lunedì 24 febbraio 2020 08:26

Ex 1.5.10 Find the gen sol of

#6 $y'' + y = \frac{1}{\cos t}$.

Sol: We've a second order non homog eqn.
Let's first compute a fund ~~syst~~ of sols. The charact. eqn is

$$\lambda^2 + 1 = 0 \implies \lambda = \pm i = 0 \pm i1$$

$$\implies w_1 = e^{0t}\cos(1t) = \cos t, \quad w_2 = \sin t$$

Gen sol is

$$y(t) = c_1 \cos t + c_2 \sin t + U(t)$$

where

$$U(t) = \left(-\int \frac{w_2}{W} f \, dt\right) w_1 + \left(\int \frac{w_1}{W} f \, dt\right) w_2$$

where
$$f(t) = \frac{1}{\cos t},$$
$$W(t) = \det \begin{bmatrix} w_1 & w_2 \\ w_1' & w_2' \end{bmatrix} = \det \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \equiv 1$$

$$= \left(+\int \frac{-\sin t}{1} \cdot \frac{1}{\cos t} \, dt\right) \cos t + \left(\int \frac{\cancel{\cos t}}{1} \cdot \frac{1}{\cancel{\cos t}} \, dt\right) \sin t$$

$$= \left(\log|\cos t|\right) \cos t + t \sin t. \quad \blacksquare$$

#2. $y'' - y' + y = e^t$.

Sol: Let's first det the fund syst for hom eqn

$$y'' - y' + y = 0$$

Characteristic eqn: $\lambda^2 - \lambda + 1 = 0$ $\quad \Delta = (-1)^2 - 4\cdot 1\cdot 1$
$$= -3$$

$$\Rightarrow \lambda_{1,2} = \frac{+1 \pm \sqrt{-3}}{2} = \frac{1 \pm i\sqrt{3}}{2} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$\Rightarrow$ Fund syst: $w_1 = e^{t/2}\cos\left(\frac{\sqrt{3}}{2}t\right)$

$$w_2 = e^{t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)$$

Gen sol of NH eqn is

$$y(t) = c_1 w_1(t) + c_2 w_2(t) + U(t) \qquad c_1, c_2 \in \mathbb{R}$$

where

$$U(t) = \left(-\int \frac{w_2}{W} f\right) w_1 + \left(\int \frac{w_1}{W} f\right) w_2$$

$$\int \frac{w_2}{W} f \, dt = \int \frac{e^{t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)}{\frac{\sqrt{3}}{2}\cancel{e^t}} \cdot \cancel{e^t} \, dt = \frac{2}{\sqrt{3}} \int e^{t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)$$

$$W = \det\begin{bmatrix} w_1 & w_2 \\ w_1' & w_2' \end{bmatrix} = \frac{\sqrt{3}}{2} e^t$$

$$W = \det \begin{bmatrix} \cdots & \cdots \\ w_1' & w_2' \end{bmatrix} = \cdots \frac{\cdots}{2} \cdots$$

$$\int \underbrace{e^{t/2}}_{(2e^{t/2})'} \sin\left(\frac{\sqrt{3}}{2}t\right) dt = 2e^{t/2} \sin\left(\frac{\sqrt{3}}{2}t\right) - \frac{\sqrt{3}}{\cancel{2}} \int \cancel{2} \underbrace{e^{t/2}}_{(2e^{t/2})'} \cos\left(\frac{\sqrt{3}}{2}t\right)$$

$$= 2e^{t/2} \sin \frac{\sqrt{3}}{2}t - \sqrt{3}\left[2e^{t/2} \cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{\sqrt{3}}{\cancel{2}} \int \cancel{2} e^{t/2} \sin \frac{\sqrt{3}}{2}t\right]$$

$$= 2e^{t/2}\left[\sin \frac{\sqrt{3}}{2}t - \sqrt{3} \cos\left(\frac{\sqrt{3}}{2}t\right)\right]$$

$$- 3 \int e^{t/2} \sin\left(\frac{\sqrt{3}}{2}t\right) dt$$

$$\Rightarrow \int e^{t/2} \sin\left(\frac{\sqrt{3}}{2}t\right) dt = \frac{1}{2} e^{t/2}\left[\sin \frac{\sqrt{3}}{2}t - \sqrt{3} \cos \frac{\sqrt{3}}{2}t\right]$$

