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Ex 2.5.6

\# 5
$$\lim_{(x,y,z)\to\infty_3} \underbrace{\left[x^2+y^2+z^4-xz\right]}_{f(x,y,z)}$$

Sol: Notice that

$$f(x,0,0) = x^2 \xrightarrow[|x|\to+\infty]{} +\infty$$

Let's see if lim is $+\infty$. In spherical coords

$$f(x,y,z) = \rho^2(\sin\varphi)^2\underbrace{\left((\cos\theta)^2+(\sin\theta)^2\right)}_{=1} + \rho^4(\cos\varphi)^4 - \rho^2\cos\varphi\sin\varphi\cos\theta$$

$$\begin{cases} x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\varphi \end{cases}$$

$$= \rho^2(\sin\varphi)^2 + \rho^4(\cos\varphi)^4 - \rho^2\frac{1}{2}\sin(2\varphi)\cos\theta$$

$$\geq \rho^2(\sin\varphi)^2 + \underset{\underset{\rho^2}{\vee\!\shortparallel}}{\rho^4}(\cos\varphi)^4 - \frac{1}{2}\rho^2$$

$\rho\to+\infty$

$\rho\geq 1$

$$\geq \rho^2\underbrace{\left((\sin\varphi)^2+(\cos\varphi)^4\right)}_{\geq c \overset{?}{>} 1/2} - \frac{\rho^2}{2}$$

$$\geq \underset{+}{\left(c-\tfrac{1}{2}\right)}\rho^2 \longrightarrow +\infty$$

$\dot{\vee}$

$$g(\varphi) = (\sin\varphi)^2 + (\cos\varphi)^4 \qquad g'(\varphi) = 2\sin\varphi\cos\varphi + -4(\cos\varphi)^3\sin\varphi$$

$$c \quad = 2\sin\varphi\cos\varphi\,\big(\overset{\sin^2+\cos^2}{1} - 2(\cos\varphi)^2\big)$$

$$= \sin(2\varphi)\big((\sin\varphi)^2 - (\cos\varphi)^2\big)$$

$$g'(\varphi) = \sin(2\varphi)\,(\sin\varphi - \cos\varphi)(\sin\varphi + \cos\varphi)$$

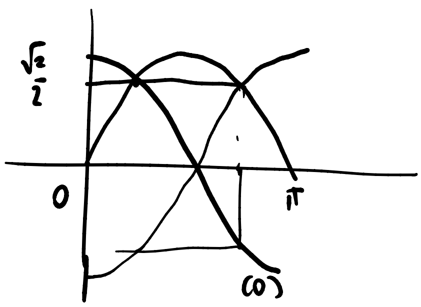
$$g'(\varphi) = 0 \iff \sin(2\varphi) = 0 \;\vee\; \sin\varphi = \cos\varphi \;\vee\; \sin\varphi = -\cos\varphi$$

$\varphi \in [0,\pi]$

$$\sin(2\varphi)=0 \iff 2\varphi = 0, \pi, 2\pi \iff \boxed{\varphi = 0, \pi/2, \pi}$$

$$\sin\varphi = \cos\varphi \iff \varphi = \pi/4$$

$$\sin\varphi = -\cos\varphi \iff \varphi = \tfrac{3}{4}\pi$$



$$g(0) = 1 \qquad g(\pi) = +1$$

$$g\left(\tfrac{\pi}{2}\right) = 1$$

$$g\left(\tfrac{\pi}{4}\right) = \left(\tfrac{\sqrt{2}}{2}\right)^2 + \left(\tfrac{\sqrt{2}}{2}\right)^4 = \tfrac{1}{2} + \tfrac{1}{4}$$

$$g\left(\tfrac{3}{4}\pi\right) = \left(\tfrac{\sqrt{2}}{2}\right)^2 + \left(-\tfrac{\sqrt{2}}{2}\right)^4 = \tfrac{1}{2} + \tfrac{1}{4} \quad \Rightarrow \quad \min_{\varphi\in[0,\pi]} g(\varphi) = \tfrac{1}{2} + \tfrac{1}{4}$$

