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Second Order Linear Eqns

Recall that a first order lin. eqn has the form

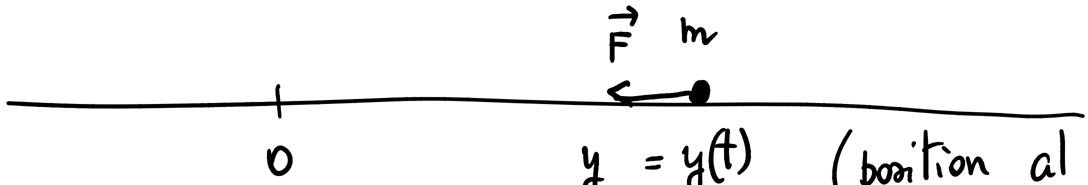
$$y' = a(t)y + b(t)$$

We will now consider eqns of type

$$\boxed{y'' = a(t)y' + b(t)y + f(t)}$$

where coeffs $a = a(t)$, $b = b(t)$, $f = f(t)$ are known.

Example In Physics, precisely in Class. Mech, the motion of particles / bodies is driven by this kind of eqns.



$y = y(t)$ (position at time t)

Newton's second law: $\boxed{\vec{F} = m \cdot \vec{a}}$

In general forces may depend on

- position
- velocity
- time

$$\vec{F} = \vec{F}(t, y(t), y'(t))$$

$$\vec{a} = y''(t)$$

– time

$$\vec{a} = y''(t)$$

$$\Rightarrow \qquad m y'' = F(t, y(t), y'(t))$$

For inst:

$$F_{el} = F(y(t)) = -\hat{k}\, y(t)$$

$$F_{fr} = -\check{\nu}\, y'(t)$$

$$m y'' = -ky - \nu y'$$

$$\Updownarrow$$

$$y'' = \underbrace{-\frac{k}{m}}_{b(t)} y \underbrace{-\frac{\nu}{m}}_{a(t)} y' + \underset{f(t)}{0}$$

$\boxtimes$

We will focus on eqns of type

$$y'' = a y' + b y + f(t), \qquad \text{where } a, b \in \mathbb{R} \text{ (const)}$$

$$\Updownarrow$$

$$\boxed{y'' + a y' + b y = f(t)}$$

$$\boxed{y'' + ay' + by = f(t)}$$

Homogeneous: $y'' + ay' + by = 0$

Non homog: $y'' + ay' + by = f(t)$

We will start by solving the homogeneous eqn:

$$y'' + ay' + by = 0 \qquad a, b \in \mathbb{R}$$

$$D = \frac{d}{dt} \qquad D^2 y + aDy + by = 0$$

$$\underbrace{(D^2 + aD + b)}\, y = 0$$

We call characteristic polynomial

$$\lambda^2 + a\lambda + b$$

and characteristic eqn

$$\boxed{\lambda^2 + a\lambda + b = 0}$$

$$\begin{array}{ccc} (y'' & + ay' & + by = 0) \\ \updownarrow & \updownarrow & \updownarrow \\ \lambda^2 & \lambda^1 & \lambda^0 \end{array}$$

$$\Delta = a^2 - 4b$$

This quantity determines how do we solve the

This quantity determines how do we solve the diff eqn:

Case $\Delta > 0$

Eqn $\lambda^2 + a\lambda + b = 0$ has two real roots $\lambda_1 \neq \lambda_2$

$$\lambda^2 + a\lambda + b = (\lambda - \lambda_1)(\lambda - \lambda_2) = 0$$

$$\Rightarrow \quad (D^2 + aD + b)y = 0 \qquad \text{(eqn)}$$

$$\boxed{(D - \lambda_1)\underbrace{(D - \lambda_2)y}_{w} = 0} \qquad \begin{cases} (D - \lambda_1)w = 0 \\ (D - \lambda_2)y = w \end{cases}$$

$$(D - \lambda_1)w = 0 \iff w' - \lambda_1 w = 0$$

$$w' = \lambda_1 w$$

$$\Rightarrow \quad w = Ce^{\int \lambda_1 \, dt} = Ce^{\lambda_1 t}$$

$$\Rightarrow (D - \lambda_2)y = w = Ce^{\lambda_1 t}$$

$$\iff y' - \lambda_2 y = Ce^{\lambda_1 t}$$

$$\iff y' = \lambda_2 y + Ce^{\lambda_1 t}$$

$$\Leftrightarrow \quad y(t) = e^{\int \lambda_2\, dt}\left[\int C e^{\lambda_1 t} \cdot e^{-\int \lambda_2\, dt}\, dt + \tilde{C}\right]$$

