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Ex 1.5.6

#1 Solve the CP

$$\begin{cases} y' = \dfrac{y^2 - y - 2}{3} \arcsin t \\ y(0) = 3 \end{cases}$$

Sol: Clearly the eqn is a sep vars eqn

$$y' = \underbrace{\frac{1}{3}(\arcsin t)}_{a(t)} \underbrace{(y^2 - y - 2)}_{f(y)}$$

$y \equiv C$ is const sol $\Leftrightarrow$ $0 = \left(\frac{1}{3}\arcsin t\right)(C^2 - C - 2)$

$$\Leftrightarrow \quad C^2 - C - 2 = 0$$

$$\Delta = (-1)^2 - 4 \cdot 1 \cdot (-2) = +9$$

$$C_{1,2} = \frac{+1 \pm \sqrt{9}}{2} = \frac{1 \pm 3}{2}$$

$$= 2, -1$$

$y \equiv 2$, $y \equiv -1$ are the unique const sols.

The sol of our CP is not one of two const sols $\Rightarrow$ y can be determined by sep of vars

$$\Leftrightarrow \quad \frac{y'(t)}{y^2(t) - y(t) - 2} = \frac{1}{3} \arcsin t$$

$$\Leftrightarrow \quad \int \frac{y'(t)}{y(t)^2 - y(t) - 2}\, dt = \int \frac{1}{3} \arcsin t \, dt + C$$

$$\underbrace{\int \arcsin t \, dt}_{\substack{\uparrow \\ 1\cdot \\ = (t)'}} = t \arcsin t - \underbrace{\int t \frac{1}{\sqrt{1-t^2}}\, dt}_{= +\frac{1}{2}\int (1-t^2)^{-1/2}\,\underbrace{(-2t)}_{=(1-t^2)'} \; = \; \frac{1}{\cancel{2}} \cancel{\frac{(1-t^2)^{1/2}}{1/2}}}$$

$$\int f^{\alpha} f' = \begin{cases} \alpha = -1 & \log|f| \\ \alpha \neq -1 & \dfrac{f^{\alpha+1}}{\alpha+1} \end{cases}$$

$$= t \arcsin t - \sqrt{1-t^2}.$$

$$\int \frac{y'(t)}{y(t)^2 - y(t) - 2}\, dt \underset{\substack{u = y(t) \\ du = y'(t)\,dt}}{=} \int \frac{du}{u^2 - u - 2}$$

$$u^2 - u - 2 = (u-2)(u+1)$$

$$\frac{1}{u^2-u-2} = \frac{1}{(u-2)(u+1)} = \left(\frac{1}{u-2} - \frac{1}{u+1}\right)\frac{1}{3}$$

$$\Rightarrow \int \frac{1}{u^2-u-2}\,du = \frac{1}{3}\left(\int \frac{1}{u-2}\,du - \int \frac{1}{u+1}\,du\right)$$

$$= \frac{1}{3}\left(\log|u-2| - \log|u+1|\right)$$

$$= \frac{1}{3}\log\left|\frac{u-2}{u+1}\right|$$

$$= \frac{1}{3}\log\left|\frac{y(t)-2}{y(t)+1}\right|$$

Therefore we obtained the implicit form of the non const sols

$$\boxed{\frac{1}{3}\log\left|\frac{y-2}{y+1}\right| = \frac{1}{3}\left(t\arcsin t - \sqrt{1-t^2}\right) + C \qquad C\in\mathbb{R}}$$

Imposing the passage cond $y(0) = 3$

$$\frac{1}{3}\log\frac{1}{4} = \frac{1}{3}\left(0\,\underbrace{\arcsin 0}_{}^{0} - \sqrt{1}\right) + C$$

$$\frac{1}{3}\log\frac{1}{4} = \frac{1}{3}\big(\underbrace{0\arcsin 0 - \sqrt{1}}_{=-1/3}\big) + C$$

$$C = \frac{1}{3}\left(1 - \log 4\right)$$

Let's finally det. y explicitely:

$$\left|\frac{y-2}{y+1}\right| = e^{t\arcsin t - \sqrt{1-t^2} + 3C}$$

$$\Leftrightarrow \quad \overset{+}{\frac{y-2}{y+1}} = \overset{?}{\pm}\, \underset{+}{e}^{t\arcsin t - \sqrt{1-t^2} + 3C}$$

$+$

Because $y(0) = 3$ $\quad = 1/4$

$$\Downarrow$$

$$\frac{y-2}{y+1} = +\, e^{t\arcsin t - \sqrt{1-t^2} + 3C}$$

$$= e^{\boxed{t}}$$

$$y - 2 = e^{\boxed{t}}(y+1)$$

$$y\left(e^{\boxed{t}} - 1\right) = -2 - e^{\boxed{t}}$$

$$y = -\quad 2 + e^{\boxed{t}} \qquad 2 + e^{t\arcsin t - \sqrt{1-t^2} + 3C}$$

$$y = -\frac{2+e^{\boxed{t}}}{e^{\square}-1} = -\frac{2+e^{t\arcsin t-\sqrt{1-t^2}+3C}}{e^{t\arcsin t-\sqrt{1-t^2}+3C}-1}$$

