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Ex 1.5.1

#1 Find gen sol of

$$y' + (\cos t) y = \frac{1}{2} \sin(2t) \qquad t \in \mathbb{R}$$

Sol: The eqn is a first ord. lin non homog eqn

$$y' = \underbrace{-(\cos t)}_{a(t)} y + \underbrace{\frac{1}{2}\sin(2t)}_{b(t)}$$

Clearly $a, b \in \mathcal{C}(\mathbb{R})$ and the gen sol is

$$y(t) = e^{\int a(t)\,dt} \left(\int b(t)\, e^{-\int a(t)\,dt}\, dt + C \right)$$

$$= e^{\int -\cos t\,dt} \left(\int \frac{1}{2} \sin(2t)\, e^{+\int +\cos t\,dt}\, dt + C \right)$$

$$= e^{-\sin t} \left[\frac{1}{\cancel{2}} \int \sin(2t)\, e^{\sin t}\, dt + C \right]$$

$$\sin(2t) = \cancel{2} \sin t \underbrace{\cos t\, e^{\sin t}}$$

$$\int \sin t \left(e^{\sin t} \right)'$$

$$= e^{-\sin t}\Big[\sin t\; e^{\sin t} - \int \underset{\overset{\shortparallel}{(e^{\sin t})'}}{\cos t\; e^{\sin t}}\,dt + C\Big]$$

$$= e^{-\sin t}\Big[\sin t\; e^{\sin t} - e^{\sin t} + C\Big]$$

$$= \sin t - 1 + C e^{-\sin t}. \qquad \square$$

#9. $y' = \frac{2t}{t^2+1}\,y + 2t(t^2+1) \qquad t \in \mathbb{R}$

We have a first order lin eq $y' = ay + b$

with $a(t) = \frac{2t}{t^2+1} \in \mathcal{C}(\mathbb{R})$, $b(t) = 2t(t^2+1) \in \mathcal{C}(\mathbb{R})$

The gen sol is

$$y = e^{\underset{\overset{\shortparallel}{\log(t^2+1)}}{\int \frac{2t}{t^2+1}\,dt}}\left(\int 2t(t^2+1)\, e^{-\int \frac{2t}{t^2+1}\,dt}\,dt + C\right)$$

$$= (t^2+1)\Big[\int 2t\,\cancel{(t^2+1)}\;\underset{\overset{\shortparallel}{\frac{1}{\cancel{t^2+1}}}}{e^{-\log(t^2+1)}}\,dt + C\Big]$$

$$= (t^2+1)\left[\, t^2 + C \,\right]. \qquad C \in \mathbb{R} \qquad ▨$$

#8. $y' + (\cos t)y = (\cos t)^2 \qquad t \in \mathbb{R}.$

Sol: Here we have $y' = ay + b$ with

$$\left.\begin{aligned} a(t) &= -\cos t \\ b(t) &= (\cos t)^2 \end{aligned}\right. \quad \in \mathscr{C}(\mathbb{R})$$

The gen sol is

$$y(t) = e^{-\int \cos t \, dt}\left(\int (\cos t)^2 e^{\int \cos t} dt + C \right)$$

$$= e^{-\sin t}\left[\int e^{\sin t} (\cos t)^2 \, dt + C \right]$$

$$\int e^{\sin t} (\cos t)^2 = \int \underset{\overset{\|}{(e^{\sin t})'}}{e^{\sin t} \cos t} \cdot \cos t \qquad \int e^{t^2}$$

$$= e^{\sin t} \cos t + \int e^{\sin t} \sin t \, dt$$

?

1.5.4 Consider the eqn

$$y' = (-\sin t)y + \sin t \qquad t\in\mathbb{R}$$

i) Find gen int

ii) Are there sols s.t. $\lim\limits_{t\to+\infty} y(t) \in \mathbb{R}$?

iii) Solve CP $y(\pi/2) = 1$.