⋮

too long.... ☒

CP for 2nd order eqns

What is the appropriate form for the CP for an eqn of type

$$y'' + ay' + by = f(t)$$

We know that the gen sol has form

We know that the gen ...

$$y(t) = c_1 w_1(t) + c_2 w_2(t) + U(t) \qquad c_1, c_2 \in \mathbb{R}$$

$$\left[\begin{aligned} y(t_0) &= y_0 = c_1 w_1(t_0) + c_2 w_2(t_0) + U(t_0) \\ y(t_1) &= y_1 = c_1 w_1(t_1) + c_2 w_2(t_1) + U(t_1) \end{aligned}\right]$$

$$y'(t_0) = v_0$$

Thm: The CP

$$\text{CP} \quad \begin{cases} y'' + ay' + by = f(t) \\ y(t_0) = y_0 \\ y'(t_0) = v_0 \end{cases}$$

has a unique sol $\forall \, (t_0, y_0, v_0)$

Proof: Indeed, $\boxed{y = c_1 w_1 + c_2 w_2 + U}$ is sol of CP

$$\Leftrightarrow \begin{cases} c_1 w_1(t_0) + c_2 w_2(t_0) + U(t_0) = y_0 & (y(t_0) = y_0) \\ c_1 w_1'(t_0) + c_2 w_2'(t_0) + U'(t_0) = v_0 & (y'(t_0) = v_0) \end{cases}$$

$$\Leftrightarrow \begin{cases} c_1 w_1(t_0) + c_2 w_2(t_0) = y_0 - U(t_0) \\ c_1 w_1'(t_0) + c_2 w_2'(t_0) = v_0 - U'(t_0) \end{cases} \qquad (*)$$

$$\begin{bmatrix} w_1(t_0) & w_2(t_0) \end{bmatrix}$$

(*) has a unique sol $\Leftrightarrow$ $\underbrace{\det \begin{bmatrix} w_1(t_0) & w_2(t_0) \\ w_1'(t_0) & w_2'(t_0) \end{bmatrix}}_{W(t_0)} \neq 0$

(wronskian) (true!) ∎

Ex 1.5.11.

#1. $\begin{cases} y'' - y = t \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$

Sol: We first compute the gen int of NH eq

$$y'' - y = t$$

Ch. eqn: $\cancel{\lambda^2 - \lambda = 0}$ $\quad \lambda^2 - 1 = 0$

$$\Updownarrow$$

$$\lambda = \pm 1$$

$$\Rightarrow w_1 = e^t, \quad w_2 = e^{-t}$$

$\Rightarrow$ gen sol is

$$y(t) = c_1 e^t + c_2 e^{-t} + U(t)$$

where

$$U(t) = \left(- \int \frac{w_2}{W} f \right) w_1 + \left(\int \frac{w_1}{W} f \right) w_2$$

$$\int \frac{w_2}{W} f =$$

$$W = \det \begin{bmatrix} w_1 & w_2 \\ w_1' & w_2' \end{bmatrix} = \det \begin{bmatrix} e^{t} & e^{-t} \\ e^{t} & -e^{-t} \end{bmatrix} = -1 - 1 = -2$$

$$= \int \frac{e^{-t}}{-2} t \, dt = + \frac{1}{2} \int \underbrace{(-e^{-t})}_{= (e^{-t})'} t \, dt$$

$$= \frac{1}{2} \left[e^{-t} t + \int -e^{-t} \, dt \right]$$

$$= \frac{1}{2} e^{-t} (t+1).$$

$$\int \frac{w_1}{W} f = \int \frac{e^{t}}{-2} t = -\frac{1}{2} \int e^{t} t \, dt$$

$$= -\frac{1}{2} \left[e^{t} t - \int e^{t} \right] = -\frac{1}{2} e^{t} (t-1)$$

$$\Rightarrow y(t) = c_1 e^{t} + c_2 e^{-t} - \frac{1}{2} \cancel{e^{-t}} (t+1) \cancel{e^{t}} - \frac{1}{2} \cancel{e^{t}} (t-1) \cancel{e^{-t}}$$

$$- \frac{1}{2} 2t = -t$$

$$\boxed{y(t) = c_1 e^{t} + c_2 e^{-t} - t.}$$ $\qquad y' = c_1 e^{t} - c_2 e^{-t} - 1$

