Exercises on Multiple Integrals

Ex. 5.5.6

Compute the volume of

#1 $\{(x,y,z) \in \mathbb{R}^3 : 9(1-\sqrt{x^2+y^2})^2 + 4z^2 \leq 1\}$

Sol: The domain has a symmetry around z-axis.
We can plot a section on the plane $x=0$

$$D \cap \{x=0\} = \{ 9(|y|-1)^2 + 4z^2 \leq 1 \}$$

$$= \{ \cancel{4} z^2 \leq \frac{1-9(|y|-1)^2}{4} \}$$

Now $\frac{1-9(|y|-1)^2}{4} \geq 0 \iff 1-9(|y|-1)^2 \geq 0$

$$\iff (|y|-1)^2 \leq 1/9$$

$$\iff -\frac{1}{3} \leq |y|-1 \leq 1/3$$

$$\iff \frac{2}{3} \leq |y| \leq \frac{4}{3}$$

$$\iff -\frac{4}{3} \leq y \leq -\frac{2}{3}, \quad \frac{2}{3} \leq y \leq \frac{4}{3}$$

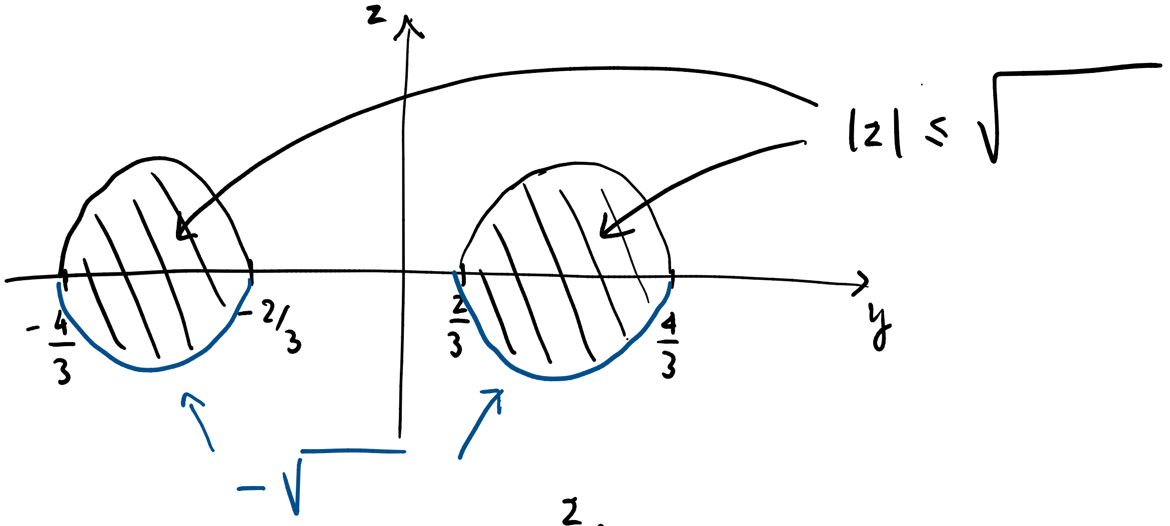
For these y:

$$|z| \leq \sqrt{\frac{1-9(|y|-1)^2}{4}}$$

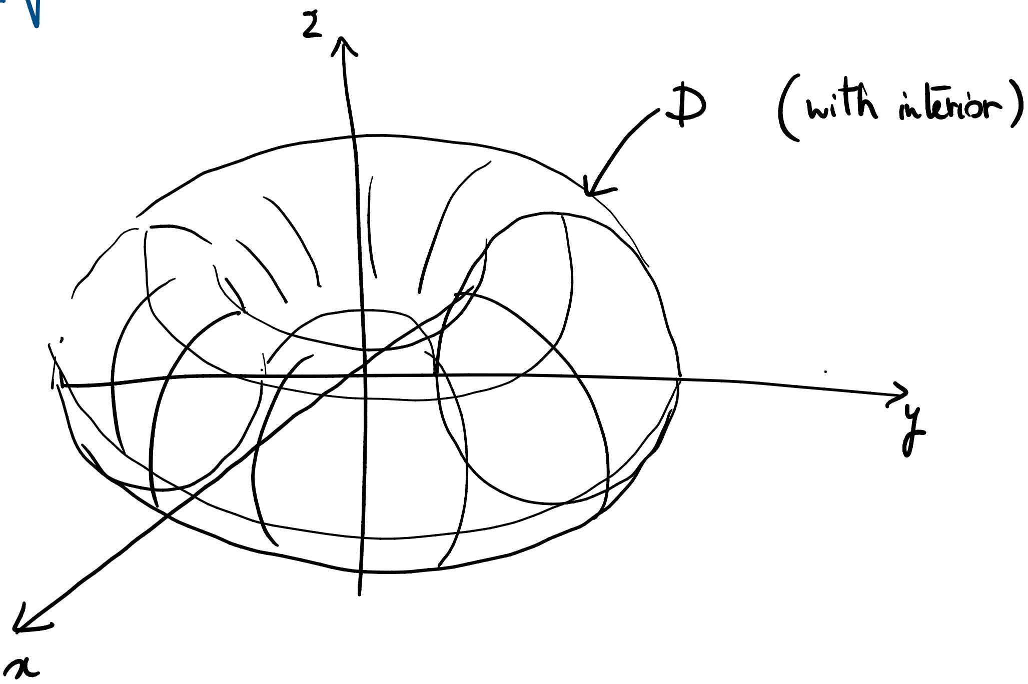
$$4$$

$$\Leftrightarrow \quad -\sqrt{} \leq z \leq \sqrt{}$$

Let's plot $\quad z = +\sqrt{\dfrac{1-9(|y|-1)^2}{4}}$



Therefore:



To compute $\quad \operatorname{Vol} D = \displaystyle\int_D 1 \, dx\,dy\,dz$

we use cylindrical coords:

$$= \int_{9(\rho-1)^2+4z^2 \leq 1} \rho \, d\rho \, d\theta \, dz$$

$$\begin{cases} x = \rho\cos\theta \\ \ldots \end{cases}$$

$$\begin{cases} x = \rho\cos\vartheta \\ y = \rho\sin\vartheta \\ z = z \end{cases} \qquad \begin{aligned} &9(\rho-1)^2 + 4z^2 \leq 1 \\ &0 \leq \rho, \quad z \in \mathbb{R} \\ &0 \leq \vartheta \leq 2\pi \end{aligned}$$

$$\overset{RF}{=} \int_{\substack{9(\rho-1)^2+4z^2 \leq 1 \\ 0 \leq \rho,\ z \in \mathbb{R}}} \rho \underbrace{\int_0^{2\pi} d\vartheta}_{\overset{\shortparallel}{2\pi}} \, d\rho\, dz$$

$$= 2\pi \int_{\substack{9(\rho-1)^2+4z^2 \leq 1 \\ (\rho \geq 0)}} \rho \, d\rho\, dz$$

$(\rho \geq 0) \leftarrow$ never forget this.

$$\overset{RF}{=} 2\pi \int \left(\int \rho \, d\rho \right) dz \overset{(*)}{=}$$

$\uparrow \; ? \; \uparrow$

$$9(\rho-1)^2 + 4z^2 \leq 1 \iff 9(\rho-1)^2 \leq 1 - 4z^2$$

$\Downarrow$

we must impose

$1 - 4z^2 \geq 0$

$\Updownarrow$

$z^2 \leq 1/4$

$\Updownarrow$

otherwise there're not ρ: $|z| \leq 1/2$

Thus, for $|z| \leq 1/2$, $\quad (\rho-1)^2 \leq \dfrac{1-4z^2}{9}$

$\Updownarrow$

Thus, for ... 9

$$\Updownarrow$$

$$|\rho - 1| \le \sqrt{\frac{1-4z^2}{9}} = \frac{1}{3}\sqrt{1-4z^2}$$

$$\Updownarrow$$

$$-\frac{1}{3}\sqrt{} \le \rho - 1 \le \frac{1}{3}\sqrt{}$$

$$\Updownarrow$$

$$1 - \frac{1}{3}\sqrt{1-4z^2} \le \rho \le 1 + \frac{1}{3}\sqrt{1-4z^2}$$

Is this the range for ρ? We should check if $\rho \ge 0$! Thus

$$1 - \frac{1}{3}\sqrt{1-4z^2} \overset{?}{\ge} 0$$

$$\Leftrightarrow \quad 3 \ge \sqrt{1-4z^2}$$

$$\Leftrightarrow \quad 9 \ge 1 - 4z^2 \quad \Leftrightarrow \quad 4z^2 \ge -8 \quad \text{yes!}$$

