First Order Linear Eqns

An eqn of type

$$\boxed{y'(t) = a(t)y(t) + b(t)} \quad (EQ)$$

is called first order linear eqn.

If $b(t) \equiv 0$ $\left(y'(t) = a(t)y(t) \right)$ the eqn is said to be homogeneous

if $b \neq 0 \Rightarrow$ non homogeneous.

Functs $a = a(t)$, $b = b(t)$ are supposed to be known,

$$a, b \in \mathcal{C}(I) \quad I \subset \mathbb{R} \text{ interval.}$$

Homogeneous Case

We consider first the case of eqns like

$$y'(t) = a(t)y(t)$$

Idea: $y'(t) = a(t)y(t) \overset{y \neq 0}{\Longleftrightarrow} \dfrac{y'(t)}{y(t)} = a(t)$

Idea: $\quad y'(t) = a(t)y(t) \iff \dfrac{y'(t)}{y(t)} = a(t)$

$$\frac{y'(t)}{y(t)} = (\log|y(t)|)'$$

$$\iff \quad (\underbrace{\log|y(t)|}_{\text{is a primitive of } a(t)})' = a(t)$$

$t \in I$

$$\iff \quad \log|y(t)| = \int a(t)\,dt + c, \qquad c \in \mathbb{R}$$

$$\iff \quad |y(t)| = e^{\int a(t)\,dt + c}$$

$$= e^{c}\, e^{\int a(t)\,dt}$$

$$|y(t)| = \underbrace{C\, e^{\int a(t)\,dt}}_{+} \qquad C > 0$$

$$\iff \quad y(t) = \pm C\, e^{\int a(t)\,dt}$$

$$\boxed{y(t) = C\, e^{\int a(t)\,dt} \qquad C \in \mathbb{R}\setminus\{0\}}$$

Are these all the possible sols of

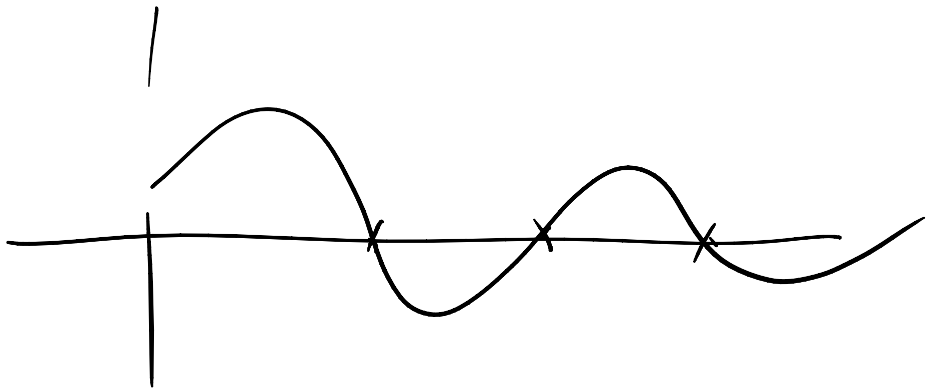
$$y'(t) = a(t)\,y(t) \;?$$

<u>NO</u>: also $y(t) \equiv 0$ is a sol.

$$\left(y' \equiv 0 = a(t)\,y(t) \equiv 0 \right)$$

$$\Rightarrow \quad \boxed{y(t) = C\,e^{\int a(t)\,dt} \qquad C \in \mathbb{R}}$$

are sols of $y' = ay$.



