mercoledì 26 febbraio 2020 08:25

Ex 2.5.2

#4 $$\lim_{(x,y)\to\vec{0}} \frac{y^2 - xy}{x^2+y^2}.$$

<u>Sol:</u> Here $f(x,y) = \dfrac{y^2 - xy}{x^2+y^2}$ defd on $D = \mathbb{R}^2 \setminus \{\vec{0}\}$

<u>Sections:</u> $$f(x,0) = \frac{0}{x^2} \equiv 0 \xrightarrow[x\to 0]{} 0$$

$$f(0,y) = \frac{y^2}{y^2} \equiv 1 \xrightarrow[y\to 0]{} 1$$

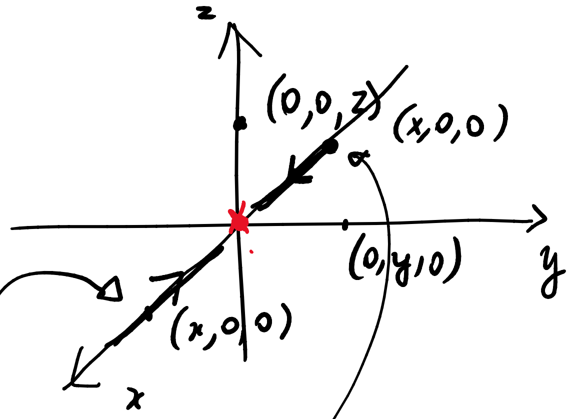
$\Rightarrow$ limit does not exists. $\blacksquare$

#5. $$\lim_{(x,y,z)\to\vec{0}} \frac{x+y^2+z^3}{\sqrt{x^2+y^2+z^2}}$$

<u>Sol:</u> Here $f(x,y,z) := \dfrac{x+y^2+z^3}{\sqrt{x^2+y^2+z^2}}$ defd

$$D = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \neq 0\}$$

$$= \mathbb{R}^3 \setminus \{\underbrace{(0,0,0)}_{\vec{0}}\}$$



<u>Sections:</u>

$$f(x,0,0) = \frac{x}{\sqrt{x^2}} = \frac{x}{|x|}$$

$$= \operatorname{sgn} x$$

$$= \operatorname{sgn} x$$

$$\lim_{x \to 0+} f(x,0,0) = 1 \quad , \quad \lim_{x \to 0-} f(x,0,0) = -1$$

$\Rightarrow$ limit does not $\exists$ ■

2.5.3

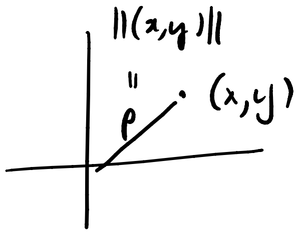
\# 3. $\displaystyle\lim_{(x,y) \to \vec{0}} \frac{x^3 - y^3}{x^2 + y^2}$.

Here $f(x,y) = \dfrac{x^3 - y^3}{x^2 + y^2}$ defd on $D = \mathbb{R}^2 \setminus \{\vec{0}\}$.

Sections:

$$f(x,0) = \frac{x^3}{x^2} = x \xrightarrow[x \to 0]{} 0$$

$$f(0,y) = \frac{-y^3}{y^2} = -y \xrightarrow[y \to 0]{} 0$$



Notice that

$$f(\rho\cos\theta, \rho\sin\theta) =$$

$$= \frac{\rho^3(\cos\theta)^3 - \rho^3(\sin\theta)^3}{\rho^2}$$

$$= \rho\left((\cos\theta)^3 - (\sin\theta)^3\right)$$

$$\Rightarrow 0 \leq \underbrace{|f(x,y) - 0|} = |f(x,y)|$$

$$= \rho \,\left|(\cos\theta)^3 - (\sin\theta)^3\right|$$

$$= \rho \, \left| (\cos\theta)^3 - (\sin\theta)^3 \right|$$

$$\leqslant \rho \left(|\cos\theta|^3 + |\sin\theta|^3 \right)$$

$$\leqslant 2\rho \longrightarrow 0 \qquad (x,y) \to \vec{0}$$

$$\Rightarrow \lim_{(x,y)\to\vec{0}} f = 0. \qquad \blacksquare$$

Some remarks after this example: imagine we guess that

$$\lim_{(x,y)\to\vec{0}} f(x,y) = \ell$$

How can we prove this:

$$|f(x,y) - \ell| = |f(\rho\cos\theta, \rho\sin\theta) - \ell|$$

$$\leqslant \lambda(\rho) \quad \text{where} \quad \lambda(\rho) \underset{\rho\to 0}{\longrightarrow} 0$$

$$\Rightarrow \quad 0 \leqslant |f(x,y) - \ell| \leqslant \lambda(\underbrace{\|(x,y)\|}_{\downarrow \, 0}) \longrightarrow 0$$

so when $(x,y) \longrightarrow \vec{0} \iff \|(x,y)\| \longrightarrow 0$

$$\Rightarrow \quad f(x,y) - \ell \underset{(x,y)\to\vec{0}}{\longrightarrow} 0.$$

Similarly, if we have

$$0 \leq |f(x,y,z) - \ell| \leq \lambda(\|(x,y,z)\|)$$

with $\lambda = \lambda(\rho)$ s.t. $\lambda(\rho) \to 0$ if $\rho \to 0$

$$\Downarrow$$

$$\lim_{(x,y,z) \to \vec{0}} f = \ell .$$

__Ex.__ $\displaystyle \lim_{(x,y,z) \to \vec{0}} \frac{x^4 + x^2y^2 - z^3}{\sqrt{x^2+y^2+z^2}}$

__Sol:__ Here $\displaystyle f(x,y,z) = \frac{x^4 + x^2y^2 - z^3}{\sqrt{x^2+y^2+z^2}}$, $\sqrt{x^2+y^2+z^2} = \|(x,y,z)\|$

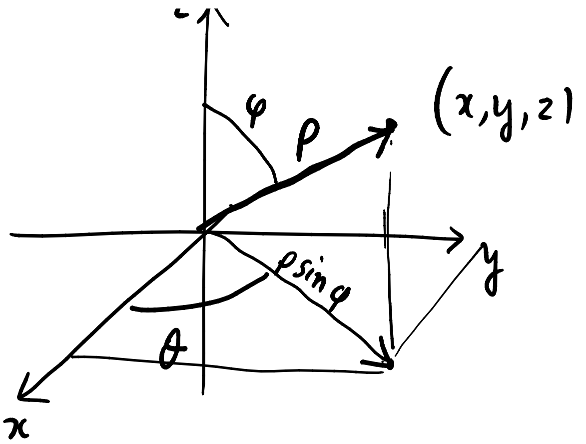
defd on $D = \mathbb{R}^3 \setminus \{\vec{0}\}$

__Sects:__ $\displaystyle f(x,0,0) = \frac{x^4}{\sqrt{x^2}} = \frac{x^4}{|x|} = \frac{x}{|x|} \cdot x^3$

$$= (\operatorname{sgn} x)\, x^3 \xrightarrow[x \to 0]{} 0$$

Guess $\boxed{\lim_{\vec{0}} f = 0}$. To check this we compute

$$0 \leq |f(x,y,z) - 0| = |f(x,y,z)|$$



(spherical coords)

$$\rho = \|(x,y,z)\|$$

$$\begin{cases} x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\varphi \end{cases}$$

$$0 \le \varphi \le \pi, \quad 0 \le \theta \le 2\pi$$

$$= \left| f(\rho\sin\varphi\cos\theta,\ \rho\sin\varphi\sin\theta,\ \rho\cos\varphi) \right|$$

$$= \left| \frac{\rho^4(\sin\varphi)^4(\cos\theta)^4 + \rho^4(\sin\varphi)^4(\cos\theta)^2(\sin\theta)^2 - \rho^3(\cos\varphi)^3}{\rho} \right|$$

$$= \rho^2 \left| -(\cos\varphi)^3 + \rho\Big[\underbrace{(\sin\varphi)^4}_{\le 1}\underbrace{(\cos\theta)^4}_{\le 1} + (\sin\varphi)^4(\cos\theta)^2(\sin\theta)^2 \Big] \right|$$

$$\le \rho^2 \Big[\underbrace{|\cos\varphi|^3}_{\le 1} + \rho\Big[\underbrace{\qquad}_{\le 1} + \underbrace{\qquad}_{\le 1} \Big]\Big]$$

$$\le \rho^2 [1 + 2\rho] =: \lambda(\rho) \to 0 \quad \text{when } \rho \to 0.$$

$$\Rightarrow \lim_{\vec{0}} f = 0. \quad \blacksquare$$

2.5.3

#5 $\lim \dfrac{xy}{\ldots}$.

