Limit for a function of several vars

Let $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m \ (\mathbb{R})$

$$f = f(x_1, \dots\dots, x_d)$$

Goal: To define

$$\boxed{\lim_{\vec{x} \to \vec{x}_0} f(\vec{x}) = \ell \in \mathbb{R} \cup \{\pm\infty\}.}$$

Let's refresh the case $f = f(x) : D \subset \mathbb{R} \longrightarrow \mathbb{R}$.

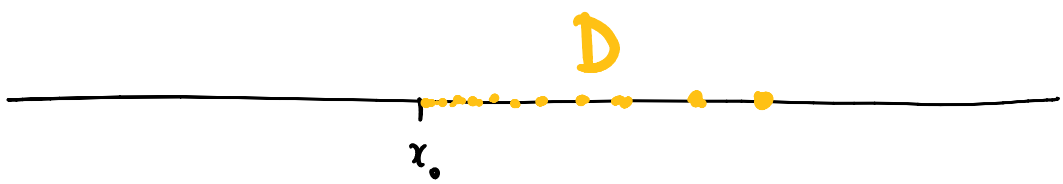
To def

$$\lim_{x \to x_0} f(x) = \ell$$

we need x_0 be an accumulation point for D

Def: We say that $x_0 \in \mathbb{R} \cup \{\pm\infty\}$ is accumulation pt. for D if

$$\exists (x_n) \subset D \setminus \{x_0\} : x_n \longrightarrow x_0 .$$



Def: Let $f: D \subset \mathbb{R} \to \mathbb{R}$, $x_0 \in \text{Acc } D$ ($\equiv$ set of all acc. pts for D)

We say that

$$\exists \lim_{x \to x_0} f(x) = \ell \in \mathbb{R} \cup \{\pm\infty\}$$

if

$$\forall (x_n) \subset D \setminus \{x_0\} : \quad x_n \to x_0 \implies f(x_n) \to \ell.$$

Def: Let $f: D \subset \mathbb{R} \to \mathbb{R}$, $x_0 \in D \cap \text{Acc } D$. We say that f is cont at pt x_0 if

$$\lim_{x \to x_0} f(x) = f(x_0)$$

If f is cont at every $x_0 \in D$ we say f is cont on D and we write $f \in \mathcal{C}(D)$.

We can now extend these defs to the case of functions

$$\vec{f} = \vec{f}(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$$

$$\vec{x} = (x_1, x_2, \dots x_d)$$

$$\vec{f}(\vec{x}) = (f_1(x_1 \dots x_d), f_2(x_1 \dots x_d), \dots, f_m(x_1 \dots x_d))$$

Example:

$$\vec{f}(x,y,z) = (\overset{f_1}{xy}, \overset{f_2}{y+z}) \qquad \vec{f} : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

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$$\vec{f}(x,y,z) = (\underset{f_1}{x+y}, \underset{f_2}{y+z}, \underset{f_3}{x+z}) \qquad \vec{f}: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

Def: Let $D \subset \mathbb{R}^d$. We say that $x_0 \in \mathbb{R}^d \cup \{\infty_d\}$ is accumulation point for D if

$$\exists \, (\vec{x}_n) \subset D \setminus \{\vec{x}_0\} \; : \; \vec{x}_n \longrightarrow \vec{x}_0 .$$

Def: Let $\vec{f}: D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$, $\vec{x}_0 \in \operatorname{Acc} D$

We say that

$$\exists \lim_{\vec{x} \longrightarrow \vec{x}_0} \vec{f}(\vec{x}) = \vec{\ell} \in \mathbb{R}^m \cup \{\infty_m\}$$

if

$$\forall (\vec{x}_n) \subset D \setminus \{\vec{x}_0\} : \vec{x}_n \longrightarrow \vec{x}_0 \implies \vec{f}(\vec{x}_n) \longrightarrow \vec{\ell}$$

Def: Let $\vec{f}: D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$, $\vec{x}_0 \in D \cap \operatorname{Acc} D$

We say that f is cont at x_0 if

$$\lim_{\vec{x} \to \vec{x}_0} \vec{f}(\vec{x}) = \vec{f}(\vec{x}_0).$$

If this holds $\forall x_0 \in D$ we write $f \in \mathscr{C}(D)$.