$$\leadsto \quad f(x,y,z) \geq \rho^2 \underbrace{\big((\sin\varphi)^2 + (\cos\varphi)^4\big)}_{\geq\, 1/2 + 1/4} - \frac{\rho^2}{2}$$

$$\geq \rho^2\left(\tfrac{1}{2} + \tfrac{1}{4}\right) - \frac{\rho^2}{2} = \frac{1}{4}\rho^2 = \lambda(\rho) \xrightarrow{\rho\to+\infty} +\infty$$

$$\Rightarrow \exists \lim_{(x,y,z) \to \infty_3} f = +\infty. \quad \blacksquare$$

$$\boxed{ab \leq \frac{a^2+b^2}{2}}$$
$$\Updownarrow$$
$$a^2+b^2-2ab \geq 0$$
$$\Updownarrow$$
$$(a-b)^2 \geq 0$$

Alternative sol:

$$x^2+y^2+z^4-xz$$

$$xz \leq \frac{1}{2}(x^2+z^2)$$

$$\geq x^2+y^2+z^4-\frac{1}{2}(x^2+z^2)$$

$$= \frac{1}{2}x^2+y^2+z^4-\frac{z^2}{2}$$

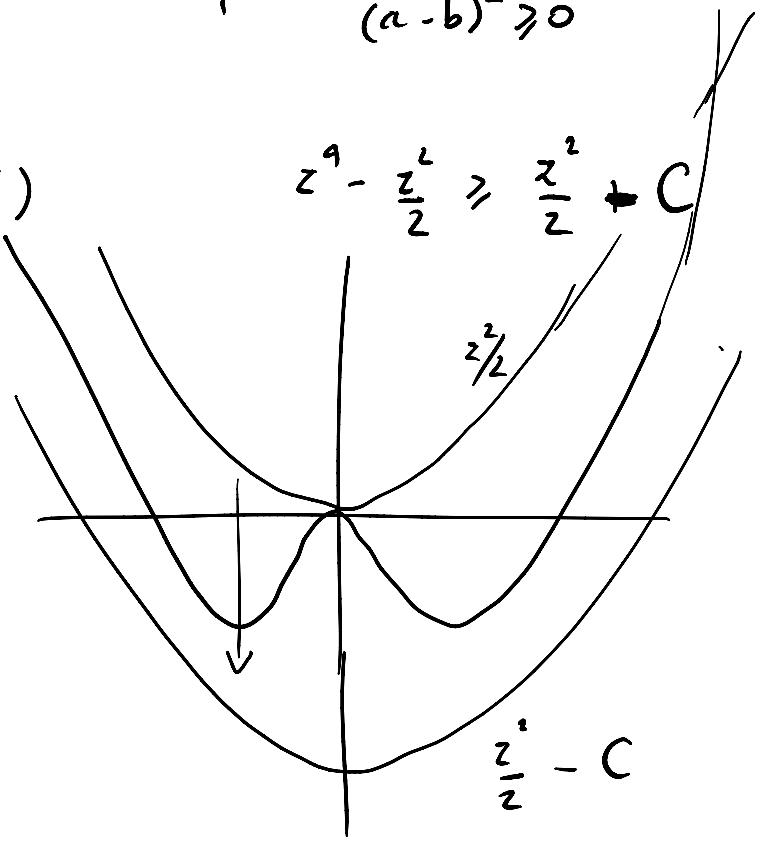
$$\geq \frac{1}{2}x^2+y^2+\frac{1}{2}z^2-C$$