#5 $\displaystyle\lim_{(x,y)\to\vec{0}} \frac{xy}{|x|+|y|}$.

Sol: Here $f(x,y) := \dfrac{xy}{|x|+|y|}$ def'd on $D = \mathbb{R}^2 \setminus \{\vec{0}\}$

Sect$_x$: $f(x,0) = \dfrac{x\cdot 0}{|x|} \equiv 0 \xrightarrow[x\to 0]{} 0$

If $\lim_{\vec{0}} f$ exists it must be 0.

$$0 \le |f(x,y) - 0| = |f(x,y)|$$

$$= \left| f(\rho\cos\theta, \rho\sin\theta) \right|$$

$$= \left| \frac{\rho^2 \cos\theta\sin\theta}{|\rho\cos\theta| + |\rho\sin\theta|} \right|$$

$$= \frac{\cancel{\rho^2}\,|\cos\theta\sin\theta|}{\cancel{\rho}\,(|\cos\theta| + |\sin\theta|)}$$

$$= \rho \cdot \frac{\overbrace{|\cos\theta\sin\theta|}^{\le 1}}{|\cos\theta| + |\sin\theta|}$$

$\dfrac{1}{\theta}$

$$\le \rho \, \frac{1}{|\cos\theta| + |\sin\theta|}$$

$$\frac{1}{} \le C \iff |\cos\theta| + |\sin\theta| \ge 1$$

$$\frac{1}{|\cos\theta|+|\sin\theta|} \leq C \iff 2 \geq \underbrace{|\cos\theta|+|\sin\theta|}_{\text{Call this } g(\theta)} \geq \frac{1}{C} > 0$$

$g \in \mathcal{C}([0,2\pi]) \overset{\text{Weierstrass}}{\Longrightarrow} g$ has a min

$$|\cos\theta| + |\sin\theta| = g(\theta) \geq \underbrace{g(\hat{\theta})} \quad \forall \theta \in [0,2\pi]$$

if $g(\hat{\theta}) > 0$ we're done: but $g(\hat{\theta}) = 0 \iff$

$$\iff \begin{cases} \cos\hat{\theta} = 0 \\ \sin\hat{\theta} = 0 \end{cases} \text{ imposs.}$$

$$\Longrightarrow \quad |f(x,y) - 0| \leq C\rho =: \lambda(\rho) \xrightarrow[\rho \to 0]{} 0$$

$$\Rightarrow \quad \lim_{(x,y)\to\vec{0}} f = 0. \qquad \blacksquare$$

#4: $\displaystyle \lim_{(x,y)\to\vec{0}} \frac{x\sqrt{|y|}}{\sqrt[3]{x^4+y^4}}$.

Sol: Here $f(x,y) := \dfrac{x\sqrt{|y|}}{(x^4+y^4)^{1/3}}$ defd on $D = \mathbb{R}^2 \setminus \{\vec{0}\}$.

Sects: $f(x,0) = \dfrac{0}{x^{4/3}} \equiv 0 \xrightarrow[x\to 0]{} 0$.

Now

$$|f(x,y) - \ell| = |f(x,y)| = \frac{|x|\sqrt{|y|}}{(x^4+y^4)^{1/3}}$$

$$\underset{\substack{x=\rho\cos\theta \\ y=\rho\sin\theta}}{=} \frac{\rho^{3/2}|\cos\theta|\sqrt{|\sin\theta|}}{\left(\rho^4\left((\cos\theta)^4+(\sin\theta)^4\right)\right)^{1/3}}$$

$$= \frac{\rho^{3/2}}{\rho^{4/3}} \frac{\overset{\leq 1}{|\cos\theta|}\overset{\leq 1}{\sqrt{|\sin\theta|}}}{\left((\cos\theta)^4+(\sin\theta)^4\right)^{1/3}}$$

$$\frac{\rho^{3/2}}{\rho^{4/3}} = \rho^{1/6} \to 0$$

$$\leq \rho^{1/6} \underbrace{\frac{1}{\left((\cos\theta)^4+(\sin\theta)^4\right)^{1/3}}}_{\leq C}$$

$$\Updownarrow$$

$$g(\theta) := \left((\cos\theta)^4+(\sin\theta)^4\right)^{1/3} \geq 1/C > 0$$

$$g \in \mathscr{C}([0,2\pi]) \overset{W}{\Longrightarrow} \exists\, \hat\theta \in [0,2\pi] \quad g(\theta) \geq g(\hat\theta) \quad \forall\theta$$