Now, to solve CP we need

$$\begin{cases} c_1 + c_2 = 0 & y(0)=0 \\ c_1 - c_2 - 1 = 0 & y'(0)=0 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \cancel{2}c_1 = 1/\cancel{2} \\ c_2 = -1/2 \end{cases} \Rightarrow y = \frac{1}{2}e^{t} - \frac{1}{2}e^{-t} - t. \quad \blacksquare$$

1.5.12 Consider the eqn

$$y'' - y' = te^{t} \qquad t \in \mathbb{R}$$

i) Gen int $\qquad$ ii) Are there sols : $\lim_{t \to +\infty} y(t) \in \mathbb{R}$?

iii) Solve CP $\quad y(0)=1,\ y'(0)=0$

Sol: i) The hom eqn is

$$y'' - y' = 0$$

Its charact eqn is $\lambda^2 - \lambda = 0 \Rightarrow \lambda(\lambda-1)=0$

$$\Rightarrow \lambda = 0, 1$$

$$\Rightarrow \quad w_1 = 1, \quad w_2 = e^{t}$$

The gen sol of NH eqn is

$$y(t) = c_1 \cdot 1 + c_2 e^t + U(t) \quad \text{where}$$

$$U(t) = \left(-\int \frac{\cancel{e^t}}{\cancel{e^t}} \cdot t e^t\right) \cdot 1 + \left(\int \frac{1}{\cancel{e^t}} \cdot t \cancel{e^t}\right) e^t$$

$$W(t) = \det \begin{bmatrix} 1 & e^t \\ 0 & e^t \end{bmatrix} = e^t$$

$$= -e^t(t-1) + \frac{t^2}{2} e^t = e^t\left(\frac{t^2}{2} - t + 1\right)$$

$$\Rightarrow \quad y = c_1 + c_2 e^t + e^t\left(\frac{t^2}{2} - t + 1\right).$$

ii) $\exists\, c_1, c_2 \;:\; \lim\limits_{t \to +\infty} \Big(c_1 + c_2 e^t + \underbrace{e^t\left(\frac{t^2}{2} - t + 1\right)}_{\sim e^t \frac{t^2}{2} \to +\infty} \Big) \in \mathbb{R}$

$$= +\infty$$

$\forall c_1, c_2 \Rightarrow \nexists$ sols with $\lim\limits_{+\infty} y \in \mathbb{R}$

iii) $\begin{cases} y(0) = 1 \\ y'(0) = 0 \end{cases} \iff \begin{cases} c_1 + c_2 + \cancel{1} = \cancel{1}\,0 \\ c_2 + 1 - 1 = 0 \end{cases}$ $\quad y' = c_2 e^t + e^t\left(\frac{t^2}{2} - t + 1\right) + e^t(t-1)$

$\begin{cases} c_2 = 0 \\ c_1 = 0 \end{cases} \Rightarrow \quad y = U.$ ☒

1.5.16 $\quad y'' + 4y' + 4y = \dfrac{e^{-2t}}{t^2}$. $\quad (*)$

i) Find gen int

ii) Are there sols : $\exists \lim_{t\to 0+} y(t)$?

Sol: The homog eqn associated to $(*)$ is

$$y'' + 4y' + 4 = 0$$

$\Rightarrow$ charact. eqn $\cdot \lambda^2 + 4\lambda + 4 = 0$

$$(\lambda+2)^2 = 0$$

$$\Rightarrow \lambda = -2 \qquad (\Delta = 0)$$

$\Rightarrow w_1 = e^{-2t} \qquad w_2 = te^{-2t}$

Therefore the gen sol is

$$y = c_1 e^{-2t} + c_2 te^{-2t} + U(t)$$

where

$$U(t) = \left(-\int \frac{t\cancel{e^{-2t}}}{\cancel{e^{-4t}}} \frac{\cancel{e^{-2t}}}{t^2}\, dt \right) e^{-2t} + \left(\int \frac{\cancel{e^{-2t}}}{\cancel{e^{-4t}}} \frac{\cancel{e^{-2t}}}{t^2} \right) te^{-2t}.$$

Δ

$$\Big[\ e^{-2t} \qquad te^{-2t} \qquad \Big]$$

$$W(t) = \det \begin{bmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t}(-2t+1) \end{bmatrix}$$

$$= e^{-4t}\left[\cancel{-2t}+1\right] + \cancel{2te^{-4t}}$$

$$= e^{-4t}$$

$$= -\left(\int \frac{1}{t}\right) e^{-2t} + \left(\int \frac{1}{t^2}\right) te^{-2t}$$

$$= -\left(\log|t|\right) e^{-2t} + \left(-\frac{1}{\cancel{t}}\right)\cancel{t}e^{-2t}$$

$$= -e^{-2t}\left(+\log|t| + 1\right)$$

$\Rightarrow$ gen int $\quad y = c_1 e^{-2t} + c_2 te^{-2t} - e^{-2t}\left(\log|t|+1\right)$

ii) Are there sols : $\exists \lim_{t\to 0+} y(t)$?

