-1}(\vec{x})$ and

$$\phi : \mathbb{R}^d \longrightarrow \mathbb{R}^d$$

$\phi \in \mathcal{C}$ and $\phi' \in \mathcal{C}$, $\phi^{-1} \in \mathcal{C}$, $(\phi^{-1})' \in \mathcal{C}$

Rmk: Here ϕ', $(\phi^{-1})'$ are jacobian matrices,
$d \times d$

Thm: Assume, $\phi, \phi^{-1} \in \mathcal{C}^1$ $\left(\phi, \phi^{-1} \in \mathcal{C},\ \phi', (\phi^{-1})' \in \mathcal{C} \right)$

Then

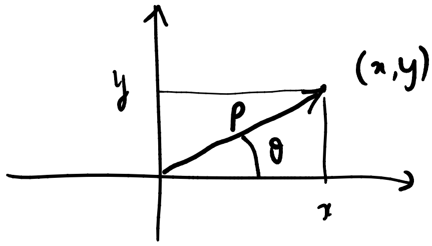
$$\int_D f(\vec{x})\, d\vec{x} \underset{\substack{\vec{y} = \phi(\vec{x}) \\ \vec{x} = \phi^{-1}(\vec{y})}}{=} \int_{\phi(D)} f(\phi^{-1}(\vec{y})) \left| \det (\phi^{-1})'(\vec{y}) \right| d\vec{y}$$

image of D through map ϕ

Special Changes of Variables

I) Polar Coords.

$$\int_D f(x,y)\,dx\,dy =$$



$$\phi^{-1}\begin{cases}\boxed{x=\rho\cos\theta.}\\ y=\rho\sin\theta.\end{cases}\qquad \phi\begin{cases}\rho=\sqrt{x^2+y^2}\\ \theta = \text{more complicate}\end{cases}$$

$$= \int_{\phi(D)} f(\rho\cos\theta,\ \rho\sin\theta)\cdot\left|\det(\phi^{-1})'(\rho,\theta)\right|\,d\rho\, d\theta$$

$$(\phi^{-1})' = \begin{array}{c} \\ \\ \end{array}\begin{array}{cc} \partial_\rho & \partial_\theta \\ \end{array}$$

$$(\phi^{-1})' = \begin{bmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{bmatrix}$$

$$\left|\det(\phi^{-1})'\right| = \left|\det \downarrow \right| = \left|\rho(\cos\theta)^2+\rho(\sin\theta)^2\right|$$

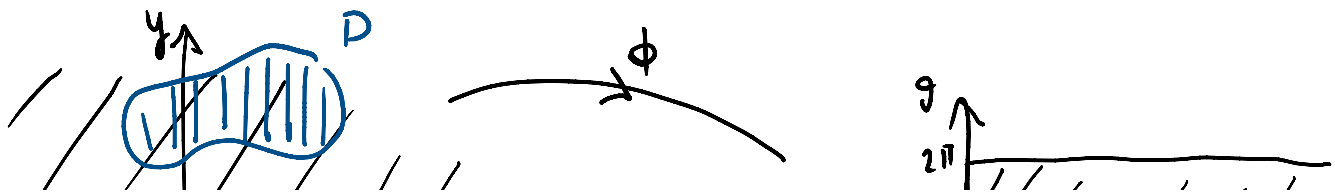
$$= |\rho| \overset{\rho>0}{=} \rho$$

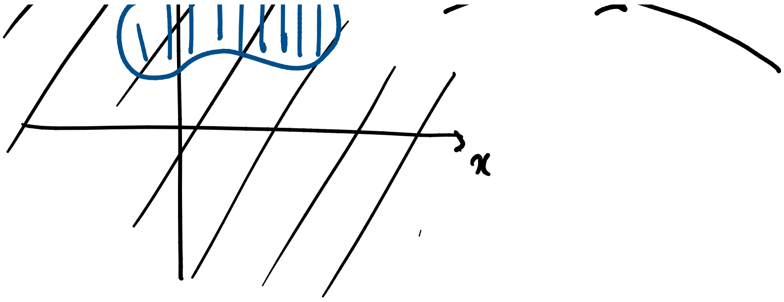
$$\boxed{\left|\det(\phi^{-1})'(\rho,\theta)\right| = \rho}$$

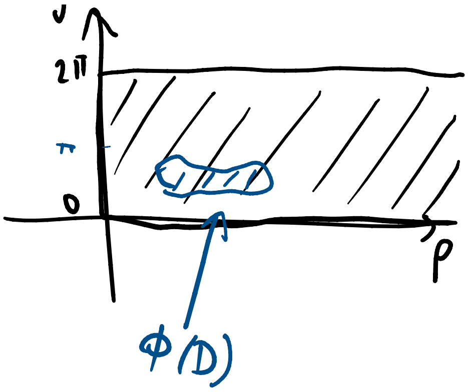
$$\Rightarrow \boxed{\int_D f(x,y)\,dx\,dy = \int_{\phi(D)} f(\rho\cos\theta,\rho\sin\theta)\,\rho\,d\rho\,d\theta}$$

Notice that $\phi : (x,y) \longrightarrow (\rho,\theta)$

$$\mathbb{R}^2 \longmapsto [0,+\infty[\times[0,2\pi]$$







Ex 5.3.4.

Compute $\displaystyle\int_{\mathbb{R}^2} e^{-\sqrt{x^2+y^2}}\,dx\,dy$

$$\underset{\substack{x=\rho\cos\theta\\ y=\rho\sin\theta}}{=} \int_{\substack{\rho\geq 0\\ 0\leq\theta\leq 2\pi}} \underbrace{e^{-\sqrt{\rho^2}}}_{\substack{\parallel\ \rho>0\\ e^{-\rho}}} \cdot \rho\, d\rho\, d\theta$$

$$= \int_{\substack{\rho\geq 0\\ 0\leq\theta\leq 2\pi}} \rho\, e^{-\rho}\, d\rho\, d\theta$$

$$\overset{RF}{=} \int_0^{+\infty} \int_0^{2\pi} \rho e^{-\rho}\, d\theta\, d\rho$$

$$= 2\pi \int_0^{+\infty} \rho\, \underbrace{e^{-\rho}}_{\substack{\parallel\\ \partial_\rho(-e^{-\rho})}} d\rho = 2\pi\Big[\underbrace{-e^{-\rho}\rho\,\Big|_0^{+\infty}}_{\substack{\parallel\\ 0}} - \int_0^{+\infty} 1\cdot(-e^{-\rho})\,d\rho \Big]$$

$$= 2\pi \int_0^{+\infty} \underbrace{e^{-\rho}}_{\substack{\parallel\\ \partial_\rho(-e^{-\rho})}} d\rho = 2\pi\left[-e^{-\rho}\right]_0^{+\infty} = 2\pi\left[0-(-1)\right] = 2\pi \quad \blacksquare$$

Ex. 5.5.7 #2 Compute $\displaystyle\int_{x^2+y^2\leq 4} \sqrt{4-x^2-y^2}\,dx\,dy$

Sol: $\displaystyle\int_{x^2+y^2\leq 4} \sqrt{4-(x^2+y^2)}\,dx\,dy$