<u>Thm</u>: All the sols of

$$y'(t) = a(t)\,y(t)$$

are

$$y(t) = C\,e^{\int a(t)\,dt} \qquad C \in \mathbb{R}$$

$$y(t) = C e^{\int a(t)\,dt}, \qquad C \in \mathbb{R}.$$

(general sol. / general integral)

Proof: Assume y be a sol of

$$y' = ay \qquad \left(\begin{array}{l} a = a(t) \\ y = y(t) \end{array} \right)$$

$$\Updownarrow$$

$$\boxed{y' - ay = 0}$$

Define $A := \int a$ $\qquad A(t) = \int a(t)\,dt$

$$\boxed{A' = a}$$

$$e^{-A}(y' - ay) = 0$$

$$\parallel$$

$$e^{-A} y' \boxed{- e^{-A} A'} y = 0$$

$$e^{-A} y' + (e^{-A})' y = 0$$

$$\Uparrow$$

$$\Updownarrow$$

$$\left(e^{-A} y\right)' = 0 \quad t \in I$$

interval

$$\Updownarrow$$

$$e^{-A} y = C \quad \text{constant}$$

$$\Updownarrow$$

$$y = C e^{A} = C e^{\int a(t)\,dt} \quad \blacksquare$$

Non homogeneous Case

Let's consider now the eqn

$$y' = ay + b \quad (*) \qquad \begin{pmatrix} a = a(t) \\ b = b(t) \\ y = y(t) \end{pmatrix}$$

Proposition: Suppose we know just a special sol $\bar{U}$ of $(*)$. Then the general sol of $(*)$ is

$$\boxed{y(t) = C e^{\int a(t)\,dt} + \bar{U}(t), \quad C \in \mathbb{R}.}$$

Proof: Because $\bar{U}$ is sol of $(*)$

$$U' = aU + b$$

If y is any other sol of $(*)$

$$y' = ay + b$$

$$\Rightarrow \quad y' - U' = (ay + \cancel{b}) - (aU + \cancel{b})$$

$$\boxed{(y - U)' = a(y - U)}$$

$\Rightarrow \quad y - \bar{U}$ is sol of homog eq

$$\Rightarrow \quad y - U = C e^{\int a(t)\,dt}$$

$$\Rightarrow \quad y = C e^{\int a(t)\,dt} + U. \qquad \blacksquare$$

How can we determine a particular/special sol of

$$y' = ay + b \quad ?$$

Lagrange variation of const. formula :

Lagrange variation of const. formula:

$$U(t) = C(t)\, e^{\int a(t)\,dt}$$

let's det $C = C(t)$ imposing that U be a sol of $y' = ay + b$.

If we impose $U' = aU + b$

$$\Updownarrow$$

$$\left(C\cdot e^{\int a}\right)' = a\,C\,e^{\int a} + b \quad ?$$

$$\parallel$$

$$C'\,e^{\int a} + C\left(e^{\int a}\right)'$$

$$\parallel$$

$$C'\,e^{\int a} + C\,e^{\int a}\cdot a$$

$$C'\,e^{\int a} + \cancel{a\,C\,e^{\int a}} = \cancel{a\,C\,e^{\int a}} + b$$

$$\Leftrightarrow \quad C'\,\underset{\neq 0}{e^{\int a}} = b$$

$$\Leftrightarrow \boxed{C' = b e^{-\int a}}$$

$$\Rightarrow C(t) = \int \left(b(t) e^{-\int a(t)\,dt} \right) dt + \cancel{c}$$

Conclusion:

$$U(t) = \underbrace{\left(\int b(t) e^{-\int a(t)\,dt}\, dt \right)}_{C(t).} e^{\int a(t)\,dt}$$

Final conclusion:

Thm: Let $a, b \in \mathcal{C}(I)$ $I \subset \mathbb{R}$ interval. The general sol of

$$y' = ay + b \qquad \begin{pmatrix} a = a(t) \\ b = b(t) \\ y = y(t) \end{pmatrix}$$

is

$$y(t) = C e^{\int a(t)\,dt} + \left(\int b(t) e^{-\int a(t)\,dt}\, dt \right) e^{\int a(t)\,dt}$$

$$y(t) = C e^{\int a(t)\,dt} + \left(\int b(t)\, e^{-\int a(t)\,dt}\, dt \right) e^{\int a(t)\,dt}$$

$$= e^{\int a(t)\,dt} \left(C + \int \left(b(t)\, e^{-\int a(t)\,dt} \right) dt \right)$$