Therefore

$$\overset{(*)}{=} 2\pi \int_{-1/2}^{1/2} \left(\int_{1-1/3\sqrt{}}^{1+1/3\sqrt{}} \rho \, d\rho \right) dz$$

$$\| $$

$$\left[\frac{\rho^2}{2} \right]_{1-1/3\sqrt{}}^{1+1/3\sqrt{}}$$

$$= \pi \int_{-1/2}^{1/2} \left(1 + \frac{1}{3}\sqrt{}\right)^2 - \left(1 - \frac{1}{3}\sqrt{}\right)^2 dz$$

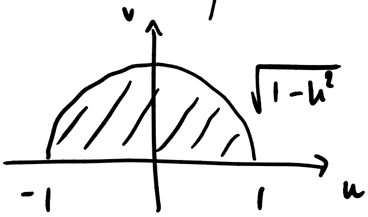
$$\cancel{1} + \cancel{\frac{1}{9}(1-4z^2)} + \frac{2}{3}\sqrt{} - \left(\cancel{1} + \cancel{\frac{1}{9}(1-4z^2)} - \frac{2}{3}\sqrt{}\right)$$

$$= \pi \int_{-1/2}^{1/2} \frac{4}{3} \sqrt{1 - \underset{(2z)^2}{\underbrace{4z^2}}}\, dz$$

$$= \frac{2}{3}\pi \int_{-1}^{1} \underset{\frac{1}{2}\pi \cdot 1^2}{\underbrace{\sqrt{1-u^2}\, du}} = \frac{\pi^2}{3}.$$

$u = 2z$
$du = 2\,dz$

This is the standard int. leading to the area of $1/2$ disk:

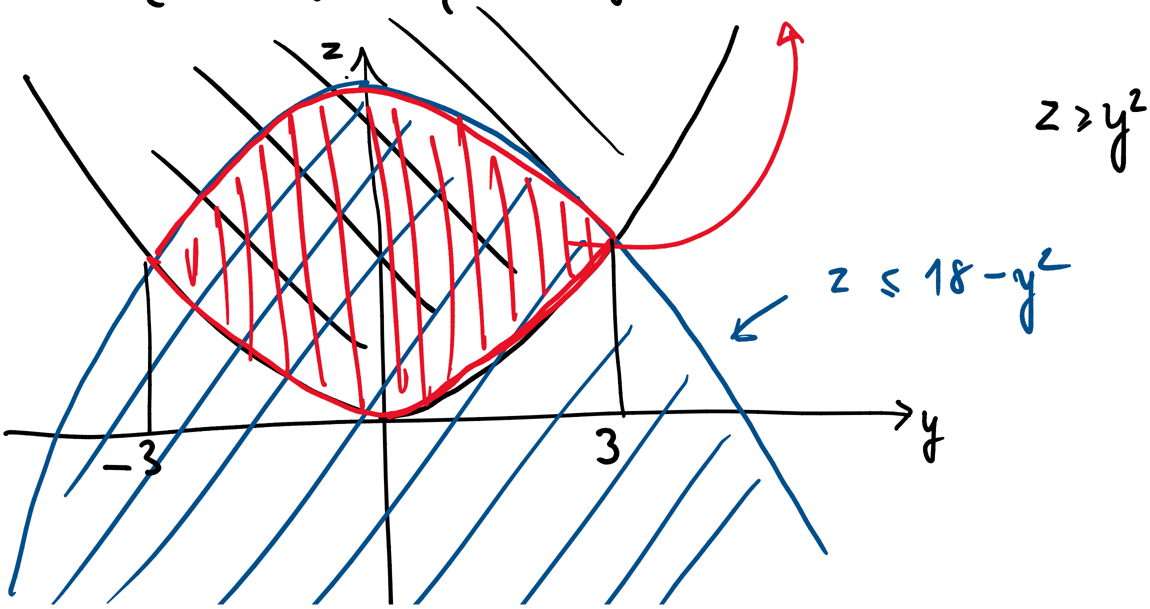


$\blacksquare$

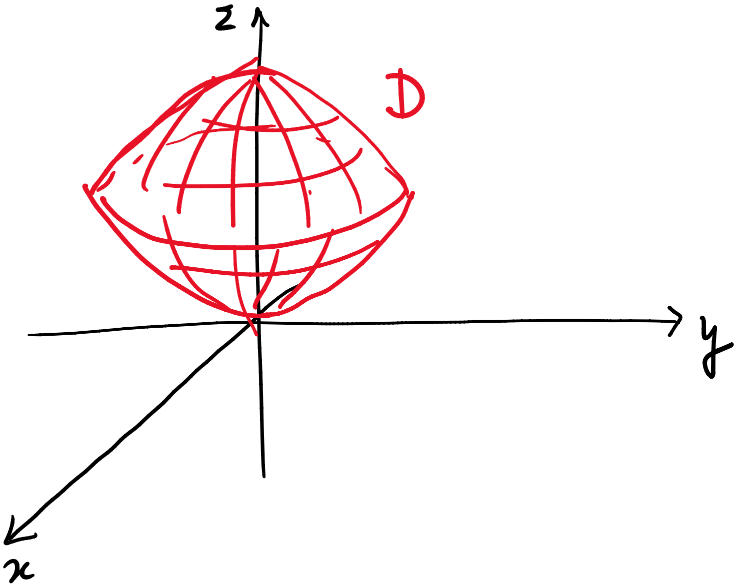
#8. $\{ (x,y,z) \in \mathbb{R}^3 : z \geq x^2 + y^2, \; z \leq 18 - x^2 - y^2 \}$

Sol: Also in this case we've a domain invariant for rotations around z-axis.

$$D \cap \{x = 0\} = \{ z \geq y^2, \; z \leq 18 - y^2 \}$$



$$y^2 = 18 - y^2 \iff 2y^2 = 18 \iff y^2 = 9 \iff y = \pm 3$$



$$\mathrm{Vol}\, D = \int_D 1 \, dx\, dy\, dz$$

$$\underset{\text{cyl coords}}{=} \int_{\substack{z \geq \rho^2, \\ z \leq 18 - \rho^2 \\ \rho \geq 0,\, 0 \leq \theta \leq 2\pi}} \rho \, d\rho\, d\theta\, dz$$

$$\overset{RF}{=} 2\pi \int_{\underbrace{\rho^2 \leq z \leq 18 - \rho^2}_{< ?}} \rho \, d\rho\, dz$$

$$\rho^2 \leq 18 - \rho^2 \iff \rho^2 \leq 9 \iff |\rho| \leq 3$$

$$\overset{\rho \geq 0}{\iff} 0 \leq \rho \leq 3$$

$$\overset{RF}{=} 2\pi \int_0^3 \left(\int_{\rho^2}^{18-\rho^2} \rho \, dz \right) d\rho$$