$\Updownarrow$

$\exists c_1, c_2 : \quad \lim_{t \to 0+} \left(\underset{\downarrow\, 1}{c_1 e^{-2t}} + \underset{\downarrow\, 0}{c_2 te^{-2t}} \overbrace{- \underset{\swarrow\, 1}{e^{-2t}}\left(\underset{\downarrow\, -\infty}{\log|t|}+\underset{\downarrow\, -\infty}{1}\right)}^{\nearrow +\infty}\right)$

exists

$= +\infty \quad \forall c_1, c_2.$

☑

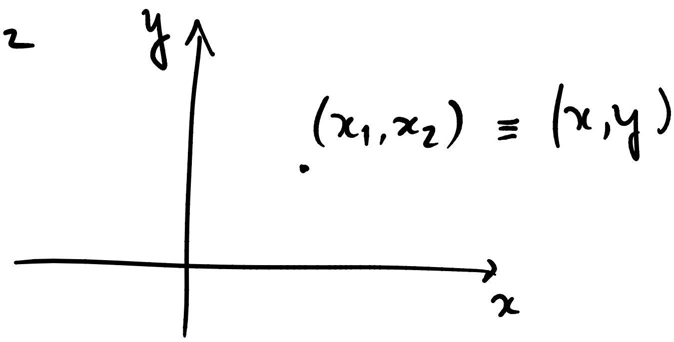
lunedì 24 febbraio 2020 09:45

Basic Topology on $\mathbb{R}^d$

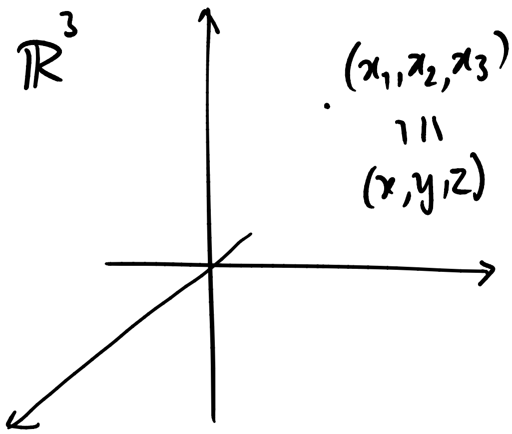
Let

$$\mathbb{R}^d = \{ \overbrace{(x_1, \dots, x_d)}^{\vec{x}} : x_1, x_2, \dots x_d \in \mathbb{R} \}$$

$\mathbb{R}^2$



$\mathbb{R}^3$



$\mathbb{R}^d$ is a <u>vector space</u> on which are defined

sum $\quad \vec{x} + \vec{y} := (x_1 + y_1, x_2 + y_2, \dots, x_d + y_d)$

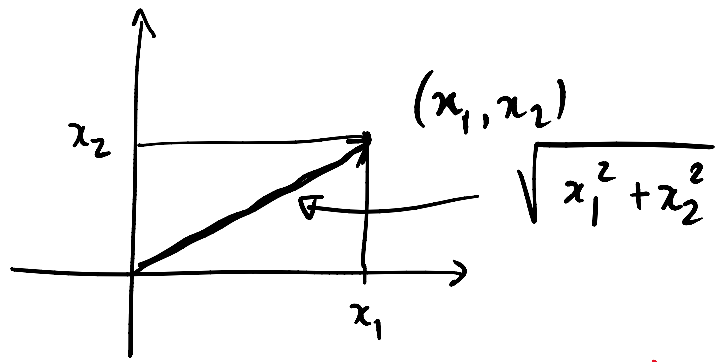
$\vec{x} = (x_1 \dots x_d)$, $\quad \vec{y} = (y_1, \dots y_d)$

product $\quad \lambda \vec{x} := (\lambda x_1, \lambda x_2, \dots, \lambda x_d)$.

$\lambda \in \mathbb{R}$, $\quad \vec{x} = (x_1 \dots x_d)$

$\vec{0} = (0, 0, \dots, 0)$.