Notice $g(\hat\theta) \geq 0$. If $g(\hat\theta) = 0 \Rightarrow \begin{cases} \cos\hat\theta = 0 \\ \sin\hat\theta = 0 \end{cases}$ impossible

FALSE

$$g(\hat\theta) > 0 \Longleftarrow$$

$$\Rightarrow \quad 0 \leq |f(x,y) - 0| \leq \rho^{1/6} C =: \lambda(\rho) \xrightarrow[\rho \to 0]{} 0$$

$\blacksquare$

Let's generalize the previous method. Given

$$f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$$

$$x_0 \in \operatorname{Acc} D \cap \mathbb{R}^d$$

We want to prove $\displaystyle\lim_{\vec{x} \to \vec{x}_0} f(\vec{x}) = \ell \in \mathbb{R}$

Suppose that we prove

$$0 \leq |f(\vec{x}) - \ell| \leq \lambda(\|\vec{x} - \vec{x}_0\|)$$

with $\lambda(\rho) \longrightarrow 0 \quad \rho \to 0$

$$\Rightarrow \lim_{\vec{x} \to \vec{x}_0} f(x) = \ell.$$

Do 2.5.5 #2, 3, 6

<u>Ex 2.5.6</u>

#1 $\displaystyle\lim_{(x,y) \to \infty_2} (x^3 + xy^2 - y^2)$

Here $f(x,y) = x^3 + xy^2 - y^2$ defd on $D = \mathbb{R}^2$

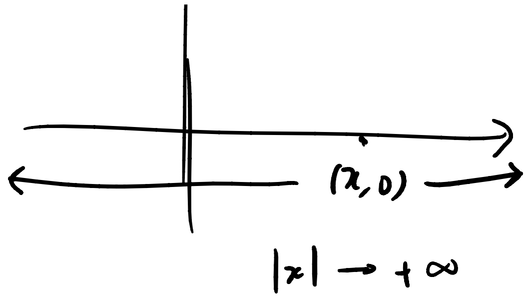
<u>Sects:</u>

$+\infty$

$f(\ldots) = x^3 \nearrow \ldots \; x \to +\infty$

[illegible]:

$$f(x,0) = x^3 \quad \begin{array}{l} \xrightarrow[x \to +\infty]{} +\infty \\ \xrightarrow[x \to -\infty]{} -\infty \end{array}$$



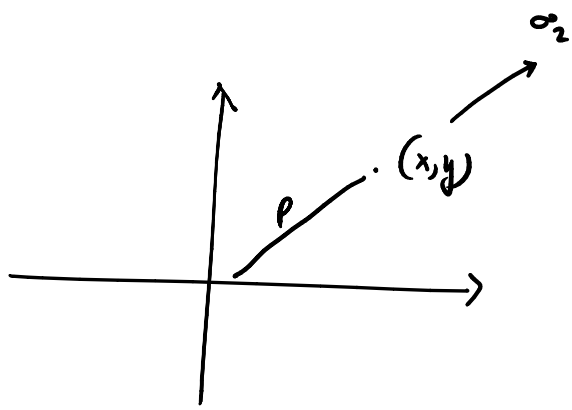
$$\Rightarrow \quad \nexists \lim_{(x,y) \to \infty_2} f$$

3 $\displaystyle \lim_{(x,y) \to \infty_2} \big(\underbrace{x^2y^2 + x^2 + y^2 - xy}_{f(x,y)} \big)$