Ex 1.5.8 Consider the CP

$$\begin{cases} y' = y(1-y^2) \\ y(0) = 1/2 \end{cases}$$

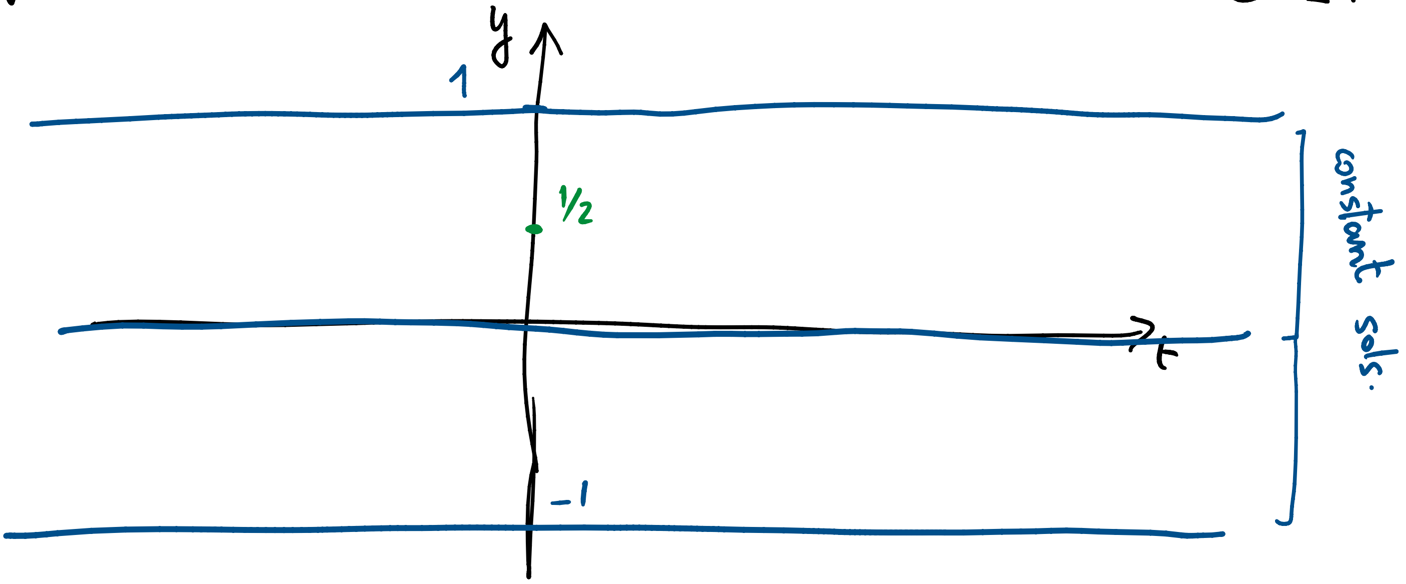
i) Det the implicit form for the sol

ii) Is it true that the sol is defined $\forall t \in \mathbb{R}$?

i) The eqn $y' = y(1-y^2)$ is a sep vars eqn

$$= \underset{a(t)}{1} \cdot \underset{f(y)}{y(1-y^2)}$$

$y \equiv C$ is a sol $\iff 0 = C(1-C^2) \iff \begin{matrix} C=0 \\ C=\pm 1 \end{matrix}$



Because of the passage cond, the sol of our CP is not constant $\Rightarrow$ it must be determined by sep. of vars $\Leftrightarrow$

$$\frac{y'}{y(1-y^2)} = 1$$

$$\Updownarrow$$

$$\int \frac{y'(t)}{y(t)(1-y(t)^2)}\,dt = \int 1\,dt + C \qquad C \in \mathbb{R}$$

$$= t + C$$

$$\int \frac{y'}{y(1-y^2)}\,dt \underset{\substack{u = y(t) \\ du = y'(t)dt}}{=} \int \frac{1}{u(1-u^2)}\,du$$

$$u(1-u)(1+u)$$

$$\frac{1}{u(1-u)(1+u)} = \frac{A}{u} + \frac{B}{1-u} + \frac{C}{1+u}$$

$$\int \frac{1 \cancel{-u^2} \cancel{+u^2}}{u(1-u^2)}\,du = \int \frac{1}{u} - \frac{1}{2}\,\frac{-2u}{1-u^2}\,du$$

$$= \log|u| - \frac{1}{2}\log|1-u^2| \qquad (1-u^2)' = -2u$$

$$= \log \frac{|u|}{\sqrt{|1-u^2|}}$$

$$= \log \frac{|y|}{\sqrt{|1-y^2|}}$$

$$\Rightarrow \quad \boxed{\log \frac{|y|}{\sqrt{|1-y^2|}} = t + C} \quad \text{(implicit form.)}$$