$$\geq \frac{1}{2}y^2$$

$$\geq \frac{1}{2}(x^2+y^2+z^2)-C$$

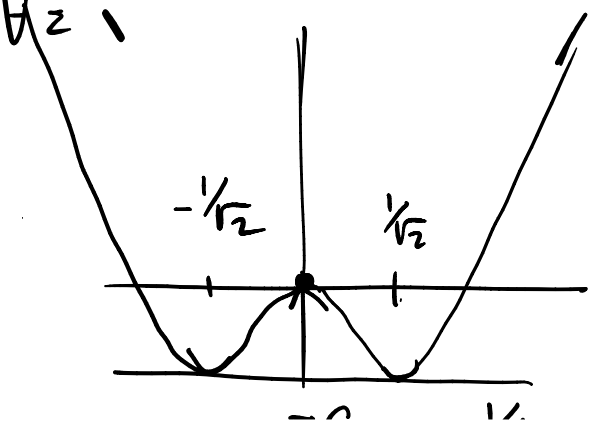
$$= \frac{1}{2}\rho^2-C \longrightarrow +\infty \quad \rho \to +\infty \quad \blacksquare$$

$$z^4-\frac{z^2}{2} \geq \frac{z^2}{2} + C$$



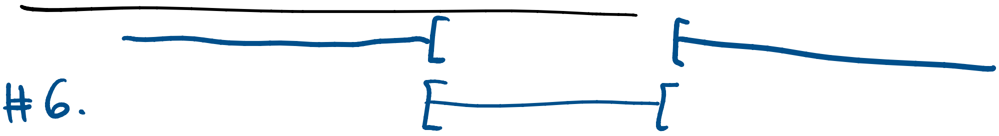
$$z^4-\frac{z^2}{2} \overset{?}{\geq} \frac{z^2}{2}-C$$
$$\Updownarrow$$
$$h(z) = \underbrace{z^4-z^2}\geq -C$$
$$h' = 4z^3-2z = 2z(2z^2-1)$$

$\forall z$



$h' = 4z^3 - 2z = 2z(2z^2 - 1)$

$-c = -1/4$



Do 2.5.6 #6.

Def: A set $O \subset \mathbb{R}^d$ is said to be <u>open</u> if O^c is <u>closed</u>.

Prop: Any set defined through <u>strict inequalities</u> involving cont functs is <u>open</u>. That is: if

$$O = \{\vec{x} \in \mathbb{R}^d : f_1(\vec{x}) > 0, f_2(\vec{x}) > 0, \dots, f_m(\vec{x}) > 0\}$$

where $f_1, f_2, \dots f_m \in \mathcal{C}(\mathbb{R}^d) \Longrightarrow O$ is open.

Example:

- $d = 1$

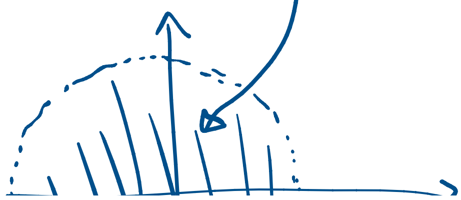
$$]a, b[= \{x \in \mathbb{R} : a < x < b\}$$
$$= \{x \in \mathbb{R} : \underbrace{x > a}_{x - a > 0}, \underbrace{x < b}_{b - x > 0}\}$$

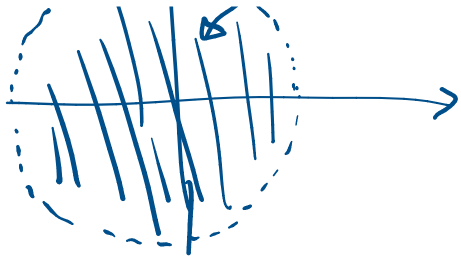
is open

- $d = 2$

$$\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < r^2\}$$

open disk





Weierstrass thm

Recall that the following fund result holds:

Thm: If $f \in \mathcal{C}([a,b]) \Rightarrow f$ has min / max on $[a,b]$

that is $\exists\ x_{min}, x_{max} \in [a,b]$:

$$f(x_{min}) \leq f(x) \leq f(x_{max}) \qquad \forall x \in [a,b]$$

x_{min} / x_{max} are called global (or absolute) min/max pt.

while

$f(x_{min}) / f(x_{max})$ are called min/max value for f on $[a,b]$.