$$= 2\pi \int_0 \left(\int_{\rho^2} \rho \, dz \right) d\rho$$

$$= 2\pi \int_0^3 \rho \,(18 - 2\rho^2)\, d\rho$$

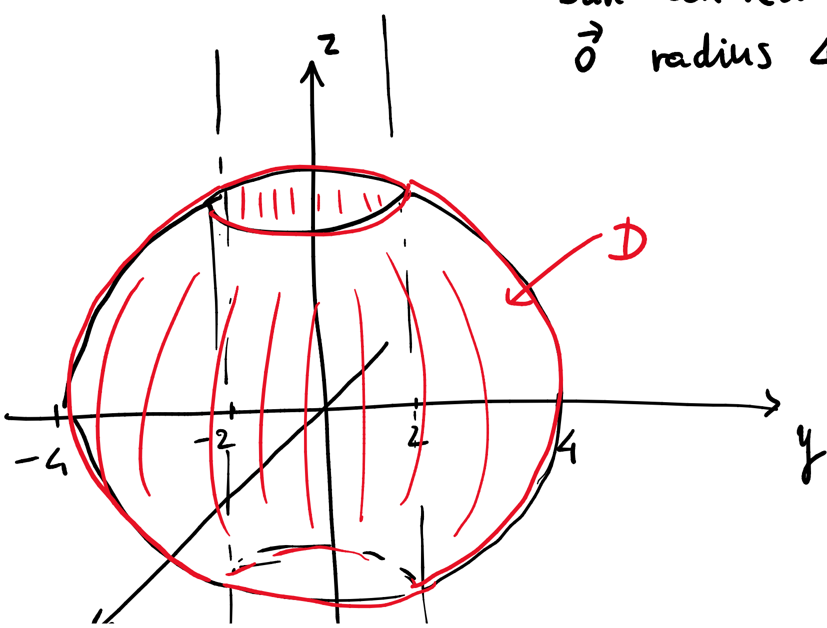
$$= 2\pi \int_0^3 18\rho - 2\rho^3 \, d\rho$$

$$= 2\pi \left[18 \left.\frac{\rho^2}{2}\right|_0^3 - 2 \left.\frac{\rho^4}{4}\right|_0^3 \right]$$

$$= 2\pi \left[\underbrace{9 \cdot (9 - 0) - \frac{1}{2}(3^4 - 0)}_{81 \; - \; \frac{81}{2} \; = \; \frac{81}{2}} \right]$$

$$= 81 \cdot \pi. \qquad \blacksquare$$

#6. $\{ (x,y,z) \in \mathbb{R}^3 : \underbrace{x^2 + y^2 + z^2 \leq 16}_{\text{ball centred at } \vec{0} \text{ radius } 4}, \; \underbrace{x^2 + y^2 \geq 4}_{\text{exterior of a cyl. with axis the z-axis radius 2.}} \}$



$$\mathrm{Vol}\, D = \int_D 1 \, dx\,dy\,dz$$

$$\underset{\text{cyl coords}}{=} \int_{\substack{\rho^2+z^2 \leq 16 \\ \rho^2 \geq 4 \\ 0 \leq \theta \leq 2\pi}} \rho \, d\rho\, d\theta\, dz$$

$\rho^2 \geq 4 \underset{\rho \geq 0}{\Leftrightarrow} \rho \geq 2$

$z^2 \leq \underbrace{16 - \rho^2}_{\text{must be } \geq 0} \Rightarrow 16 - \rho^2 \geq 0 \Leftrightarrow \rho^2 \leq 16 \underset{\rho \geq 0}{\Leftrightarrow} 0 \leq \rho \leq 4$

$2 \leq \rho \leq 4$

$$\overset{RF}{=} 2\pi \int_2^4 \left(\int_{-\sqrt{16-\rho^2}}^{\sqrt{16-\rho^2}} \rho \, dz \right) d\rho$$

$$= \frac{\cancel{4}\pi}{-3} \int_2^4 -3\rho \cdot \cancel{2} \sqrt{16-\rho^2} \, d\rho$$

$\sqrt{16-\rho^2} = (16-\rho^2)^{1/2}$

$$\partial_\rho (16-\rho^2)^{3/2} = \frac{3}{\cancel{2}} (16-\rho^2)^{1/2} \cdot (-\cancel{2}\rho)$$

$$= -\frac{4}{3}\pi \int_2^4 \partial_\rho (16-\rho^2)^{3/2} \, d\rho$$

$$= -\frac{4}{3}\pi \int_2 \partial_\rho (16-\rho^2)^{3/2}\, d\rho$$

$$\| $$

$$\left[(16-\rho^2)^{3/2} \right]_2^4$$

$$= -\frac{4}{3}\pi \left[0 - 12^{3/2} \right] = \frac{4}{3} 12^{3/2} \pi. \quad \blacksquare$$

#7 (modified) $\left\{ (x,y,z) \in \mathbb{R}^3 : x \geq \sqrt{y^2+z^2},\ x^2+y^2+z^2 \leq 1 \right\}$

(no significative changes.)