Sol i) We have a first ord lin eqn $y' = ay + b$

with $a(t) := -\sin t$, $b(t) = \sin t$, $a, b \in \mathcal{C}(\mathbb{R})$

Gen int is

$$y(t) = e^{\int -\sin t}\left(\int \sin t\, e^{\int \sin t}\, dt + C\right)$$

$$= e^{\cos t}\Big[\underset{\overset{\|}{\left(e^{-\cos t}\right)'}}{\int \sin t\, e^{-\cos t}\, dt} + C\Big]$$

$$= e^{\cos t}\Big[e^{-\cos t} + C\Big]$$

$$= 1 + C e^{\cos t}. \qquad C\in\mathbb{R}.$$

ii) The question means:

$$\exists\, C : \quad \lim \left(1 + C e^{\overset{\cancel{\uparrow}}{\cos t}}\right) \in \mathbb{R} \quad ?$$

$\exists\, C : \quad \lim_{t\to+\infty} \left(1 + C\, \underset{\not\downarrow}{e^{\cos t}}\right) \in \mathbb{R}$!

If $C \neq 0$, limit does not $\exists$

If $C = 0$ limit $= 1$

The unique sol for which $\lim_{t\to+\infty} y \in \mathbb{R}$ is $y \equiv 1$.

iii) To solve CP $y\left(\frac{\pi}{2}\right) = 1$ we need to det C s.t.

$$1 = y\left(\frac{\pi}{2}\right) = 1 + C e^{\cos \pi/2} = 1 + C$$

$$\Rightarrow \quad C = 0 \quad \Longrightarrow \quad y \equiv 1. \qquad \blacksquare$$

Separable Variables Eqns

A first order separable variables eqn is a diff eqn of type

$$\boxed{y'(t) = a(t) \cdot f(y(t))}$$

where

- $a = a(t)$ $t \in I$, $a \in \mathcal{C}(I)$

- $f = f(y)$ $\quad f : \mathbb{R} \longrightarrow \mathbb{R}$, $\quad f \in \mathcal{C}(\mathbb{R})$

Examples:

- first order lin homog. eqns : $y'(t) = a(t)\, y(t)$

  Here $f(y) = y$ $\qquad = a(t)\, f(y(t))$

- $y'(t) = (\sin t)\, y(t)^2$

  Here $a(t) = \sin t$ $\qquad f(y) = y^2$

  $y' = \sin t \; y^2$

- $y' = \dfrac{e^t}{\sqrt{1+y^2}}$ $\qquad a(t) = e^t$

  $= e^t \cdot \dfrac{1}{\sqrt{1+y^2}}$ $\qquad f(y) = \dfrac{1}{\sqrt{1+y^2}}$

- (logistic eqn) $\qquad y' = k\, y(M-y)$

  $a(t) = k$

  $f(y) = y(M-y)$

So why this type of eqns is called "sep vars"

This is because a generic first order eqn

$$y' = g(t, y(t))$$

For sep vars eqns

$$y' = \underbrace{\boxed{a(t)}}_{\text{dep on } t}\underbrace{\boxed{f(y(t))}}_{\text{dep on } y}$$

So for ins:

$y' = a(t)y + b(t)$ is sep vars $\Longleftarrow$ $b(t) \equiv 0$

$y' = e^t y^2 + \sin t\, y = y(e^t y + \sin t)$ is not sep vars.