To det C for the sol of our CP, we impose

$$y(0) = \tfrac{1}{2} \quad \Rightarrow \quad C = \log \frac{\tfrac{1}{2}}{\sqrt{1-\tfrac{1}{4}}} = \log \frac{1}{\cancel{2}} \frac{\cancel{2}}{\sqrt{3}} = -\log\sqrt{3}$$

Can we find the explicit form?

$$\frac{|y|}{\sqrt{|1-y^2|}} = e^{t - \log\sqrt{3}} = \frac{1}{\sqrt{3}} e^{t}$$

$$\sqrt{\left|\frac{y^2}{1-y^2}\right|} = \frac{1}{\sqrt{3}} e^{t}$$

$$\Rightarrow \left|\frac{y^2}{1-y^2}\right| = \frac{1}{3} e^{2t}$$

$$y^2 \quad \pm \tfrac{1}{3} e^{2t}$$

$$\Rightarrow \quad \frac{y^2}{1-y^2} = \pm \tfrac{1}{3} e^{2t}$$

?

Because at $t=0$ $y = 1/2$ $\Rightarrow$ $\dfrac{1/4}{1-1/4} = \pm \dfrac{1}{3}$

$\| \quad 1/3$

$\Downarrow$

$+$

$$\Rightarrow \quad \frac{y^2}{1-y^2} = \frac{1}{3} e^{2t}$$

$$\Rightarrow \quad y^2 = \frac{1}{3} e^{2t}(1-y^2) \Rightarrow y^2\left(\frac{1}{3}e^{2t}+1\right) = \frac{1}{3}e^{2t}$$

$$\Rightarrow \quad y^2 = \frac{\frac{1}{3}e^{2t}}{\frac{1}{3}e^{2t}+1} = \frac{\cancel{\frac{1}{3}}\, e^{2t}}{\frac{e^{2t}+3}{\cancel{3}}} = \frac{e^{2t}}{e^{2t}+3}$$

$$\Rightarrow \quad y = \pm\sqrt{\frac{e^{2t}}{e^{2t}+3}} = \pm \frac{e^t}{\sqrt{e^{2t}+3}}$$

?

Because $y(0) = 1/2 \Rightarrow +$

Sol: $y(t) = \dfrac{e^t}{\sqrt{e^{2t}+3}}$.

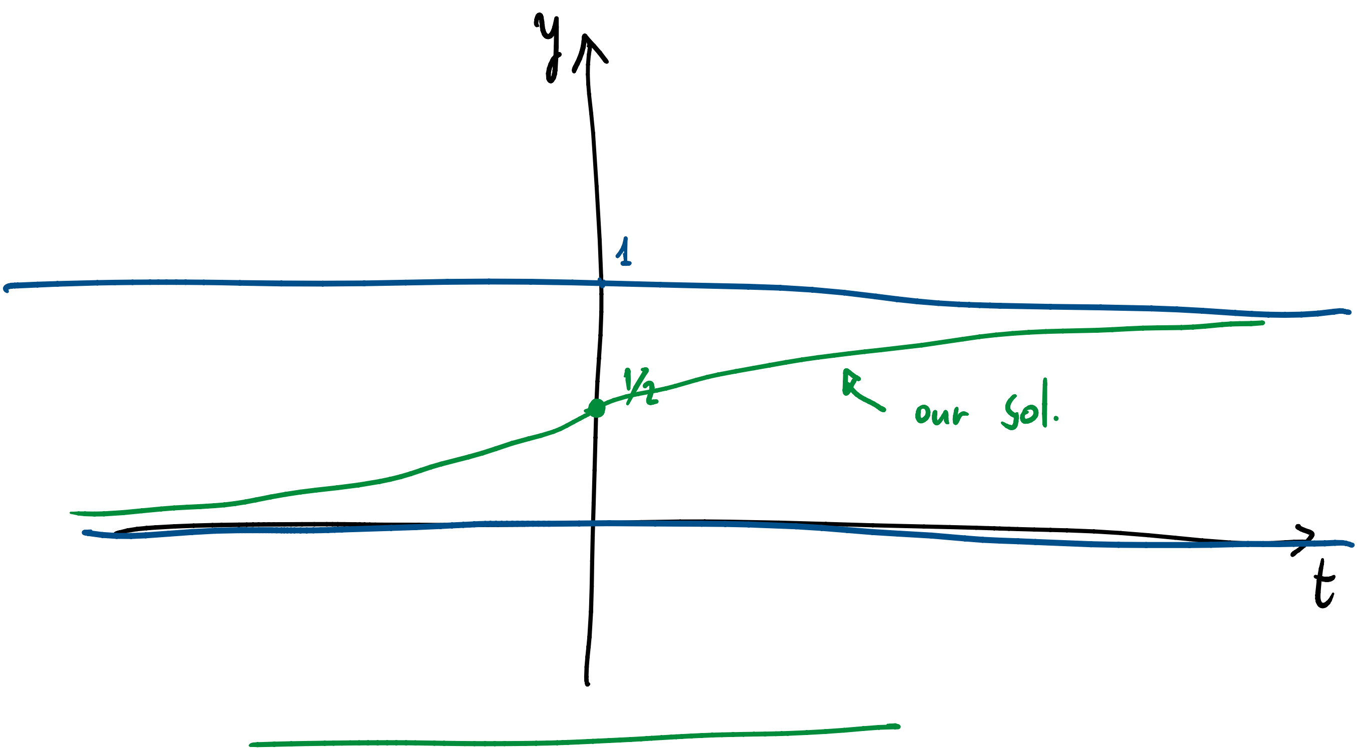
Sol: $y(t) = \dfrac{\quad}{\sqrt{e^{2t} + 3}}$.