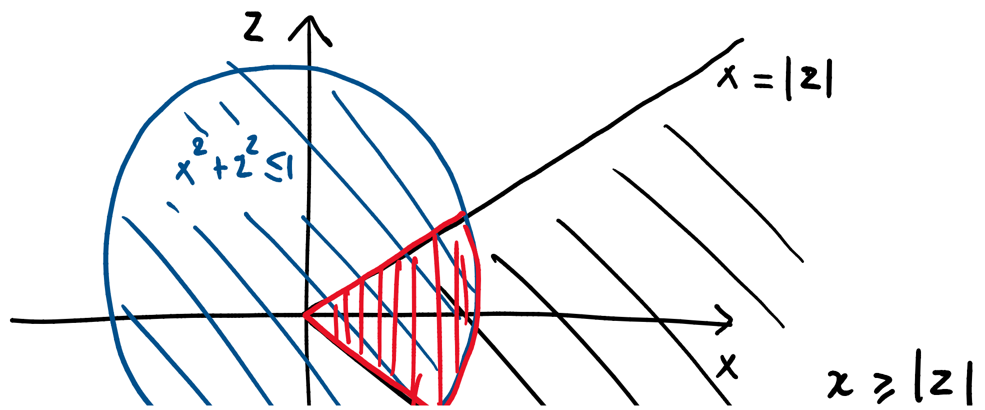
$x \geq \sqrt{y^2+z^2}$ is rotation invariant for the x-axis

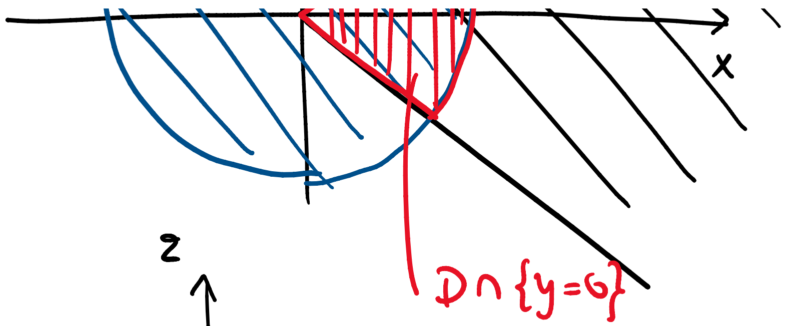
distance to x-axis.

$x^2+y^2+z^2 \leq 1$ is also rot. inv. for the x-axis.

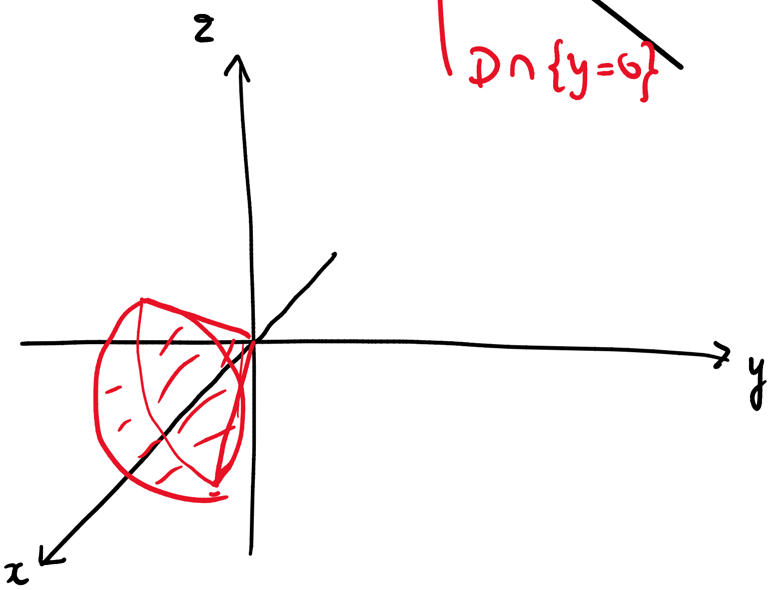
$\Rightarrow$ we plot $D \cap \{y=0\}$ (plane xz)

$$D \cap \{y=0\} = \left\{ x \geq \sqrt{z^2} = |z|,\quad x^2+z^2 \leq 1 \right\}$$