How could we solve an eqn of type

$$y' = a(t) f(y) \qquad \left(\begin{array}{l} y = y(t), \\ y' = y'(t) \end{array}\right)$$

The idea comes from the part case of lin hom. eqns

$$y' = a(t) y \qquad\qquad y' = a(t)\underline{f(y)}$$

$$\Updownarrow\ y \neq 0 \qquad\qquad \Updownarrow\ f(y) \neq 0$$

$$[\ ,\ ]' = \frac{y'}{y} = a(t) \qquad\qquad \frac{y'}{f(y)} = a(t)$$

$$\underset{\downarrow \atop \log|y(t)|}{[\quad]} = \left(\frac{y'}{y}\right) = a(t) \qquad \left|\left(\frac{y'}{f(y)}\right) = a(t)\right.$$

$$\underbrace{[\log|y|]}' = a(t) \qquad ?$$

$\downarrow$

this is <u>one</u> of the (infinite) <u>primitives</u> of $a(t)$

$$\Updownarrow$$

$$\log|y| = \int a(t)\,dt + c$$

$$c \in \mathbb{R}$$

$$\Downarrow \qquad \downarrow$$

$$|y| = \exp(\qquad)$$

Example:

$$y' = te^y$$

$$\Updownarrow = \underset{a(t)}{\boxed{t}} \cdot \underset{f(y)}{\boxed{e^y}}$$

$$\left|\overline{\frac{y'}{} = t}\right| \iff e^{-y}y' = t$$

$$\boxed{\frac{y'}{e^y} = t} \iff \underbrace{e^{-y}\,y'}_{} = t$$

$$\|$$

$$\Big[\underbrace{-e^{-y}}_{\substack{\text{is one of the} \\ \text{primitives of}}}\Big]' = t$$

$$\Rightarrow \boxed{-e^{-y} = \int t\,dt + c \qquad c \in \mathbb{R}}$$

$$-e^{-y} = \frac{t^2}{2} + C$$

$$\Rightarrow e^{-y} = -\frac{t^2}{2} - C = -\frac{t^2}{2} + C \qquad C \in \mathbb{R}$$

$$\Rightarrow -y = \log\left(-\frac{t^2}{2} + c\right)$$

$$\Rightarrow \boxed{y(t) = -\log\left(-\frac{t^2}{2} + c\right) \qquad C \in \mathbb{R}}$$

(gen. sol of the eqn)

Example $y' = 1+y^2$

$$= \underbrace{\boxed{1}}_{t}\;\underbrace{\boxed{(1+y^2)}}_{y}$$

$y' = f(y)$

$\uparrow$

$a(t) \equiv 1$

$1+y^2 \neq 0 \ \Updownarrow$

$$\frac{y'}{1+y^2} = 1$$

$$\parallel$$

$$(\text{arctg}\, y)' = 1$$

$$\Updownarrow$$

$$\boxed{\text{arctg}\, y = \int 1\, dt + c \qquad c \in \mathbb{R}}$$

$$\Updownarrow$$

$$\boxed{y(t) = \text{tg}(t+c), \quad c \in \mathbb{R}.}$$

(gen sol.)

$\blacksquare$

Assume now we have an eqn

$$y' = a(t)\, f(y)$$

with $f(y) \neq 0$ (as in prev. examples)

Then

$$y' = a(t)\,f(y) \iff \frac{y'}{f(y)} = a(t)$$

Now the key step is

$$\boxed{\frac{y'}{f(y)} = \boxed{\big[\overset{\downarrow}{G}(y)\big]' =} \; G'(y)\cdot y'}$$

$$\Rightarrow \quad G'(y) = \frac{1}{f(y)}$$

$$\Rightarrow \quad \boxed{G(y) = \int \frac{1}{f(y)}\,dy}$$

Ex: $\quad y' = \underset{a(t)}{\underset{\parallel}{t}}\,\underset{f(y)}{\underset{\parallel}{e^y}}$

$$G(y) = \int \frac{1}{f(y)}\,dy$$

$$= \int \frac{1}{e^y}\,dy = \int e^{-y}\,dy$$

$$= -e^{-y}$$

So if

$$G(y) = \int \frac{1}{f(y)}\,dy$$

ea

$$G(y) = \int \frac{}{f(y)}\, dy$$

$$\Rightarrow \quad [G(y)]' = \frac{y'}{f(y)} \overset{\text{eq}}{=} a(t)$$

$$\Rightarrow \quad \boxed{G(y(t)) = \int a(t)\,dt + C \qquad C \in \mathbb{R}}$$