Now we have a good def of limit, the pb

Now we have a good def of limit, the pb becomes: how do we compute (in practice) a limit?

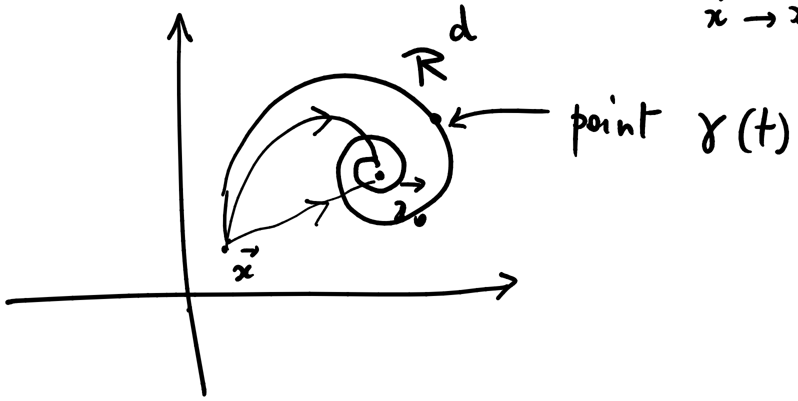
in part, can we use the techniques of calculus developed for 1-dim limits?

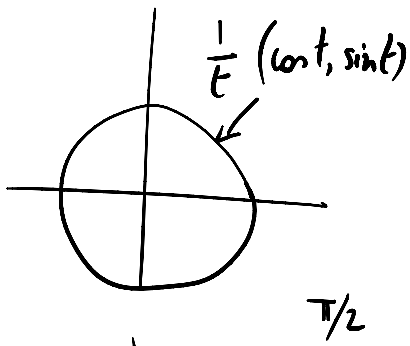
Let's focus on the case

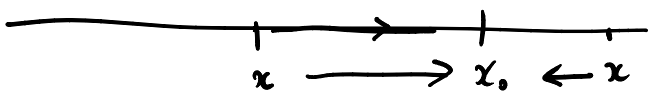
$$\boxed{f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R} \qquad (\text{real valued functs})}$$

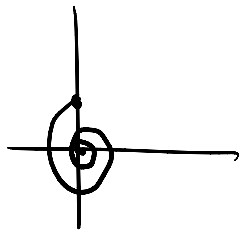
Sections:

$$\lim_{\vec{x} \to \vec{x}_0} f(\vec{x})$$









Def: A function $\vec{\gamma} = \vec{\gamma}(t) : I \subset \mathbb{R} \longrightarrow \mathbb{R}^d$

$\vec{\gamma} \in \mathcal{C}(I)$ is called

(I ↑ interval)

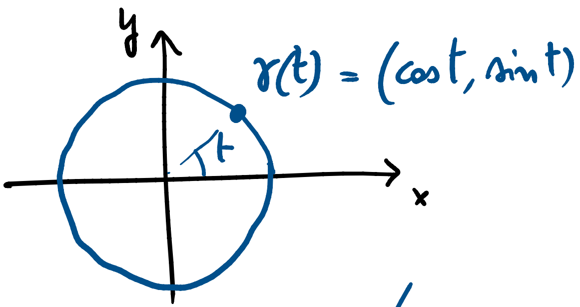
$\mathbb{R}^d$

$\gamma \in C(I)$ is called interval

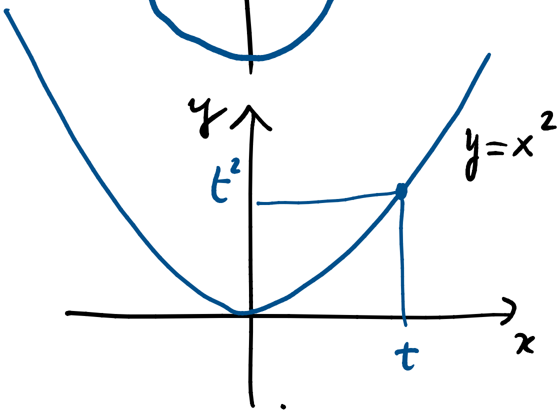
<u>curve</u> on $\mathbb{R}^d$.