Length / Norm of a vector



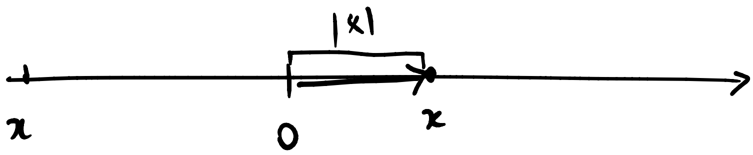
If $\vec{x} = (x_1 \dots x_d) \in \mathbb{R}^d$ we pose

$$\|\vec{x}\| := \sqrt{x_1^2 + x_2^2 + \dots + x_d^2}$$

(norm of $\vec{x}$)

Rmk: If $d=1$ $\mathbb{R}^1 \equiv \mathbb{R}$

$$\|(x)\| = \sqrt{x^2} = |x|$$



Properties of $\|\cdot\|$

1. positivity: $\|\vec{x}\| \geq 0 \quad \forall \vec{x} \in \mathbb{R}$
2. vanishing: $\|\vec{x}\| = 0 \iff \vec{x} = \vec{0}$
3. homogeneity: $\|\lambda \vec{x}\| = |\lambda| \cdot \|\vec{x}\| \quad \forall \lambda \in \mathbb{R}, \forall \vec{x} \in \mathbb{R}^d$

$$(\|\lambda \vec{x}\| = \|(\lambda x_1, \lambda x_2, \dots \lambda x_d)\|$$

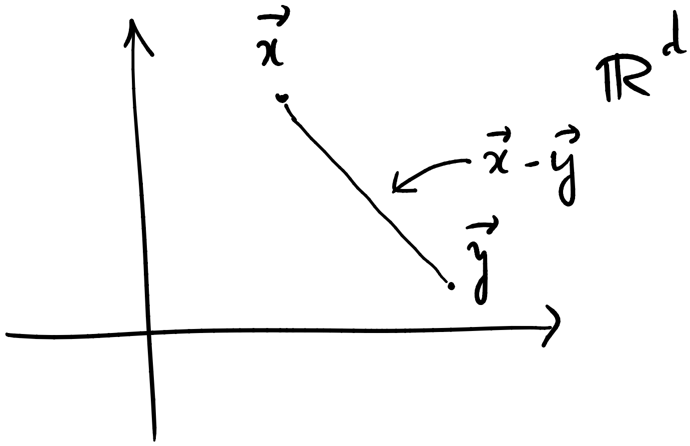
$$\sqrt{(\lambda x_1)^2 + (\lambda x_2)^2 + \dots + (\lambda x_d)^2}$$

$$= \sqrt{(\lambda x_1)^2 + (\lambda x_2)^2 + \dots + (\lambda x_d)^2}$$

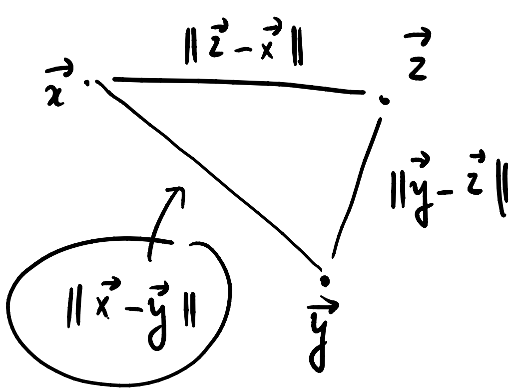
$$= \sqrt{\lambda^2 (x_1^2 + x_2^2 + \dots + x_d^2)} = |\lambda| \, \|\vec{x}\| \quad)$$

4. triangular inequality: $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\| \quad \forall \vec{x}, \vec{y} \in \mathbb{R}^d$.