ii) Clearly sol y is defined $\forall t \in]-\infty, +\infty[$

Moreover $0 < y < 1$

$y(t) \to 1 \qquad t \to +\infty$

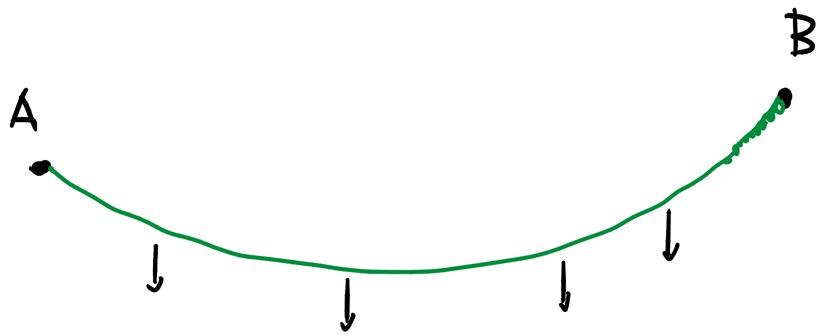
$y(t) \to 0 \qquad t \to -\infty$



Catenary Problem

Imagine we have a chain suspended between two

Imagine we have a chain hanging between two fixed points:

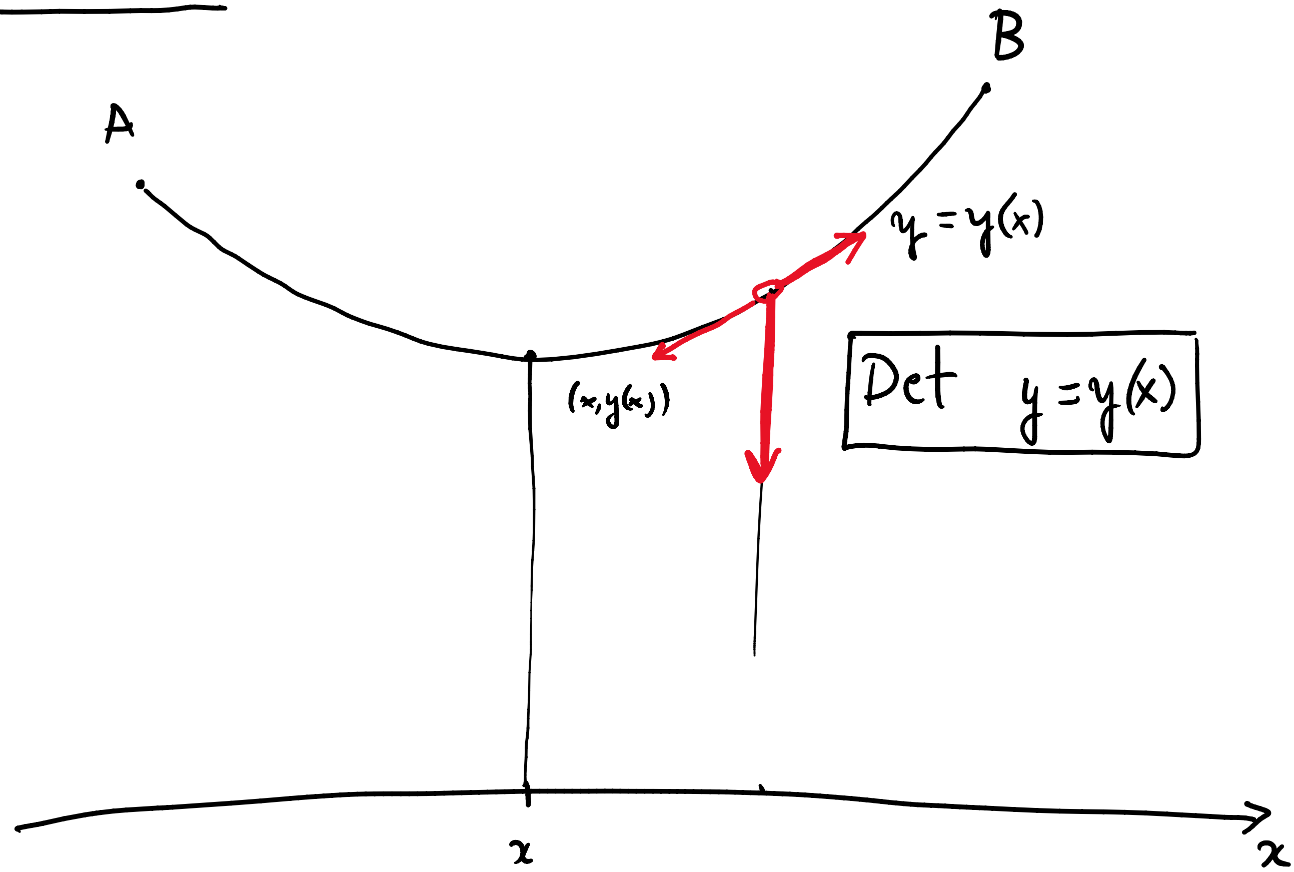


PL: determine the shape of the chain at equilibrium