<u>Examples:</u>

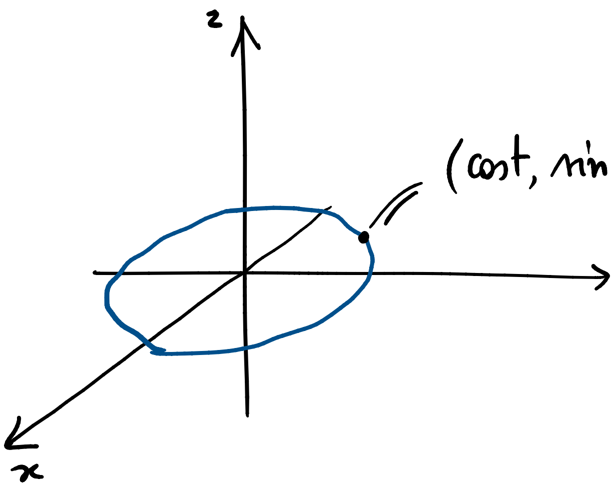
1. $\vec{\gamma} = \vec{\gamma}(t) : [0, 2\pi] \longrightarrow \mathbb{R}^2 \qquad \vec{\gamma}(t) = (\cos t, \sin t)$



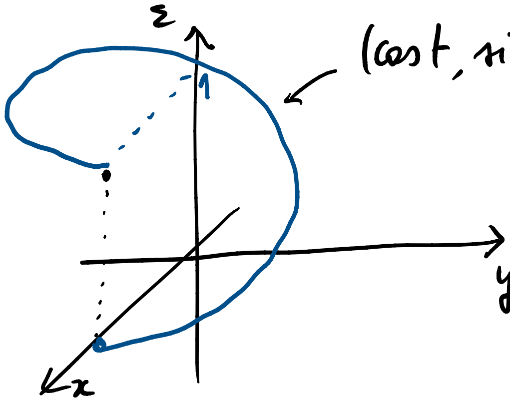
2. $\vec{\gamma}(t) = (t, t^2) \qquad t \in \mathbb{R}$

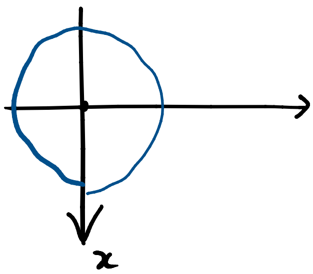


3. $(\cos t, \sin t, 0) = \vec{\gamma}(t) \qquad t \in [0, 2\pi]$

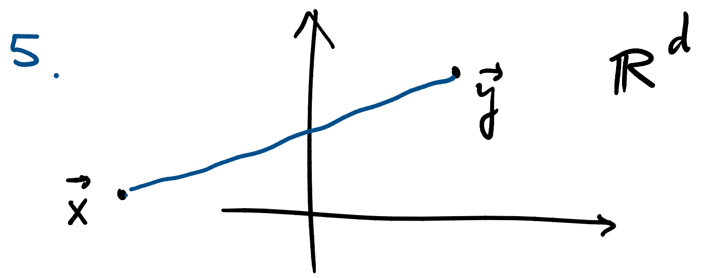


4. $(\cos t, \sin t, \frac{t}{2\pi}) =: \vec{\gamma}(t)$





5.