(implicit form for the sol)

If G is invertible

$$\boxed{y(t) = G^{-1}\left(\int a(t)\,dt + C\right) \qquad C \in \mathbb{R}}$$

(explicit sol / gen int)

Example: Solve

$$y' = e^{t}(y^2 + 2y + 2)$$

Sol: We have a sep vars eqn

$$(*) \quad y' = \underset{a(t)}{e^t} \underbrace{(y^2 + 2y + 2)}_{f(y)}$$

where $a \in \mathcal{C}(\mathbb{R})$, $f \in \mathcal{C}(\mathbb{R})$ $\quad f > 0$

$$f(y) = y^2 + 2y + 2 = y^2 + 2y + 1 + 1$$

$$= (y+1)^2 + 1 > 0$$

Now, $(*) \iff$ $\quad \dfrac{y'}{y^2+2y+2} = e^t$

$$[G(y)]' = \frac{y'}{y^2+2y+2}$$

$$\frac{y'}{f(y)} = a(t)$$

where

$$G(y) = \int \frac{1}{f(y)}\, dy = \int \frac{1}{y^2+2y+2}\, dy$$

$$= \int \frac{1}{(y+1)^2+1}\, dy = \operatorname{arctg}(y+1)$$

$$\Rightarrow \quad G(y) = \int a(t)\, dt + c = \int e^t dt + C = e^t + C$$

$$\shortparallel$$

$$\operatorname{arctg}(y+1) = e^t + c \qquad \text{(implicit form)}$$

$$\Rightarrow \quad y+1 = \operatorname{tg}(e^t + C)$$

$$\Rightarrow \quad \boxed{y = -1 + \operatorname{tg}(e^t + c) \qquad C \in \mathbb{R}} \quad \text{(explicit form)}$$

▨

Example: Solve

$$\begin{cases} y' = (e^y+1)\sqrt{t} \\ \boxed{y(1) = 1} \end{cases}$$

Sol: Here we have a sep vars eqn

$$y' = \underbrace{(e^y+1)}_{f(y)}\sqrt{t} \qquad a(t) \in \mathcal{C}([0,+\infty[)$$

$$\Updownarrow \qquad f \in \mathcal{C}(\mathbb{R}),\ f > 0$$

$$[G(y)]' = \boxed{\frac{y'}{e^y+1} = \sqrt{t}} \qquad\qquad G(y) = \int \frac{1}{e^y+1}\,dy$$

$$\Downarrow$$

$$" \ t^{1/2}$$

$$\Downarrow$$

$$\int \frac{y'(t)}{e^{y(t)}+1}\,dt = \int \overset{t^{1/2}}{\sqrt{t}}\,dt + C$$

$$= \frac{t^{3/2}}{3/2} + C$$

$$\underset{\substack{\boxed{u=y(t)}\\ du = y'(t)\,dt}}{=} \int \frac{du}{e^{u}+1} =$$

$$\int \frac{1}{e^{u}+1}\,du \underset{\substack{\boxed{v=e^{u}}\\ u=\log v\\ du=\frac{1}{v}\,dv}}{=} \int \frac{1}{(v+1)}\,\frac{1}{v}\,dv = \int\left(\frac{-1}{v+1}+\frac{1}{v}\right)dv$$

$$= \log|v| - \log|v+1|$$

$$= \log\left|\frac{v}{v+1}\right| = \log\left|\frac{e^{y}}{e^{y}+1}\right|$$

$$\Longrightarrow \boxed{\log\left|\frac{e^{y}}{e^{y}+1}\right| = \frac{2}{3}t^{3/2}+C}$$ implicit form