$$= e^{\lambda_2 t}\left[C \int e^{\lambda_1 t} e^{-\lambda_2 t}\, dt + \tilde{C}\right]$$

$$(*) \quad y(t) = e^{\lambda_2 t}\left[C \int e^{(\lambda_1 - \lambda_2) t}\, dt + \tilde{C}\right]$$

$$\lambda_1 - \lambda_2 \neq 0$$

$$\int e^{(\lambda_1-\lambda_2)t}\, dt = \frac{e^{(\lambda_1-\lambda_2)t}}{\lambda_1 - \lambda_2}$$

$$= \frac{C}{\lambda_1 - \lambda_2} e^{\lambda_1 t} + \tilde{C} e^{\lambda_2 t} \equiv c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

$$c_1, c_2 \in \mathbb{R}$$

Gen sol in case $\Delta > 0$: $\boxed{y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \quad c_1, c_2 \in \mathbb{R}.}$

__Example:__ $y'' + 5y' - 6y = 0$

The charact. eqn is $\lambda^2 + 5\lambda - 6 = 0$

$\Delta = 25 + 24 = 49 > 0$

$$\lambda_{1,2} = \frac{-5 \pm \sqrt{49}}{2} = \frac{-5 \pm 7}{2} = 1, -6$$

Gen sol: $y = c_1 e^{t} + c_2 e^{-6t}$ $\quad c_1, c_2 \in \mathbb{R}$

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$\boxed{\text{Case } \Delta = 0}$ This is the same of prev. one with $\lambda_1 = \lambda_2$. We can repeat the calc. up to $(*)$

$$y(t) = e^{\lambda_2 t}\left[c \int e^{(\lambda_1 - \lambda_2)t}\, dt + \tilde{c}\right]$$

$$(\lambda_1 = \lambda_2) = e^{\lambda_1 t}\Big[c \underbrace{\int 1\, dt}_{= t} + \tilde{c}\Big]$$

$$= e^{\lambda_1 t}\left[ct + \tilde{c}\right] = c_1 t e^{\lambda_1 t} + c_2 e^{\lambda_1 t}$$

Def: Functs

- $(e^{\lambda_1 t}, e^{\lambda_2 t})$ in case $\Delta > 0$
- $(e^{\lambda_1 t}, t e^{\lambda_1 t})$ " " $\Delta = 0$

are called fundamental solutions

$\boxed{\text{Case } \Delta < 0}$ In this case $\lambda_1, \lambda_2 \in \mathbb{C}$

$$\lambda_{1,2} = \frac{-b \pm \sqrt{\Delta<0}}{2a} = -\frac{b}{2a} \pm i \frac{\sqrt{-\Delta}}{2a}$$

$$= \alpha \pm i\omega \qquad \alpha, \omega \in \mathbb{R}$$

$$e^{\alpha \pm i\omega} = e^{\alpha} \underbrace{e^{\pm i\omega}}_{(\cos\omega + i \,\mathrm{sen}\,\omega)}$$

Repeating step 1 (case $\Delta > 0$) we obtain

$$y(t) = c_1 e^{(\alpha+i\omega)t} + c_2 e^{(\alpha-i\omega)t}$$

$$= c_1 e^{\alpha t}\big(\cos(\omega t) + i\sin(\omega t)\big) + c_2 e^{\alpha t}\big(\cos(\omega t) - i\sin(\omega t)\big)$$

$\Downarrow$

$y(t)$ is linear combination of

$$e^{\alpha t}\big(\cos(\omega t) + i\sin(\omega t)\big) +$$

$$e^{\alpha t}\big(\cos(\omega t) - i\sin(\omega t)\big)$$

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(2) $e^{\alpha t}\cos(\omega t) \rightsquigarrow \underline{e^{\alpha t}\cos(\omega t)}$ is a sol.

$\longrightarrow e^{\alpha t}\sin(\omega t)$ is a sol

$$\Rightarrow \boxed{y(t) = c_1 e^{\alpha t}\cos(\omega t) + c_2 e^{\alpha t}\sin(\omega t) \quad c_1, c_2 \in \mathbb{R}}$$

$$\Rightarrow \boxed{y(t) = c_1 e^{\alpha t} \cos(\omega t) + c_2 e^{\alpha t} \sin(\omega t)} \quad c_1, c_2 \in \mathbb{R}$$

$(e^{\alpha t} \cos(\omega t), e^{\alpha t} \sin(\omega t))$ is the fund syst of sols.