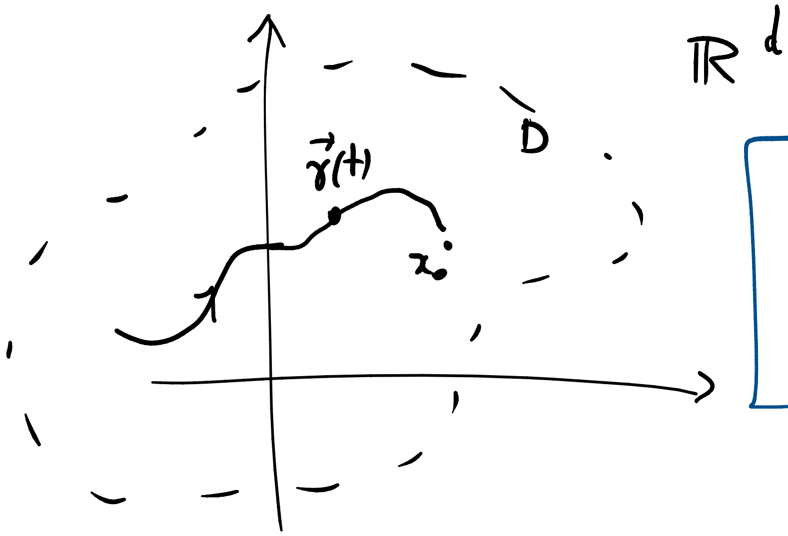


$\vec{\gamma}(t) = \vec{x} + t(\vec{y} - \vec{x})$

$t \in [0,1]$

Now let $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$, $x_0 \in \text{Acc } D$

Let $\vec{\gamma}$ be a curve contained in D



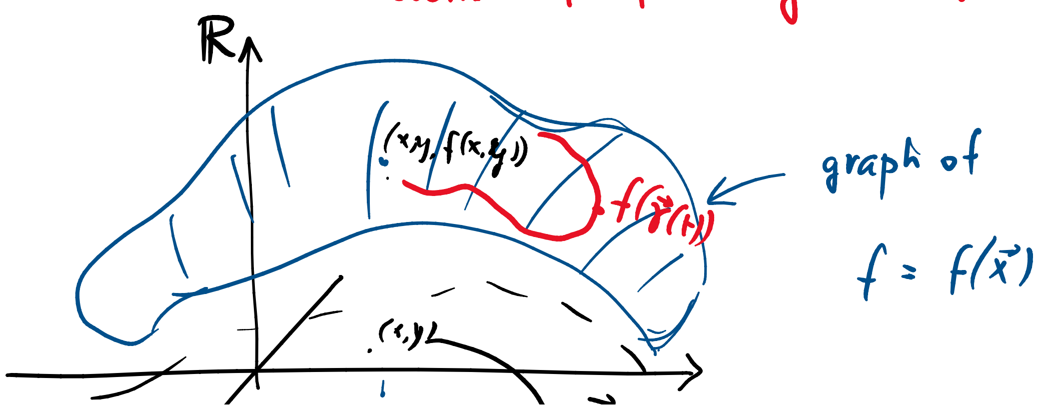
Assume that

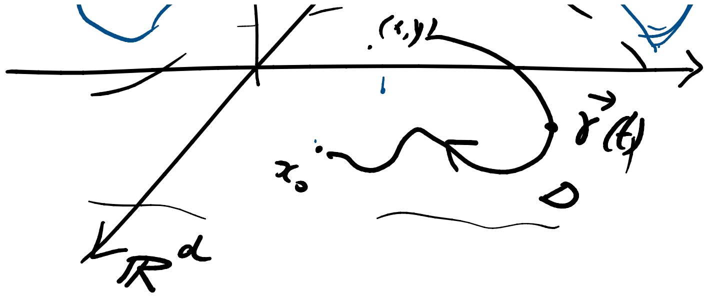
$$\boxed{\begin{array}{l} \vec{\gamma}(t) \longrightarrow \vec{x}_0 \text{ as } t \to t_0 \\ \text{and } \vec{\gamma}(t) \neq \vec{x}_0 \end{array}}$$