Sol: We have

$$f(x,0) = x^2 \xrightarrow[|x| \to +\infty]{} +\infty$$

$$f(0,y) = y^2 \xrightarrow[|y| \to +\infty]{} +\infty$$



$$(x,y) \to \infty_2 \iff \|(x,y)\| \to +\infty$$

$$\begin{aligned} f(x,y) &= f(\rho\cos\theta, \rho\sin\theta) \\ &= (\rho\cos\theta)^2(\rho\sin\theta)^2 + \rho^2 - \rho^2 \cos\theta\sin\theta \\ &= \rho^4 (\cos\theta\sin\theta)^2 + \rho^2 \left(1 - \tfrac{1}{2} 2\cos\theta\sin\theta\right) \\ &= \underbrace{\rho^4 (\cos\theta\sin\theta)^2}_{\geq 0} + \rho^2 \underbrace{\left(1 - \tfrac{1}{2}\sin(2\theta)\right)}_{\geq \frac{1}{2}} \end{aligned}$$

$$\underbrace{\qquad\qquad}_{\geq 0} \qquad\qquad \underbrace{\qquad\qquad\qquad}_{\geq 1/2}$$

$$f(x,y) \geq 0 + \tfrac{1}{2}\rho^2 = \tfrac{1}{2}\rho^2$$

$$\Leftrightarrow \quad f(x,y) \geq \frac{1}{2}\|(x,y)\|^2 \longrightarrow +\infty$$

Now as $(x,y) \to \infty_2 \Leftrightarrow \|(x,y)\| \to +\infty$

$$\Downarrow$$

$$\lim_{(x,y)\to\infty_2} f = +\infty. \quad \blacksquare$$

General idea: if

$$f(\vec{x}) \geq \lambda(\|\vec{x}\|) \quad \text{s.t} \quad \lambda(\rho) \underset{\rho\to+\infty}{\longrightarrow} +\infty$$

$$\Downarrow$$

$$\lim_{\vec{x}\to\infty_d} f(\vec{x}) = +\infty.$$

Similarly:

$$f(\vec{x}) \leq \lambda(\|\vec{x}\|) \quad \text{with} \quad \lambda(\rho) \underset{\rho\to+\infty}{\longrightarrow} -\infty$$

$$\Downarrow$$

$$\lim_{\vec{x}\to\infty_d} f(\vec{x}) = -\infty.$$

Example : $\lim\limits_{(x,y,z)\to\infty_3} \left[(x^2+y^2+z^2)^2 - xyz\right]$

Sol: $f(x,0,0) = x^4 \xrightarrow[|x|\to+\infty]{} +\infty$

If lim exists it must be $+\infty$.

To check this we pass. to spherical coords

$$\begin{cases} x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\varphi \end{cases} \qquad \boxed{x^2+y^2+z^2 = \rho^2}$$

$$f(x,y,z) = \rho^4 - \rho^3 \overbrace{(\sin\varphi)^2 \cos\theta\sin\theta\cos\varphi}^{\leq 1}$$

$$\geq \rho^4 - \rho^3 = \lambda(\rho) \xrightarrow[\rho\to+\infty]{} +\infty$$

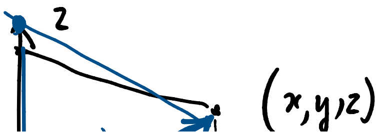
$$\Rightarrow \quad \exists \lim_{(x,y,z)\to\infty_3} f = +\infty \quad ■$$

Example 2.2.16 $\lim\limits_{(x,y,z)\to\infty_3} [\underbrace{(x^2+y^2)^2+z^2-xy}_{f(x,y,z)}]$

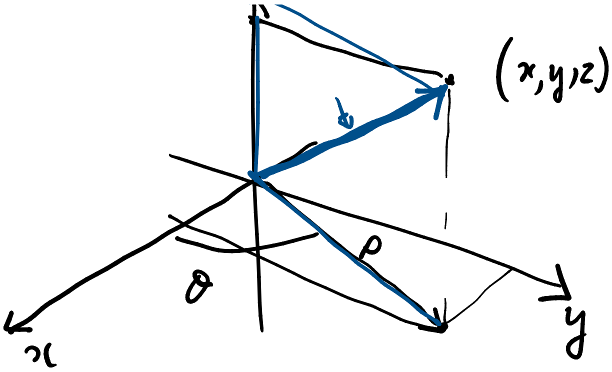
Sol: We have

$$f(x,0,0) = x^4 \longrightarrow +\infty \quad \text{if} \quad |x|\to+\infty.$$

Candidate for lim $= +\infty$.