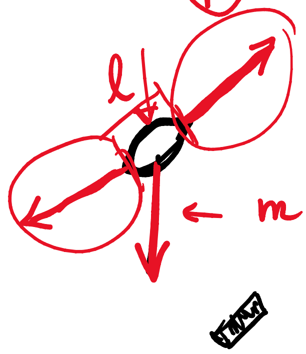
Galileo guessed that shape = arc of parabola

We will see that this is not the correct answer.

Formalization:



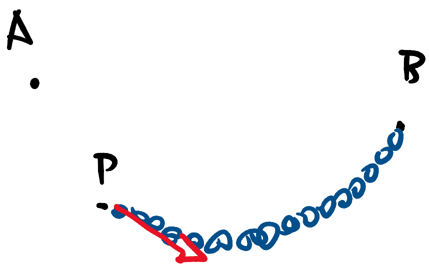
Hypothesis: we assume the chain be homogeneous with density of mass ρ (mass per unit of length)



$\leftarrow m\vec{g} = \rho l \vec{g}$

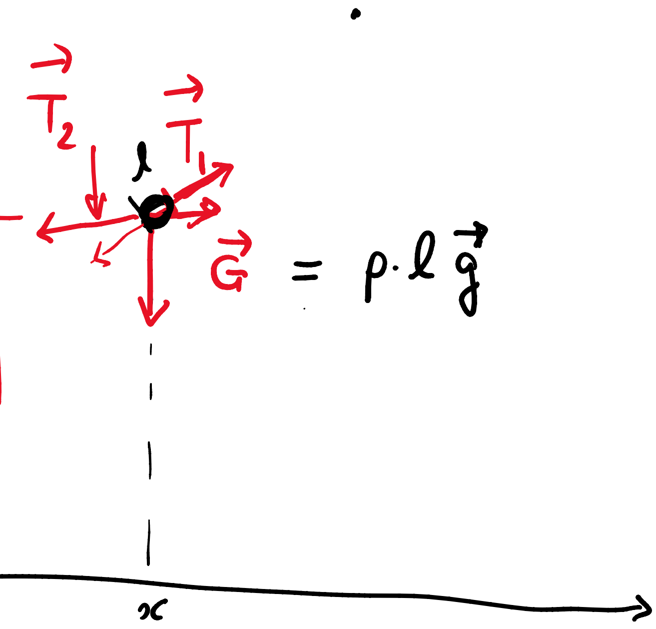
if we have a piece of chain of length l its mass is ρl