Prop: Assume $\lim_{\vec{x} \to \vec{x}_0} f(\vec{x}) = \ell$

Then

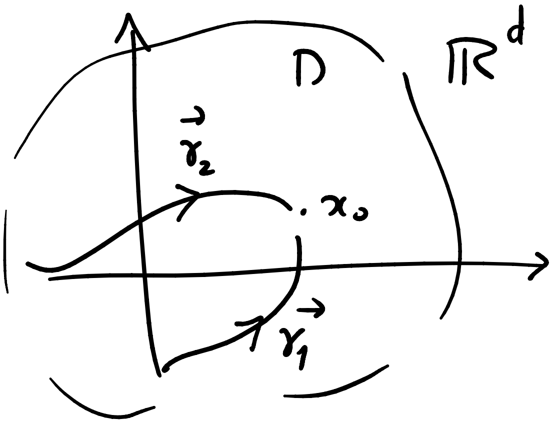
$$\lim_{t \to t_0} \underbrace{f(\vec{\gamma}(t))}_{\text{section of } f \text{ along cve } \vec{\gamma}} = \ell$$





<u>Rmk:</u> In particular if $\exists\, \vec{\gamma}_1, \vec{\gamma}_2$ cvs on D,

$$\vec{\gamma}_1(t),\ \vec{\gamma}_2(t) \to \vec{x}_0$$
$$\neq \vec{x}_0 \qquad \neq \vec{x}_0$$



such that

$$f(\vec{\gamma}_1(t)) \longrightarrow \ell_1$$
$$f(\vec{\gamma}_2(t)) \longrightarrow \ell_2 \quad \neq$$

$$\Rightarrow \nexists \lim_{\vec{x} \to \vec{x}_0} f(\vec{x}).$$

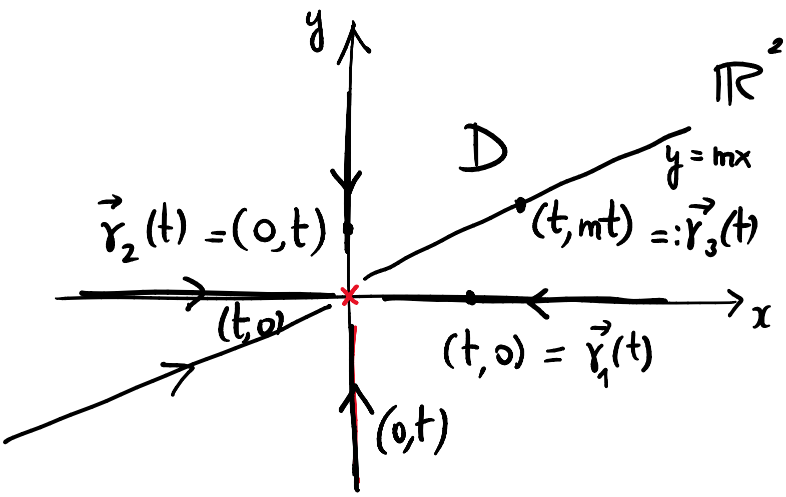
<u>Example 2.2.6</u>

Show that
$$\lim_{(x,y) \to 0_2} \frac{xy}{x^2+y^2}$$
doesn't $\exists$.

<u>Sol:</u> Here $f = f(x,y) = \dfrac{xy}{x^2+y^2} : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$

where $D = \{ (x,y) \in \mathbb{R}^2 : x^2+y^2 \neq 0 \} = \mathbb{R}^2 \setminus \{\vec{0}\}$

$(x^2+y^2 = 0 \iff (x,y) = (0,0) = \vec{0} \equiv 0_2)$. Clearly



$\vec{0} \in \mathrm{Acc}\, D$

$\Downarrow$

$\lim\limits_{(x,y)\to\vec{0}} f$ makes sense

If $\vec{\gamma}_1(t) = (t,0)$, $t \in \mathbb{R}\setminus\{0\}$, $t \to 0$ of course $(t,0) \to \vec{0}$

$$f(\vec{\gamma}_1(t)) = f(t,0) = \frac{t \cdot 0}{t^2 + 0^2} = \frac{0}{t^2} \equiv 0 \xrightarrow[t\to 0]{} 0$$

$$\Rightarrow \exists \lim_{t\to 0} f(\vec{\gamma}_1(t)) = 0.$$

Is this sufficient to conclude $\lim\limits_{(x,y)\to\vec{0}} f = 0$?