Cylindrical Coords



Cylindrical Coords

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z \end{cases}$$

$\rho \geq 0, \quad \theta \in [0, 2\pi], \quad z \in \mathbb{R}$

$\|(x,y,z)\|^2 = \rho^2 + z^2 = \|(\rho, z)\|^2$

To check that lim $= +\infty$ we write f in cyl. coords:

$$f(x,y,z) = f(\rho\cos\theta, \rho\sin\theta, z)$$

$$= (\rho^2)^2 + z^2 - \rho^2\cos\theta\sin\theta$$

$$= \rho^4 + z^2 - \rho^2\cos\theta\sin\theta$$

$$\geq \rho^4 + z^2 - \rho^2$$

$$= \rho^4 - \rho^2 + z^2$$

$$\geq \rho^2$$

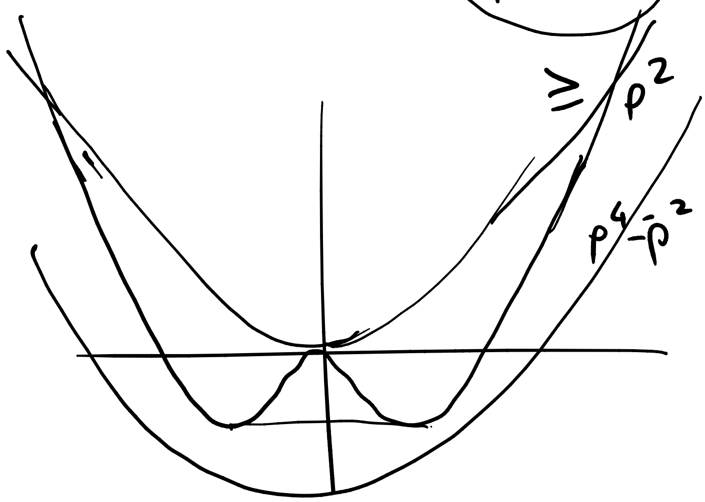
$\rho^4 - \rho^2 \overset{?}{\geq} \rho^2$

$\Updownarrow$

$\rho^4 \geq 2\rho^2$

$\Updownarrow$

$\rho^2 \geq 2$



$\exists\, C:$

$$\rho^4 - \rho^2 \geqslant \rho^2 - C$$

$$\Rightarrow \quad f(x,y,z) \geqslant \rho^2 - C + z^2 = \underset{\substack{\| \\ \|(x,y,z)\|^2}}{\rho^2 + z^2} - C$$

$$\geq \|x,y,z\|^2 - C \xrightarrow[\|(x,y,z)\| \to +\infty]{} +\infty .$$

$\blacksquare$

Do 2.5.6 # 5, 6

4. $\lim\limits_{(x,y,z) \to \infty_3} \left(x^4 + y^4 + z^4 - xyz \right)$

Here $f(x,0,0) = x^4 \xrightarrow[|x| \to +\infty]{} +\infty$

If lim exists it is equal to $+\infty$.

Passing f to sph. coords $\quad C(\theta,\varphi)$

$$f(x,y,z) = \rho^4 \Big[(\cos\theta)^4 (\sin\varphi)^4 + (\sin\theta)^4 (\sin\varphi)^4 + (\cos\varphi)^4 \Big]$$

$$- \rho^3 \underbrace{\cos\theta \sin\theta (\sin\varphi)^2 \cos\varphi}_{\leq 1} .$$

$$\geq \rho^4 \cdot C(\theta,\varphi) - \rho^3$$

Goal $C(\theta,\varphi) \geq C_0 > 0$

$$\geq \underset{\substack{\vee \\ 0}}{C_0}\rho^4 - \rho^3 = \lambda(\rho) \xrightarrow[\rho\to+\infty]{} +\infty$$

$$C(\theta,\varphi) = (\sin\varphi)^4 \Big[\underbrace{(\cos\theta)^4 + (\sin\theta)^4}_{\geq \alpha > 0 \quad \forall\, \theta\in[0,2\pi]} \Big] + (\cos\varphi)^4$$

$$\geq \underbrace{\underset{\substack{\vee \\ 0}}{\alpha}(\sin\varphi)^4 + (\cos\varphi)^4}$$

cont, has a minimum at $\varphi = \hat{\varphi}$

and min value cannot be $= 0$

$$\geq C_0 > 0 \qquad \forall\, \theta,\varphi.$$

∎

Open and Closed sets

Def: We say that a set C is <u>closed</u> if

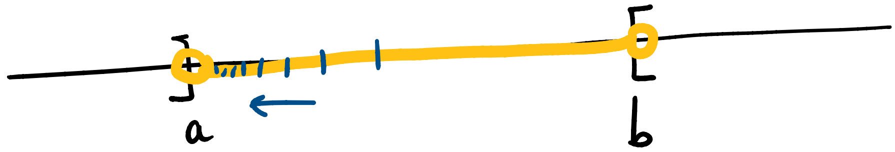
$$\forall (\vec{x}_n) \subset C : \vec{x}_n \to \vec{\ell} \Rightarrow \vec{\ell} \in C$$