$x \geq |z|$



$$\mathrm{Vol}\, D = \int_D 1 \, dx\,dy\,dz$$

$$\underset{\text{cyl. coord to } x\text{-axis:}}{=} \int_{\substack{x \geq \rho \\ x^2+\rho^2 \leq 1 \\ 0 \leq \theta \leq 2\pi}} \rho \, d\rho\, d\theta\, dx$$

$$\begin{cases} x = x \\ y = \rho \cos\theta \\ z = \rho \sin\theta \end{cases}$$

$$y^2 + z^2 = \rho^2$$

$$\overset{RF}{=} 2\pi \int_{\substack{x \geq \rho \\ x^2+\rho^2 \leq 1}} \rho \, d\rho\, dx$$

$\boxed{\rho \leq x}$, $\quad \hookrightarrow \quad x^2 \leq \underbrace{1-\rho^2}_{+} \Rightarrow \rho \leq 1$

① $\Downarrow$

$\underbrace{\qquad\qquad}_{①}$ $\qquad$ $\Updownarrow$ $\underbrace{\quad}_{+} \Rightarrow \rho \leq 1$

$$|x| \leq \sqrt{1-\rho^2}$$

$$\Updownarrow$$

$$-\underbrace{\sqrt{1-\rho^2}}_{\ominus} \leq x \leq \sqrt{1-\rho^2}$$

$$\Downarrow$$

$$\underbrace{\rho \quad \leq x \leq \sqrt{1-\rho^2}}_{\leq ?}$$

$$\Leftrightarrow \quad \rho \leq \sqrt{1-\rho^2} \quad \Leftrightarrow \quad \rho^2 \leq 1-\rho^2 \quad \Leftrightarrow \quad 2\rho^2 \leq 1$$

$$\Leftrightarrow \quad \rho^2 \leq 1/2$$

$$\Leftrightarrow \quad |\rho| \leq 1/\sqrt{2}$$

$$\overset{\rho \geq 0}{\Leftrightarrow} \quad 0 \leq \rho \leq \frac{1}{\sqrt{2}}$$

$$\overset{RF}{=} \quad 2\pi \int_0^{1/\sqrt{2}} \left(\int_\rho^{\sqrt{1-\rho^2}} \rho \; dx \right) d\rho$$

$$= 2\pi \int_0^{1/\sqrt{2}} \underbrace{\rho\left(\sqrt{1-\rho^2} - \rho\right)}_{\rho(1-\rho^2)^{1/2} - \rho^2} d\rho$$

$$2\pi\left[\; 1 \int^{1/\sqrt{2}} \quad \rho(1-\rho^2)^{1/2} \, d\rho \qquad \int^{1/\sqrt{2}} \rho^2 \, d\rho \right]$$

$$= 2\pi\left[-\frac{1}{3}\int_0^{1/\sqrt{2}} \underbrace{-3\rho(1-\rho^2)^{1/2}}_{\partial_\rho (1-\rho^2)^{3/2}} d\rho - \int_0^{1/\sqrt{2}} \underbrace{\rho^2}_{\partial_\rho \frac{\rho^3}{3}} d\rho \right]$$

$$= 2\pi\left[-\frac{1}{3}\left[(1-\rho^2)^{3/2}\right]_0^{1/\sqrt{2}} - \frac{1}{3}\left[\rho^3\right]_0^{1/\sqrt{2}} \right]$$

$$= \frac{2\pi}{3}\left[-\left(\frac{1}{2^{3/2}} - 1\right) - \frac{1}{2^{3/2}} \right]$$

$$= \frac{2}{3}\pi\left[1 - \frac{2}{2^{3/2}} \right] = \frac{2}{3}\pi\left[1 - \frac{1}{\sqrt{2}}\right] \quad \blacksquare$$