Through the concept of norm we may introduce the concept of distance between pts in $\mathbb{R}^d$:



$$\text{dist}(\vec{x}, \vec{y}) := \|\vec{x} - \vec{y}\|$$



$$\|\vec{x} - \vec{y}\| \leq \|\vec{x} - \vec{z}\| + \|\vec{z} - \vec{y}\|$$

$$\|(\vec{x}-\vec{z})+(\vec{z}-\vec{y})\| \leq \| \quad \| + \| \quad \|$$

$\vec{u}$ $\quad$ $\vec{v}$

$$\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$$

Def of limit (sequence)

A sequence of vectors $(\vec{x}_n)_{n\in\mathbb{N}} \subset \mathbb{R}^d$

$$\vec{x}_n = (x_{n,1}, x_{n,2}, \dots, x_{n,d})$$

Example: $(n, 1/n)$

$(1,1), (2, 1/2), (3, 1/3), (4, 1/4), \dots$

$\left(\frac{1}{n}, \frac{1}{n+1}\right)$ $\quad$ $(1, 1/2), (1/2, 1/3), (1/3, 1/4), \dots$

(n, n) $\quad$ $(1,1), (2,2), (3,3), \dots$

Def: (finite limit)

Given $(\vec{x}_n) \subset \mathbb{R}^d$ we say that

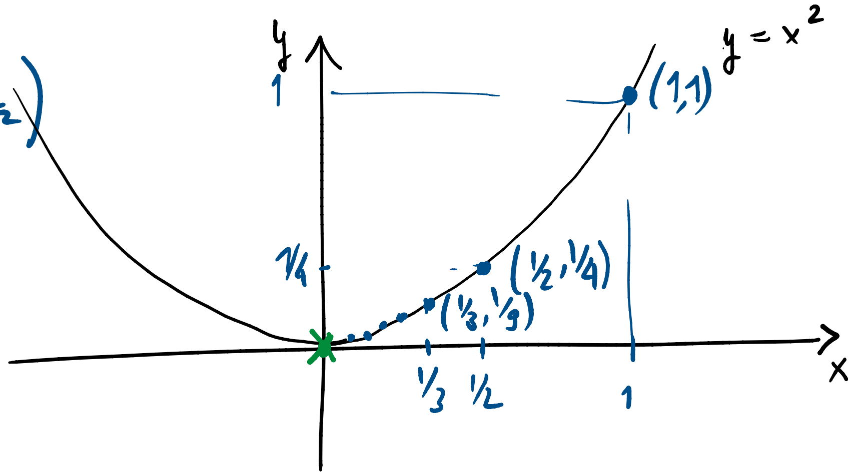
$$\vec{x}_n \longrightarrow \vec{\ell} \in \mathbb{R}^d \quad \left(\lim_{n\to\infty} \vec{x}_n = \vec{\ell}\right)$$

if

$$\forall \varepsilon > 0 \ \exists N(\varepsilon) : \ \|\vec{x}_n - \vec{\ell}\| \leq \varepsilon \quad \forall n \geq N(\varepsilon).$$

y $\quad$ $y = x^2$

Ex: $\vec{x}_n = \left(\frac{1}{n}, \frac{1}{n^2}\right) \xrightarrow{\mathbb{R}^2} \vec{0}$



Indeed

$$\|\vec{x}_n - \vec{0}\| = \|\vec{x}_n\| = \sqrt{\left(\frac{1}{n}\right)^2 + \left(\frac{1}{n^2}\right)^2}$$

$$= \underbrace{\frac{1}{n}}_{\downarrow\, 0} \sqrt{\underbrace{1 + \frac{1}{n^2}}_{\downarrow\, 1}} \longrightarrow 0$$

$$\Rightarrow \|\vec{x}_n - \vec{0}\| \longrightarrow 0 \qquad \left(\text{so } \|\vec{x}_n - \vec{0}\| \leq \varepsilon \quad \forall n \geq N(\varepsilon)\right)$$

Rmk: Clearly

$$\vec{x}_n \longrightarrow \vec{\ell} \quad \Leftrightarrow \quad \|\vec{x}_n - \vec{\ell}\| \longrightarrow 0.$$

Example: $\left(\sqrt{4 + 1/n}, 1/n\right) \longrightarrow (2, 0)$

Let's check:

$$\left\| \left(\sqrt{4+1/n}, 1/n\right) - (2,0) \right\| = \left\| \left(\sqrt{4+1/n} - 2, 1/n\right) \right\|$$

$$= \sqrt{\Big(\underbrace{\sqrt{4+\tfrac{1}{n}}}_{\to\sqrt{4}=2} - 2\Big)^2 + \Big(\underbrace{\tfrac{1}{n}}_{\to 0}\Big)^2} \longrightarrow 0$$

Prop: $\vec{x}_n \longrightarrow \vec{\ell} = (\ell_1, \dots, \ell_d) \iff x_{n,j} \longrightarrow \ell_j \quad \forall j = 1, 2, \dots, d.$