If we wish to det the sol of CP., we need to impose $y(1)=1$

$$\log \frac{e^1}{e^1+1} = \frac{2}{3} 1^{3/2} + C \implies$$

$$C = \log \frac{e}{e+1} - \frac{2}{3} .$$

Let's write the explicit sol:

$$\frac{e^y+1-1}{e^y+1} = e^{\frac{2}{3} t^{3/2} + C}$$

$$\text{//}$$

$$1 - \frac{1}{e^y+1} = . \qquad \iff \frac{1}{e^y+1} = 1 - e^{\frac{2}{3} t^{3/2} + C}$$

$$\iff e^y \cancel{+1} = \frac{1}{1 - e^{2/3 t^{3/2} + C}} - 1$$

$$\iff y(t) = \log\left(\frac{1}{1 - e^{2/3 t^{3/2} + C}} - 1 \right) \blacksquare$$

What happens if f is not always 0?

Let's consider now

$$y' = a(t)\, f(y)$$

with $a \in \mathcal{C}(I)$, $I \subset \mathbb{R}$ interval, $f \in \mathcal{C}(\mathbb{R})$

and assume

$$\exists\, y_0 \in \mathbb{R} \; : \; f(y_0) = 0.$$

Ex: $y' = y^2$ $\quad a(t) \equiv 1 \quad f(y) = y^2 \quad f(y_0) = 0 \Leftrightarrow y_0 = 0$

The main rmk is the following: the funct $y(t) \equiv y_0$ is a sol of the eqn. Indeed

$$(y(t) =)\; y \equiv y_0 \implies \underset{\overset{\|}{a(t) f(y) \;=\; a(t) \overset{\overset{0}{\shortparallel}}{f(y_0)} \;=\; 0}}{y' = 0}$$

$$\implies y \equiv y_0 \text{ is a sol of eqn.}$$

Vice versa: if $\boxed{y \equiv C}$ is a sol of $\underset{\overset{\|}{0 \;=\; a(t) f(C)}}{y' = a(t) f(y)}$

$$\Updownarrow$$

$$\boxed{f(C) = 0}$$

<u>Prop</u>: $y \equiv y_0$ is a sol for $y' = a(t) f(y) \Leftrightarrow f(y_0) = 0.$

Thm: Consider the eqn

$$y' = a(t) f(y) \quad (*)$$

with $a \in \mathcal{C}(I)$, $I \subset \mathbb{R}$ int, $\boxed{f \in \mathcal{C}^{①}(\mathbb{R})}$ $(f, f' \in \mathcal{C}(\mathbb{R}))$

Then, $y = y(t)$ is a sol of $(*)$ $\Leftrightarrow$ either

- $y \equiv y_0$ and $f(y_0) = 0$
- y is non const and can be det. by separati
  of vars $\Leftrightarrow$

$$y(t) = G^{-1}\left(\int a(t)\,dt + C \right) \quad C \in \mathbb{R}$$

$$G(y) = \int \frac{1}{f(y)}\,dy \,.$$

Do Ex 1.5.6, 7, 8.

1.5.6 #2.

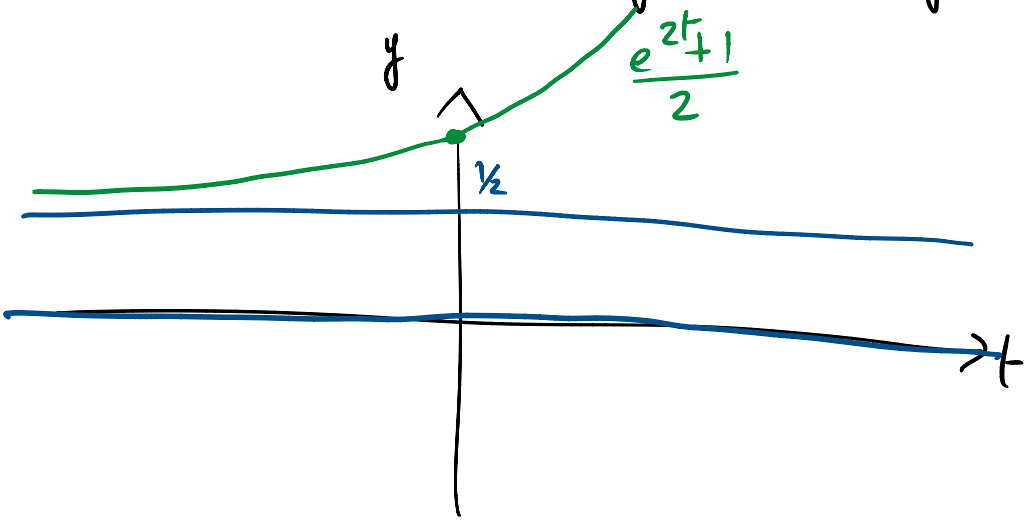
Solve
$$\begin{cases} y' = \dfrac{y\sqrt{2y-1}}{\operatorname{Ch} t} \\ \boxed{y(0) = 1} \end{cases}$$