$$C \in \mathbb{R}.$$

Example 1.2.4

Find the general sol. of

$$y'(t) - \frac{2}{t} y(t) = 1, \qquad t \in \underbrace{]0, +\infty[}_{= I}.$$

Sol: We have a first ord. lin. non hom eqn

$$y' = \underbrace{\frac{2}{t}}_{= a(t)} y + \underbrace{1}_{= b(t)}$$

$$a(t) = \frac{2}{t} \in \mathcal{C}(]0, +\infty[)$$

$$b(t) \equiv 1 \in \mathcal{C}(]0, +\infty[)$$

The gen sol of the eqn is

$$\int a \quad \Big(\quad \int \quad -\int a \quad \Big)$$

$$y(t) = e^{\int a}\left(\int b e^{-\int a} + c\right)$$

$$= e^{\int \frac{2}{t}\,dt}\left(\int 1 \cdot e^{-\int \frac{2}{t}\,dt}\,dt + C\right)$$

$$= e^{\cancel{2}\log|t|^2}\left(\int e^{+\cancel{2}\log|t|^{-2}}\,dt + C\right) \qquad t>0$$

$$= t^2\left(\int \frac{1}{t^2}\,dt + C\right)$$

$$= t^2\left(-\frac{1}{t} + C\right) = -t + Ct^2. \quad C \in \mathbb{R}.$$

Ex 1.5.1. Find the gen integral of

5. $y' - (\operatorname{tg} t)y = t^3 \qquad t \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[.$

Sol: We have a first ord. lin. non hom. eqn

$$y' = \underbrace{(\operatorname{tg} t)}_{a(t)} y + \underbrace{t^3}_{b(t)}$$

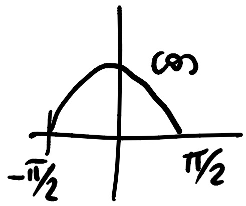
Of course $a, b \in \mathscr{C}(]-\pi/2, \pi/2[)$ and the gen int is

$$y = e^{\int a}\left(\int b e^{-\int a} + C\right)$$

$$= e^{\int \operatorname{tg} t \, dt} \left(\int t^3 e^{-\int \operatorname{tg} t \, dt} \, dt + C \right)$$

$$\int \operatorname{tg} t \, dt = -\int \frac{-\sin t}{\cos t} \, dt = -\log|\cos t|$$

$$= -\log(\cos t) \qquad t \in \left]-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right[$$



$$= e^{-\log(\cos t)} \left(\int t^3 e^{+\log(\cos t)} \, dt + C \right)$$

$$= \frac{1}{\cos t} \left(\int t^3 \cos t \, dt + C \right)$$

$$\int t^3 \cos t \, dt \, dt = \int t^3 (\sin t)' \, dt$$

$$= t^3 \sin t + \int 3t^2 \big(\overset{\overset{(\cos t)'}{\|}}{-\sin t} \big) \, dt$$

$$= t^3 \sin t + 3 \left(t^2 \cos t - \int 2t \cos t \, dt \right)$$

$$= t^3 \sin t + 3t^2 \cos t - 6 \int t \overset{\overset{(\sin t)'}{\|}}{\cos t} \, dt$$

$$= t^3 \sin t + 3t^2 \cos t - 6 \left[t \sin t - \underset{\underset{+\cos t}{\|}}{\int \sin t} \right]$$

$$= t^3\sin t + 3t^2\cos t - 6t\sin t - 6\cos t$$

$$= \frac{1}{\cos t}\left[t^3\sin t + 3t^2\cos t - 6t\sin t - 6\cos t + C\right]$$

Do Ex 1.5.1 #1, #6, #9
(do all).

Cauchy Problem

The Cauchy Pb consists in solving a diff eqn plus a passage condition / initial value cond

For a first order lin eqn this consists in

$$CP(t_0, y_0) \quad \begin{cases} y'(t) = a(t)y(t) + b(t) & t \in I \\ y(t_0) = y_0 & t_0 \in I \\ & y_0 \in \mathbb{R} \end{cases}$$

(t_0, y_0) is the initial point. (known)

Thm: In the case of first order linear eqns,

Thm: In the case of first order linear eqns, the C.P. has always a <u>unique</u> sol. $\forall (t_0, y_0)$ initial point.