$\vec{x}_n = (x_{n,1}, x_{n,2}, \dots, x_{n,d})$

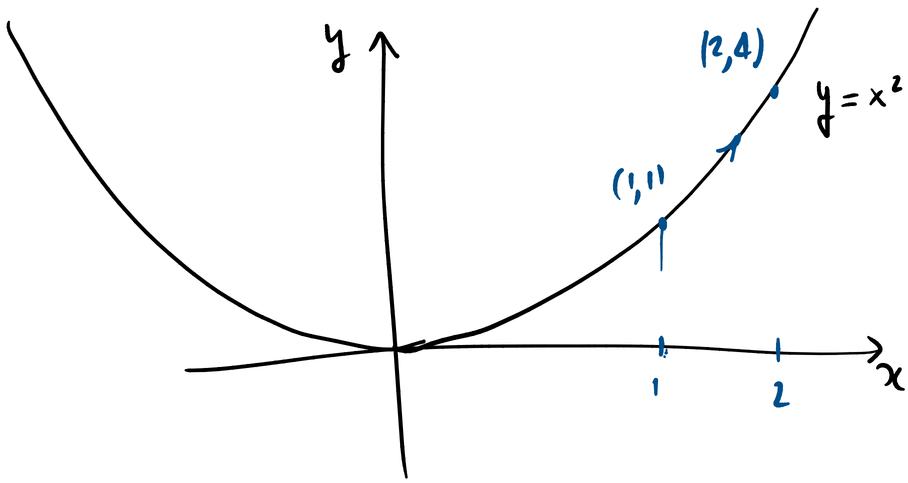
Def (Infinite limit)

We say that $\vec{x}_n \longrightarrow \infty_d$ if

$$\|\vec{x}_n\| \longrightarrow +\infty$$

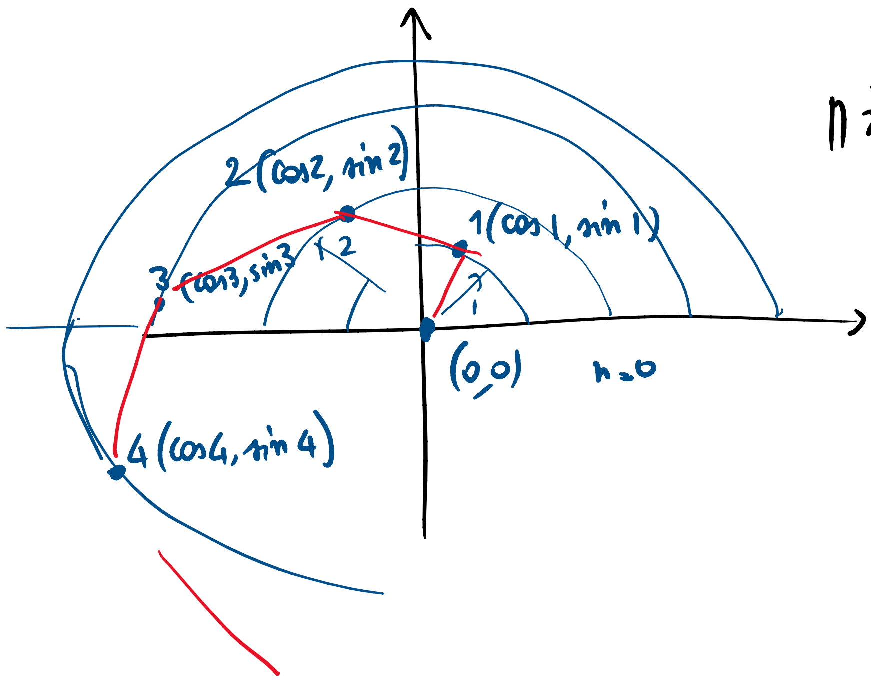
Examples:

$\vec{x}_n = (n, n^2) \qquad \|\vec{x}_n\| = \sqrt{n^2 + n^4} = n\sqrt{1+n^2} \longrightarrow +\infty$



$$\vec{x}_n = (n\cos n, n\sin n) = n(\cos n, \sin n)$$

$$\vec{x}_n = (n\cos n, n\sin n) = n(\cos n, \sin n)$$



$$\|\vec{x}_n\| = \sqrt{n^2(\sin n)^2 + n^2(\cos n)^2}$$

$$= \sqrt{n^2} = n$$

$$\downarrow$$

$$+\infty$$