No!

$$f(\vec{\gamma}_2(t)) = f(0,t) = \frac{0 \cdot t}{0^2 + t^2} = \frac{0}{t^2} \equiv 0 \longrightarrow 0$$

$$\Rightarrow \exists \lim_{t\to 0} f(\vec{\gamma}_2(t)) = 0$$

This is not yet sufficient to conclude anything except that IF $\lim f$ exists $\Longrightarrow \lim f = 0$

Let's now consider $\vec{\gamma}_3(t) = (t, mt)$ (cve $y = mx$)

Clearly $t \to 0$ $\vec{\gamma}_3(t) \to \vec{0}$ and because

$$f(\vec{\gamma}_3(t)) = f(t, mt) = \frac{t \cdot mt}{\cdots} = \frac{mt^2}{\cdots}$$

$$f(\vec{\gamma}_3(t)) = f(t, mt) = \frac{t \cdot mt}{t^2 + (mt)^2} = \frac{mt^2}{t^2 + m^2 t^2}$$

$$= \frac{m}{1+m^2} \xrightarrow[t \to 0]{} \frac{m}{1+m^2} \neq 0$$

$$\Rightarrow \lim_{t \to 0} f(\vec{\gamma}_3(t)) = \frac{m}{1+m^2}$$

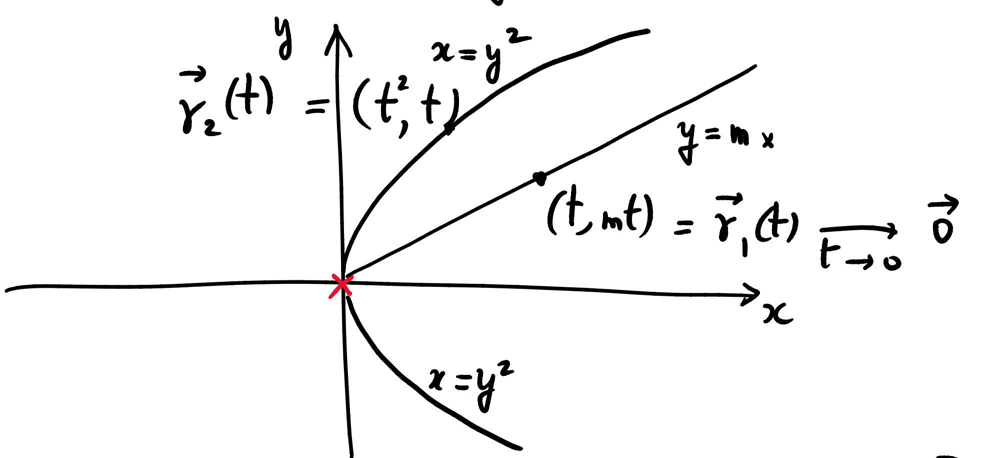
$$\Rightarrow \lim_{(x,y) \to \vec{0}} f(x,y) \text{ does not } \exists. \blacksquare$$

<u>Example</u> 2.2.7. Show that $\lim_{(x,y) \to \vec{0}} \frac{xy^2}{x^2+y^4}$ doesn't exists.

<u>Sol:</u> Here $f(x,y) = \frac{xy^2}{x^2+y^4}$ it is well defd on

$D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^4 \neq 0\} = \mathbb{R}^2 \setminus \{\vec{0}\}$, $\vec{0} \in \mathrm{Acc}\, D$

$(x^2 + y^4 = 0 \Leftrightarrow x^2 = 0, y^4 = 0 \Leftrightarrow (x,y) = \vec{0})$



$$f(\vec{\gamma}_1(t)) = f(t, mt) = \frac{t(mt)^2}{t^2 + (mt)^4} = \frac{m^2 t^3}{t^2 + m^4 t^4}$$

$$= \frac{m^2 t}{1 + m^4 t^2} \xrightarrow[t \to 0]{} 0 \quad \forall m$$

$$\Rightarrow \lim_{t \to 0} f(\vec{\gamma}_1(t)) = 0$$

$$f(0,t) = \frac{0 \cdot t^2}{0^2 + t^4} = \frac{0}{t^4} \equiv 0 \xrightarrow[t \to 0]{} 0 .$$

$$\Rightarrow \lim_{t \to 0} f(\vec{\gamma}(t)) = 0 \quad \forall \gamma \text{ straigth line to } \vec{0}$$

Is this enough to conclude $\lim_{(x,y) \to \vec{0}} f = 0$?