Ex 1.5.9. Det the gen sol of

#2 $\quad y'' - 2y' + 2y = 0.$

This is a second order lin hom eqn. The characteristic eqn is

$$\lambda^2 - 2\lambda + 2 = 0$$

$$\Delta = (-2)^2 - 4 \cdot 1 \cdot 2 = 4 - 8 = -4 < 0$$

Sols of ch. eqns are $\lambda_{1,2} = \dfrac{2 \pm \sqrt{-4}}{2} = \dfrac{2 \pm i2}{2}$

$$= \underbrace{1}_{\alpha} \pm i \underbrace{1}_{\omega}$$

$\Rightarrow$ gen sol $\quad y = c_1 e^t \cos t + c_2 e^t \sin t, \quad c_1, c_2 \in \mathbb{R}$ ∎

Non homogeneous case

We now consider a non homog eqn

$$y'' + ay' + by = f(t) \qquad \text{(NH)}$$

$$y'' + ay' + by = f(t) \qquad (NH)$$

Prop: Let (w_1, w_2) be a fund syst of sols for

$$y'' + ay' + by = 0 \qquad (H)$$

Let U be a particular solution of (NH). Then the gen sol of (NH) is

$$\boxed{y(t) = c_1 w_1(t) + c_2 w_2(t) + U(t), \quad c_1, c_2 \in \mathbb{R}}$$

Proof: Suppose U be a sol of (NH)

$$U'' + aU' + bU = f(t)$$

If y is any other sol of (NH) $-$

$$y'' + ay' + by = f(t)$$

$$\Rightarrow \quad y'' - U'' + a(y' - U') + b(y - U) = 0$$

$$\parallel$$

$$(y-U)'' + a(y-U)' + b(y-U) = 0$$

$\Rightarrow$ $y - U$ solves (H) $\Rightarrow$

$$y - U = c_1 w_1 + c_2 w_2 \qquad \blacksquare$$

To det competely the gen sol of (NH) we just need to det. a particular solution $\bar{U}$ of (NH)

Idea: Lagrange variation of constants formula

Take

$$U(t) = c_1(t)w_1(t) + c_2(t)w_2(t)$$

variable coeff.

Let's impose U be a sol of (NH)

$$U'' + aU' + bU = f(t)$$

Notice that

$$U' = (c_1 w_1 + c_2 w_2)'$$

$$= c_1' w_1 + c_1 w_1' + c_2' w_2 + c_2 w_2'$$

We impose

$$(1) \quad \boxed{c_1' w_1 + c_2' w_2 \equiv 0}$$

$$\Rightarrow \quad U' = c_1 w_1' + c_2 w_2'$$

$$\Rightarrow \quad U'' = (c_1 w_1' + c_2 w_2')'$$

$$\Rightarrow \quad U'' = (c_1 w_1' + c_2 w_2')'$$

$$= c_1' w_1' + c_1 w_1'' + c_2' w_2' + c_2 w_2''$$

So U is sol of (NH) $\Leftrightarrow$

$$\Big(\underbrace{c_1' w_1' + c_1 w_1'' + c_2' w_2' + c_2 w_2''}_{U''}\Big) + \underbrace{a(c_1 w_1' + c_2 w_2')}_{aU'} + \underbrace{b(c_1 w_1 + c_2 w_2)}_{bU} = f(t)$$

w_1 fund sol of (H) w_2 fund sol of (H)

$$c_1 \underbrace{(w_1'' + a w_1' + b w_1)}_{=0} + c_2 \underbrace{(w_2'' + a w_2' + b w_2)}_{=0} +$$

$$c_1' w_1' + c_2' w_2' = f(t)$$

$$\Leftrightarrow (2) \quad \boxed{c_1' w_1' + c_2' w_2' = f(t)}$$

<u>Conclusion</u>: $U = c_1 w_1 + c_2 w_2$ is sol of (NH)