$\vec{T}$ = force exercised by part $\overline{PB}$ of the chain



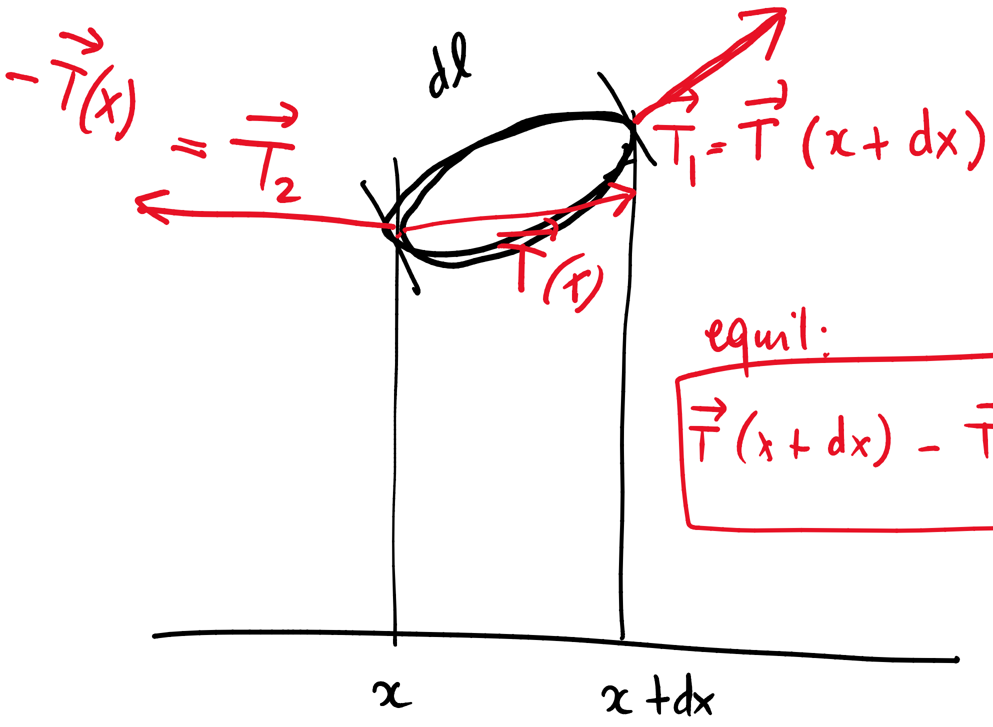
B

A .



$$\boxed{\vec{T}_1 + \vec{T}_2 + \vec{G} = \vec{0}}$$

equilibrium



equil:

$$\boxed{\vec{T}(x+dx) - \vec{T}(x) + \rho l \vec{g} = \vec{0}}$$

$$\vec{T}(x) = (\tau(x), \underline{\underline{\sigma(x)}})$$

so equilibrium becomes:

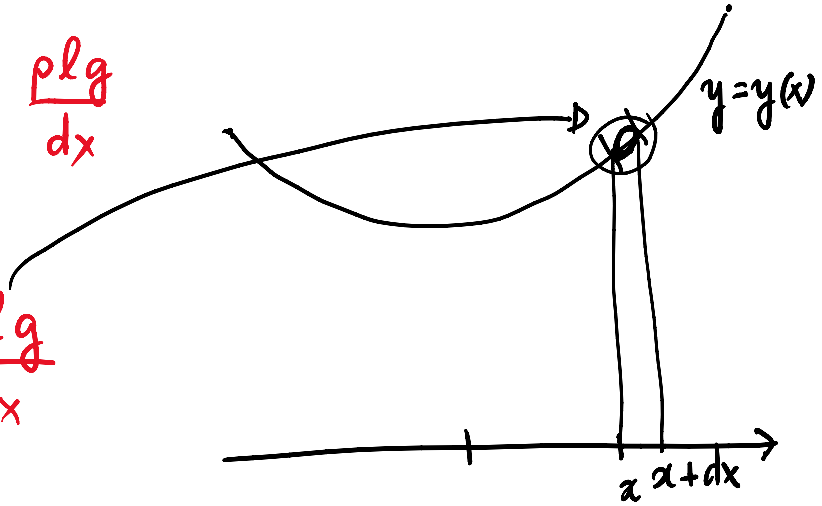
$g = 9.8 \, m/s^2.$

$$\begin{cases} \tau(x+dx) - \tau(x) + 0 = 0 \\ \sigma(x+dx) - \sigma(x) - \rho l g = 0 \end{cases}$$

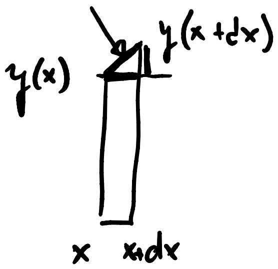
$$\Rightarrow \begin{cases} \tau(x+dx) - \tau(x) = 0 & \Rightarrow \quad \tau \quad \text{constant} \\[2ex] \dfrac{\sigma(x+dx) - \sigma(x)}{dx} = \dfrac{\rho l g}{dx} \end{cases}$$

$$\Vert$$

$$\sigma'(x) = \frac{\rho l g}{dx}$$



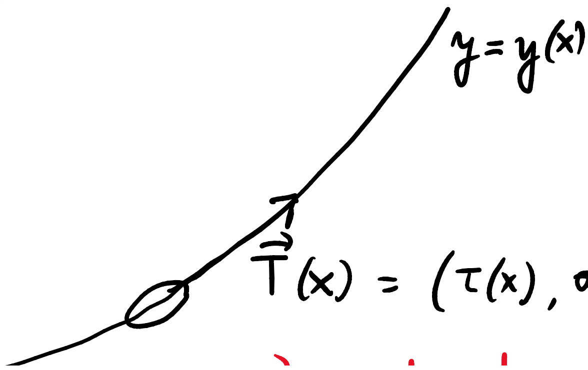
$$l = \sqrt{(dx)^2 + \left(\frac{y(x+dx) - y(x)}{dx}\right)^2 (dx)^2}$$



$$l = \sqrt{(dx)^2 + \left(y'(x) \right)^2 (dx)^2} = dx\sqrt{1 + y'(x)^2}$$

$$\Rightarrow \quad \boxed{\sigma'(x)} = \rho \frac{\cancel{dx}\sqrt{1 + y'(x)^2}\, g}{\cancel{dx}} = \rho\sqrt{1 + y'(x)^2}\, g$$