Proof: We know that the gen sol of $y' = ay + b$ has the form

$$y(t) = C e^{\int a} + U(t)$$

part sol.

$$y(t) = C e^{A(t)} + U(t) \qquad C \in \mathbb{R}$$

If we want y be a sol of $CP(t_0, y_0)$

we need

$$y(t_0) = y_0 \iff y_0 = C e^{A(t_0)} + U(t_0)$$

t_0

$$\iff C = \frac{y_0 - U(t_0)}{e^{A(t_0)}}$$

$\Rightarrow$ there's a unique possible solution. $\square$

Ex 1.2.6 Solve

$$\begin{cases} y' - \frac{2y}{1-t^2} = t & t > 1 \end{cases}$$

$$\begin{cases} y' - \dfrac{2y}{1-t^2} = t & t>1 \\ y(2) = 0 \end{cases}$$

Sol: Let's first det the gen sol. The eqn is a first order lin eqn

$$y' = \underbrace{\frac{2}{1-t^2}}_{a(t)} y + \underbrace{t}_{b(t)}$$

$a, b \in \mathcal{C}(]1,+\infty[)$

The gen sol/int is

$$y(t) = e^{\int \frac{2}{1-t^2}dt}\left(\int t \cdot e^{-\int \frac{2}{1-t^2}dt} dt + c\right)$$

$$\int \frac{2}{1-t^2}dt = \int \frac{2}{(1-t)(1+t)} dt$$

$$= \int \frac{1}{1-t} + \frac{1}{1+t} dt$$

$$= \log|1+t| - \log|1-t|$$

$$= \log\left|\frac{1+t}{1-t}\right|$$

$$= \log\left|\frac{1+t}{1-t}\right|$$

$$\Rightarrow y(t) = e^{\log\left|\frac{1+t}{1-t}\right|}\left(\int t\, e^{-\log\left|\frac{1+t}{1-t}\right|}\,dt + C\right)$$

$$= \left|\frac{1+t}{1-t}\right|\left(\int t\left|\frac{1-t}{1+t}\right|\,dt + C\right)$$

$t>1$

$$= \frac{1+t}{t-1}\left(\int t^{+1-1}\,\frac{t-1}{t+1}\,dt + C\right)$$

$$\parallel$$

$$\int t-1 - \frac{t-1}{t+1}$$

$$+ \frac{1}{t+1} - \frac{t+1-1}{t+1}$$

$$\int\left(t-2 + \frac{2}{t+1}\right)$$

$$= \frac{t+1}{t-1}\left(t^2 - 2t + 2\log|t+1| + C\right)$$

$$\overset{t>1}{=} \frac{t+1}{t-1}\left(t^2 - 2t + 2\log(t+1) + C\right)$$

Because $y(2) = 0$ to be sol of CP

we need

$$0 = \cancel{\frac{3}{1}}\left(\cancel{4}-\cancel{4} + 2\log 3 + C\right)$$

$$\Rightarrow \quad c = -2\log 3 \qquad \blacksquare$$

Do Ex 1.5.2/3/4/5

1.5.3. Consider the eqn

$$y' - (\operatorname{tg} t)y = \frac{1}{\sin t} \qquad t \in]0, \pi/2[$$

i) Find the gen int

ii) Is it true that for every sol $\lim_{t \to 0+} y(t) = -\infty$?

iii) Are there sols such that $\lim_{t \to \pi/2 -} y(t) \in \mathbb{R}$?