Let's take now $\vec{\gamma}_2(t) = (t^2, t)$

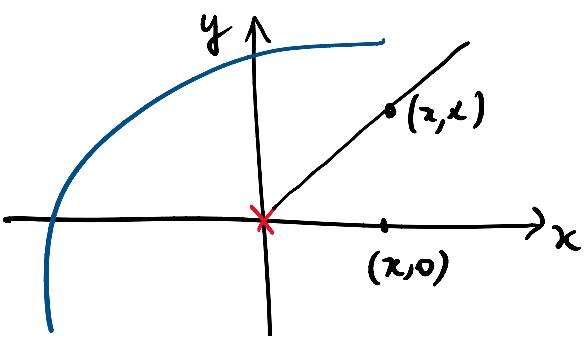
$$f(\vec{\gamma}_2(t)) = f(t^2, t) = \frac{t^2 \cdot t^2}{t^4 + t^4} = \frac{t^4}{2t^4} \equiv \tfrac{1}{2} \xrightarrow[t \to 0]{} \tfrac{1}{2}$$

$$\Rightarrow \lim_{(x,y) \to 0_2} f \text{ doesn't exists } \boxtimes$$

Do Ex 2.5.2.

#1. $\lim_{(x,y) \to \vec{0}} \frac{x^2 - y^2}{x^2 + y^2}$. (doesn't $\exists$)

Here we've $f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}$ defd on $D = \mathbb{R}^2 \setminus \{\vec{0}\}$



$$f(x,0) = \frac{x^2}{x^2} \equiv 1 \xrightarrow[x \to 0]{} 1$$

$$\#$$

$$f(x,x) = \frac{0}{2x^2} \equiv 0 \xrightarrow[x \to 0]{} 0$$

$(x,0)$ $\quad f(x,x) = \frac{0}{2x^2} = 0 \xrightarrow[x\to 0]{} 0$

$$\Downarrow$$

$$\nexists \lim_{(x,y)\to \vec{0}} f$$

Do # 2, 4, 6, 7 (difficult)

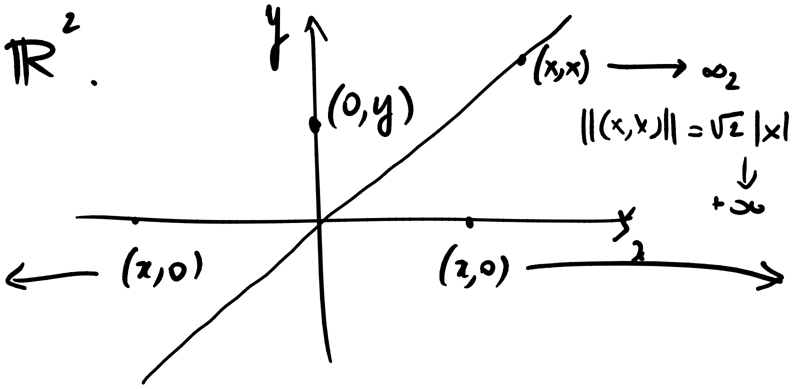
Ex 2.2.8 Show that

$$\lim_{(x,y)\to\infty_2} \left(x^2+y^2-4xy\right)$$

doesn't $\exists$.

Here $f(x,y) = x^2+y^2-4xy$ def'd on $D = \mathbb{R}^2$

and clearly $\infty_2 \in \operatorname{Acc} \mathbb{R}^2$.