To finish notice that



$$\vec{T}(x) = (\tau(x), \sigma(x))$$

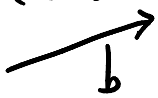
$T(x) = (\tau(x), \sigma(x))$

$\vec{T}$ is tg to $y = y(x)$

$\Updownarrow$

$\vec{T}$ has same ang coeff of $y = y(x)$

(a,b)



$\frac{b}{a}$

$$\frac{\sigma(x)}{\tau(x)} = y'(x)$$

$$\Rightarrow \boxed{\sigma(x) = y'(x)\,\underbrace{\tau(x)}_{\equiv\, T_0} = T_0\, y'(x)}$$

$$\Downarrow$$

$$\sigma'(x) = T_0\, y''(x)$$

Thus by

$$\sigma'(x) = \rho g \sqrt{1 + y'(x)^2}$$

$$\Rightarrow \boxed{T_0\, y''(x) = \rho g \sqrt{1 + y'(x)^2}}$$

By posing

$$\boxed{v(x) = y'(x)}$$

we have

$$v'(x) = \frac{\rho g}{T_0} \sqrt{1 + v(\,)^2}$$

$$\left(v' = \frac{\rho g}{T_0}\sqrt{1+v^2} = \underset{a(x)}{\frac{\rho g}{T_0}}\ \underset{f(v)}{\sqrt{1+v^2}}\ . \right)$$

This is a sep var eqn $\Updownarrow$

$$\frac{v'}{\sqrt{1+v^2}} = \frac{\rho g}{T_0}$$

$$\Rightarrow \int \frac{v'(x)}{\sqrt{1+v(x)^2}}\,dx = \int \frac{\rho g}{T_0}\,dx + C$$

$$= \frac{\rho g}{T_0}x + C$$

$u = v(x)$

$du = v'(x)\,dx$

$$= \int \frac{du}{\sqrt{1+u^2}} = Sh^{-1}(u)$$

$$= Sh^{-1}(v(x))$$

$$\Rightarrow \quad Sh^{-1}(v(x)) = \frac{\rho g}{T_0}x + C$$

$$\Rightarrow \quad v(x) = Sh\left(\frac{\rho g}{T_0}x + C\right)$$

$$\Rightarrow \quad v(x) = \mathrm{Sh}\left(\frac{\rho g}{\tau_0} x + C\right)$$

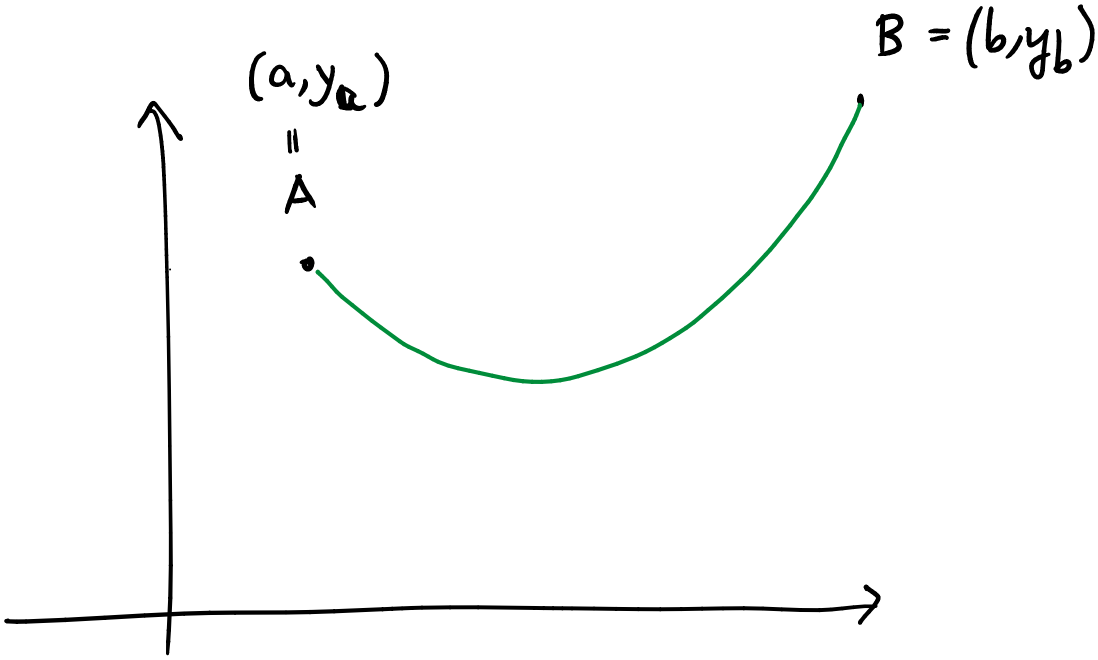
and because $y' = v$

$$\Rightarrow \quad y(x) = \int \mathrm{Sh}\left(\frac{\rho g}{\tau_0} x + C\right) dx + \tilde{C}$$