Sol: i) We have a first order linear eqn

$$y' = \underbrace{(\operatorname{tg} t)}_{a(t)} y + \underbrace{\frac{1}{\sin t}}_{b(t)} \qquad t \in]0, \pi/2[=: I.$$

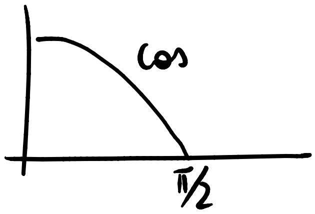
$a, b \in \mathscr{C}(I)$. The gen sol of the eqn is

$$y(t) = e^{\int \operatorname{tg} t \, dt} \left(\int \frac{1}{\sin t} e^{-\int \operatorname{tg} t} \, dt + c \right)$$

$$\int \operatorname{tg} t \, dt = -\int \frac{-\sin t}{\cos t} dt = -\log|\cos t| = -\log \cos t$$

$\cos t$

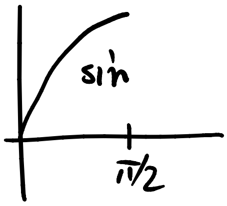
$t \in \left]0, \pi/2\right[$



$$= e^{-\log \cos t}\left[\int \frac{1}{\sin t}\, e^{\log \cos t}\, dt + c\right]$$

$$= \frac{1}{\cos t}\left[\underbrace{\int \frac{\cos t}{\sin t}\, dt}_{=\,\log|\sin t|} + C\right] \qquad t \in \left]0, \pi/2\right[$$

$$= \frac{1}{\cos t}\left[\log \sin t + C\right] \quad C \in \mathbb{R}$$



ii) We have to say if

$$\lim_{t \to 0+} y(t) = -\infty \qquad \forall \text{ sol } y.$$

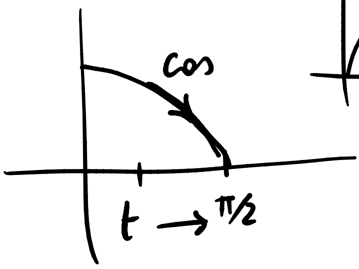
$$\Updownarrow$$

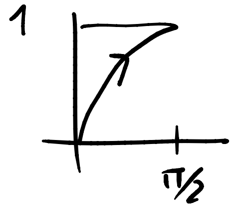
$$\lim_{t \to 0+} \underbrace{\frac{1}{\cos t}}_{\to 1} \underbrace{\left[\overbrace{\log \underbrace{\sin t}_{\to 0+}}^{-\infty} + C\right]}_{\to -\infty} = -\infty \qquad \forall C \in \mathbb{R}.$$

$1 \ | \nearrow$

$$\underset{-\infty}{\downarrow}$$

yes, it is true.





iii) $\lim_{t\to\pi/2^-} y(t) \in \mathbb{R}$?

$$\Updownarrow$$

$$\exists C : \quad \lim_{t\to\pi/2^-} \underbrace{\frac{1}{\underbrace{\cos t}_{0^+}}}_{+\infty} \underbrace{\Big[\underbrace{\log}_{0}(\underbrace{\sin t}_{1}) + C\Big]}_{C} \in \mathbb{R}$$

If $C \neq 0$ $\qquad C\cdot(+\infty) = (\operatorname{sgn} C)\infty \Rightarrow \lim_{t\to\pi/2^-} y \notin \mathbb{R}$

If $C = 0$ $\qquad 0(+\infty)$ I.F.

$$\Downarrow$$

we have $\displaystyle\lim_{t\to\pi/2^-} \overbrace{\frac{1}{\cos t}\log \sin t}^{y(t) \text{ when } C=0}$

$$= \lim_{t\to\pi/2^-} \frac{\log(\sin t)}{\cos t} \overset{\frac{0}{0}}{\underset{H}{=}}$$

$$= \lim_{t\to\pi/2^-} \frac{\frac{1}{\sin t}\cdot\cos t}{-\sin t}$$

$$\cos t \to 0$$

$$= \lim_{t \to \pi/2 -} - \frac{\cos t}{(\sin t)^2} = 0$$

$\Rightarrow$ there exists a unique sol $y(t) = \frac{1}{\cos t} \log(\sin t)$

s.t. $\lim_{t \to \pi/2 -} y(t) \in \mathbb{R}$. $\blacksquare$