$(x,0) \longrightarrow \infty_2$

$$\Updownarrow$$

$\|(x,0)\| \to +\infty$

$\|$

$\boxed{|x| \longrightarrow +\infty}$

$$f(x,0) = x^2+0^2-4x\cdot 0 = x^2 \longrightarrow +\infty$$

$$f(0,y) = 0^2+y^2-4\,0\cdot y = y^2 \xrightarrow[|y|\to+\infty]{} +\infty$$

$$f(x,x) = x^2+x^2-4xx = 2x^2-4x^2 = -2x^2 \xrightarrow[|x|\to+\infty]{} -\infty$$

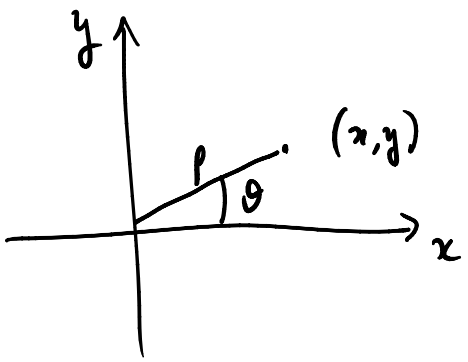
Ex 2.2.9 Compute $\lim_{(x,y)\to\vec{0}} \frac{xy^2}{x^2+y^2}$

Here $f(x,y) = \frac{xy^2}{x^2+y^2}$ dufd on $D = \mathbb{R}^2 \setminus \{\vec{0}\}$.

Sections:

$$f(x,0) = \frac{x\cdot 0^2}{x^2+0^2} \equiv 0 \xrightarrow[x\to 0]{} 0$$

$$f(0,y) = \frac{0.y^2}{0^2+y^2} \equiv 0 \xrightarrow[y\to 0]{} 0$$



$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$

$$f(x,y) = f(\rho\cos\theta, \rho\sin\theta) = \frac{\rho\cos\theta\cdot(\rho\sin\theta)^2}{\rho^2}$$

$$= \frac{\rho^3}{\rho^2}\cos\theta\,(\sin\theta)^2$$

$$\rho = \sqrt{x^2+y^2} = \|(x,y)\|$$

$$= \rho\cos\theta(\sin\theta)^2$$

$$\Rightarrow \quad |f(x,y) - 0| = |f(x,y)|$$

$$= |\rho\cos\theta\,(\sin\theta)^2|$$

$$= \rho \underbrace{|\cos\theta|\,|\sin\theta|^2}_{\leq 1} \leq \rho = \|(x,y)\|$$

$$\Rightarrow \quad \left| f(x,y) - 0 \right| \leq \|(x,y)\|$$

So when $(x,y) \to \vec{0} \Rightarrow \|(x,y)\| \to 0$

$$\Rightarrow \left| f(x,y) - 0 \right| \longleftrightarrow 0$$

$$\Rightarrow f(x,y) \to 0$$

$$\Rightarrow \lim_{(x,y)\to\vec{0}} f = 0.$$

2.5.3.

#2. $\displaystyle \lim_{(x,y)\to\vec{0}} \frac{x^2 y^3}{(x^2+y^2)^2}$.

Here $f(x,y) = \dfrac{x^2y^3}{(x^2+y^2)^2}$ def'd on $D = \mathbb{R}^2 \setminus \{\vec{0}\}$

Sections:

$f(x,0) \equiv 0 \xrightarrow[x\to 0]{} \boxed{0}$ / $f(0,y) \equiv 0 \xrightarrow[y\to 0]{} 0$

Let's prove that lim exists:

$$f(\rho\cos\theta, \rho\sin\theta) = \frac{\rho^5 (\cos\theta)^2 (\sin\theta)^3}{(\rho^2)^2}$$

$$\overline{}$$
$$(\rho^2)^2$$

$$= \rho\,(\cos\theta)^2(\sin\theta)^3$$

$$\Rightarrow 0 \le |f(x,y) - 0| = |f(x,y)| = |f(\rho\cos\theta, \rho\sin\theta)|$$

$$= \rho\,|\cos\theta|^2\,|\sin\theta|^3 \le \rho = \|(x,y)\|$$

$$\downarrow \qquad \downarrow$$
$$0 \qquad \vec{0}$$

$$\Rightarrow \quad f(x,y) \xrightarrow[(x,y)\to\vec{0}]{} 0 \qquad \blacksquare.$$

2,5.2 #3

Do 2.5.3 #1, 3, 4, 5.