Sol: $y' = \underbrace{\frac{1}{\operatorname{Ch} t}}_{a(t)} \cdot \underbrace{y\sqrt{2y-1}}_{f(y)}$ is a sep vars eqn

Notice that $y \equiv y_0$ is a sol $\Leftrightarrow f(y_0) = 0$

$$\Leftrightarrow y_0\sqrt{2y_0-1} = 0$$

$$\Leftrightarrow y_0 = 0 \vee y_0 = 1/2$$



$\Rightarrow$ because $y(0) = 1 \Rightarrow$ our sol is non const.

To find it we have to sep vars:

$$y' = \frac{1}{\operatorname{Ch} t} y\sqrt{2y-1} \Leftrightarrow \frac{y'}{y\sqrt{2y-1}} = \frac{1}{\operatorname{Ch} t}$$

$$\Leftrightarrow \boxed{\int \frac{y'(t)}{y(t)\sqrt{2y(t)-1}}\,dt = \int \frac{1}{\operatorname{ch} t}\,dt + C}$$

$$\int \frac{1}{\operatorname{ch} t}\,dt = \int \frac{1}{\frac{e^{t}+e^{-t}}{2}}\,dt = 2\int \frac{e^{t}}{e^{2t}+1}\,dt$$

$$e^{-t} = 1/e^{t}$$

$$= 2\int \frac{1}{1+(e^{t})^{2}}\cdot e^{t}\,dt = 2\operatorname{arctg}(e^{t})$$

$$\int \frac{y'(t)}{y(t)\sqrt{2y(t)-1}}\,dt = \int \frac{du}{u\sqrt{2u-1}} =$$

$u = y(t)$

$du = y'(t)\,dt$

$v = \sqrt{2u-1}$

$v^{2} = 2u-1$

$u = \frac{v^{2}+1}{2}$

$du = v\,dv$

$$= \int \frac{\cancel{v}\,dv}{\frac{v^{2}+1}{2}\cdot\cancel{v}}$$

$$= 2\int \frac{1}{1+v^{2}}\,dv = 2\operatorname{arctg} v = 2\operatorname{arctg}\sqrt{2y(t)-1}.$$

$$\Rightarrow \boxed{2\operatorname{arctg}\sqrt{2y(t)-1} = 2\operatorname{arctg}(e^{t}) + C \qquad C\in\mathbb{R}}$$

(implicit form)

For the sol of CP $y(0)=1$

$$2\operatorname{arctg}\sqrt{2.1-1} = 2\operatorname{arctg} 1 + C \Rightarrow \boxed{C=0}$$

$$\Rightarrow \cancel{2}\operatorname{arctg}\sqrt{2y(t)-1} = \cancel{2}\operatorname{arctg}(e^{t})$$

$$\Rightarrow \sqrt{2y(t)-1} = e^{t} \Rightarrow 2y-1 = e^{2t}$$

$$\Rightarrow \boxed{y(t) = \frac{e^{2t}+1}{2}}$$ (sol of CP). ∎