$C, \tilde{C} \in \mathbb{R}$

$$\boxed{y(x) = \frac{\tau_0}{\rho g} \mathrm{Ch}\left(\frac{\rho g}{\tau_0} x + C\right) + \tilde{C}}$$

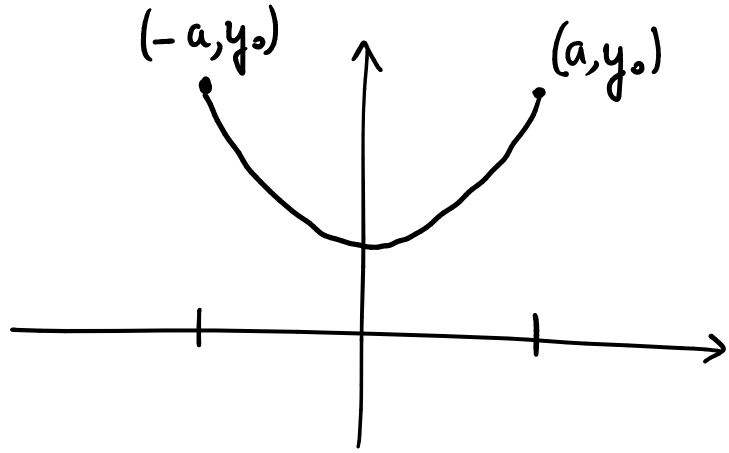
Rmk: We may notice that if



to det the specific $y = y(x)$ we have to impose

$$\begin{cases} y(a) = y_a \\ y(b) = y_b \end{cases} \rightsquigarrow C, \tilde{C}$$

<u>Example</u>



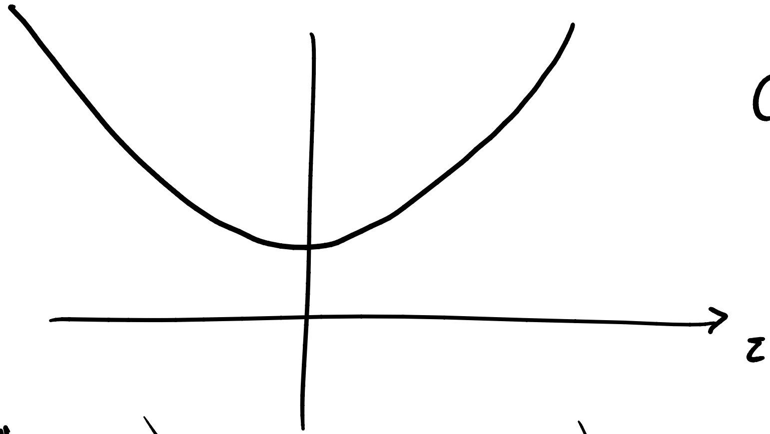
a, y_0 known

In this case we have

$$\begin{cases} y(-a) = y_0 \\ y(a) = y_0 \end{cases} \qquad (*) \begin{cases} \frac{T_0}{\rho g} \operatorname{Ch}\left(-\frac{\rho g}{T_0} a + C\right) + \tilde{C} = y_0 \\ \frac{T}{\rho g} \operatorname{Ch}\left(\frac{\rho g}{T_0} a + C\right) + \tilde{C} = y_0 \end{cases}$$

$$\cancel{\frac{T}{\rho g}} \left[\operatorname{Ch}\left(\frac{\rho g}{T_0} a + C\right) - \operatorname{Ch}\left(-\frac{\rho g}{T_0} a + C\right) \right] = 0$$

$$\Longleftrightarrow \quad \operatorname{Ch}\left(\frac{\rho g}{T_0} a + C\right) = \operatorname{Ch}\left(-\frac{\rho g}{T_0} a + C\right)$$



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$$\Leftrightarrow \quad \frac{\rho g}{\tau_0} a + \cancel{C} = -\frac{\rho g}{\tau_0} a + \cancel{C} \implies a = 0 \quad \text{NO}$$

or

$$\cancel{\frac{\rho g}{\tau_0} a} + C = -\left(-\cancel{\frac{\rho g}{\tau_0} a} + C\right)$$

$$C = -C \implies \boxed{C = 0}$$

Then, by system $(*)$

$$\boxed{\tilde{C} = y_0 - \frac{\tau_0}{\rho g} \operatorname{Ch}\left(\frac{\rho g}{\tau_0} a\right)}$$

Conclusion:

$$y = \frac{\tau_0}{\rho g} \operatorname{Ch}\left(\frac{\rho g}{\tau_0} x\right) + \tilde{C}$$