Examples:

- $d=1$ $(\mathbb{R}^d = \mathbb{R})$ An interval I is closed $\Leftrightarrow$ it contains its extremes (if finite)

$\Leftrightarrow$ it contains its extremes (if finite)

$]a,b[$ is <u>not closed</u> :



$I \ni x_n = a + 1/n \longrightarrow a \notin I$

<u>Proposition</u> Any set defined through large inequalities or equalities of cont. functs is closed.

That is: if $f_1, f_2, \dots f_m : \mathbb{R}^d \longrightarrow \mathbb{R}$ are cont functions, then

$$\{\vec{x} \in \mathbb{R}^d : f_1(\vec{x}) \geq 0, f_2(\vec{x}) \geq 0, \dots f_m(\vec{x}) \geq 0\}$$

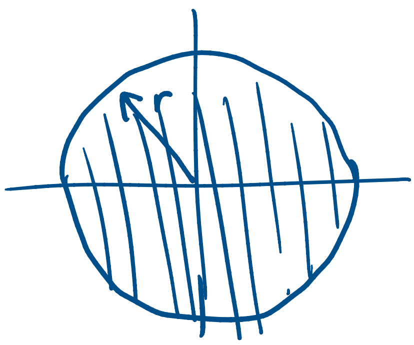
$$\{\vec{x} \in \mathbb{R}^d : f_1(\vec{x}) = 0, f_2(\vec{x}) = 0, \dots, f_m(\vec{x}) = 0\}$$

are closed sets.

<u>Example</u>

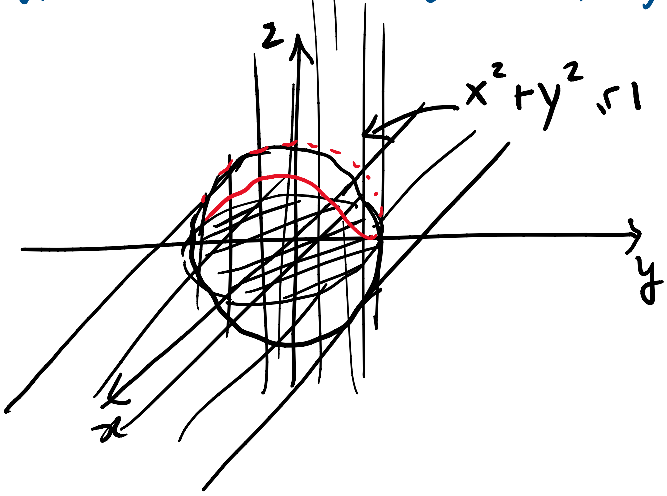
$$\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq r^2\}$$

$$\| \quad \underbrace{r^2 - (x^2+y^2)}_{f_1(x,y)} \geq 0$$



is closed.

$\{(x,y,z)\in\mathbb{R}^3 : \ x^2+y^2\leq 1 , \ y^2+z^2\leq 1\}$



it is closed.

$\{(x,y,z)\in\mathbb{R}^3 : \ x^2+y^2+z^2=1\}$ sphere

Proof: Assume for simplicity

$$C=\{\vec{x}\in\mathbb{R}^d : \ f(\vec{x})\geq 0\} \qquad f\in\mathcal{C}(\mathbb{R}^d)$$

Claim: C is closed.

We've to prove that if $(\vec{x}_n)\subset C : \vec{x}_n\longrightarrow\vec{\ell}$

$$\Downarrow ?$$

$$\vec{\ell}\in C$$

We know:

- $(\vec{x}_n)\subset C \implies f(\vec{x}_n)\geq 0$ $\qquad \vec{\ell}\in C$ $\square$
- $\vec{x}_n\to\vec{\ell} \implies f(\vec{\ell})\geq 0$ (by perm of sgn)