$\Leftrightarrow$ (1) & (2) hold

$$\Leftrightarrow \quad \begin{cases} c_1' w_1 + c_2' w_2 = 0 \\ c_1' w_1' + c_2' w_2' = f(t) \end{cases}$$

$$\Downarrow$$

$\{ \ ax + by = 0$

$$\begin{cases} ax + by = 0 \\ cx + dy = f \end{cases}$$

$$M \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{bmatrix} w_1 & w_2 \\ w_1' & w_2' \end{bmatrix}^{-1} \begin{pmatrix} 0 \\ f(t) \end{pmatrix}$$

$$\underbrace{W(t)}_{\text{wronskian}} = \det \underbrace{\begin{bmatrix} w_1 & w_2 \\ w_1' & w_2' \end{bmatrix}}_{\text{wronskian matrix}}$$

$$\begin{cases} c_1' = -\dfrac{w_2(t)}{W(t)} f(t) \\[2ex] c_2' = \dfrac{w_1(t)}{W(t)} f(t) \end{cases}$$

$$\Rightarrow \begin{cases} c_1(t) = -\displaystyle\int \frac{w_2(t)}{W(t)} f(t)\, dt \\[2ex] c_2(t) = \displaystyle\int \frac{w_1(t)}{W(t)} f(t)\, dt. \end{cases}$$

So the part sol is

$$U(t) = \left(-\int \frac{w_2(t)}{W(t)} f(t)\, dt\right) w_1(t) + \left(\int \frac{w_1(t)}{W(t)} f(t)\, dt\right) w_2(t)$$

<u>Rmk:</u> The wronskian is always $\neq 0$

For example: if $\Delta > 0$ $(w_1, w_2) = (e^{\lambda_1 t}, e^{\lambda_2 t})$

$\lambda_1 \neq \lambda_2$

$\lambda_1 \neq \lambda_2$.

Wronskian matrix $\begin{bmatrix} w_1 & w_2 \\ w_1' & w_2' \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & e^{\lambda_2 t} \\ \lambda_1 e^{\lambda_1 t} & \lambda_2 e^{\lambda_2 t} \end{bmatrix}$

Wronskian : $W(t) = \det \;// = e^{(\lambda_1+\lambda_2)t}\lambda_2 - e^{(\lambda_1+\lambda_2)t}\lambda_1$

$$= e^{(\lambda_1+\lambda_2)t}\underset{\neq 0}{(\lambda_2-\lambda_1)} \qquad \blacksquare$$

Ex. 1.4.4 Find the gen sol of

$$y'' + y' - 6y = 2e^{-t} \qquad t \in \mathbb{R}.$$

Sol: We've a 2nd order lin non hom. eqn.
To det the fund syst, let's write first the charact. eqn: for hom eqn

$$y'' + y' - 6y = 0 \longrightarrow \lambda^2 + \lambda - 6 = 0$$

$\Delta = 1^2 - 4 \cdot 1 \cdot (-6) = 25 > 0$, $\lambda_{1,2} = \frac{-1 \pm \sqrt{25}}{2} = \frac{-1 \pm 5}{2}$
$= -3, 2$

Fund syst: $(w_1, w_2) = \left(e^{-3t}, e^{2t}\right)$.

Let's compute the part sol for NH.

$$U(t) = c_1(t)\, w_1(t) + c_2(t)\, w_2(t)$$

$$c_1(t) = -\int \frac{w_2}{W} f \qquad\qquad c_2 = +\int \frac{w_1}{W} f$$

$$W(t) = \det \begin{bmatrix} w_1 & w_2 \\ w_1' & w_2' \end{bmatrix} = \det \begin{bmatrix} e^{-3t} & e^{2t} \\ -3e^{-3t} & 2e^{2t} \end{bmatrix}$$

$$= e^{-t} \cdot 2 + 3e^{-t} = 5e^{-t}$$

$$\Rightarrow c_1(t) = -\int \frac{e^{2t}}{5\cancel{e^{-t}}} \cdot 2\cancel{e^{-t}}\, dt = -\frac{\cancel{2}}{5} \int 2e^{2t}\, dt$$

$$= -\frac{1}{5} e^{2t}$$

$$c_2(t) = +\int \frac{e^{-3t}}{5\cancel{e^{-t}}} \cdot 2\cancel{e^{-t}}\, dt = -\frac{2}{5\cdot 3} \int -3e^{-3t}$$

$$= -\frac{2}{15} e^{-3t}$$

$$\Rightarrow U(t) = -\frac{3}{15} \underbrace{e^{2t} \cdot e^{-3t}}_{e^{-t}} + -\frac{2}{15} \underbrace{e^{-3t} \cdot e^{2t}}_{e^{-t}}$$

$$= -\frac{1}{3} e^{-t}$$

$$\Rightarrow \text{gen sol} \quad y(t) = c_1 e^{-3t} + c_2 e^{2t} - \frac{1}{3} e^{-t} \qquad c_1, c_2 \in \mathbb{R}$$

∎

Do 1.5.9/10/11 1.5.12 → 16.