Notation: $\min\limits_{x \in [a,b]} f(x) := f(x_{min})$, $\max\limits_{x \in [a,b]} f(x) := f(x_{max})$

Let's see the version of Weierstrass thm in the case of $f = f(\vec{x})$ $\vec{x} \in \mathbb{R}^d$.

Let

$$f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$$

Def: We say that $\vec{x}_{min} \in D$ is global/absolute min point for f on D if

$$f(\vec{x}_{min}) \leq f(\vec{x}), \qquad \forall \vec{x} \in D.$$

Similarly the concept of global max is defined.

Def: We say that a set D is compact if

D is closed and bounded

A set D is bded if

$$\exists M : \quad \|\vec{x}\| \leq M \qquad \forall \vec{x} \in D$$

Rmk: Of course, if D is bded

$$\|\vec{x}\| \leq M \iff \sqrt{x_1^2 + \cdots + x_d^2} \leq M$$

$$\forall (x_1, \dots x_d) \in D$$

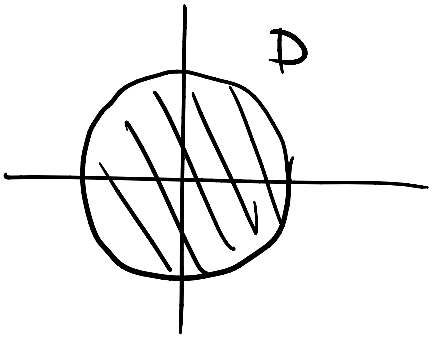
$$|x_j| = \sqrt{x_j^2} \quad (\leq \sqrt{x_1^2 + \cdots + x_d^2})$$

$$\Leftarrow) \quad |x_j| \leq M \quad \forall j = 1, \dots, d$$

$$\forall (x_1, \dots x_d) \in D$$

So, in other words, a set D is bded $\iff$ the coords of pts of D are bounded. $\square$

Ex 1 $D = \{ x^2+y^2 \le 1 \}$ (closed disk centered at $\vec{0}$, radius $= 1$)



$$\Updownarrow$$

$$\|(x,y)\|^2 \le 1 \iff \|(x,y)\| \le 1$$

$$\Downarrow$$

D bdd.

or, alternatively,

$$x^2+y^2 \le 1 \Rightarrow \begin{matrix} x^2 \le 1 \Rightarrow |x| \le 1 \\ y^2 \le 1 \Rightarrow |y| \le 1 \end{matrix} \Rightarrow \text{coords are bdd} \Rightarrow D \text{ bdd.}$$

■

Ex 2: $D = \{ (x,y,z) \in \mathbb{R}^3 : x^2+y^2 = z, \ z^2+y^2 \le 1 \}$

Here $(x,y,z) \in D \iff \begin{cases} x^2+y^2 = z \\ z^2+y^2 \le 1 \end{cases} \Rightarrow \begin{matrix} z^2 \le 1 \\ y^2 \le 1 \end{matrix} \Rightarrow \begin{matrix} |z| \le 1 \\ |y| \le 1 \end{matrix}$

$$x^2 \le x^2+y^2 = z \Rightarrow x^2 \le z \le 1 \Rightarrow x^2 \le 1 \Rightarrow |x| \le 1$$

$\Rightarrow D$ is bdd. ■

Ex 3: $D = \{ (x,y,z) \in \mathbb{R}^3 : x^2+y^2 \le 1 \}$ is not bdd: even if $|x|, |y| \le 1$, there's no cond on $z \in \mathbb{R}$. ■

Thm: (Weierstrass) Let $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$ be a continuous fuct on D compact. Then f has global min / max on D, that is

$$\exists\ \vec{x}_{min}, \vec{x}_{max} \in D : \quad f(\vec{x}_{min}) \leq f(\vec{x}) \leq f(\vec{x}_{max}) \quad \forall \vec{x} \in D.$$

Because this result does not give any algorithm to compute min/max, we need to develop some other tool as Differential Calculus to solve the pb of serching for min/max pts.

Differential Calculus

Refresh the def of derivative of a funct

$$f = f(x) \qquad x \in \mathbb{R}.$$

We have

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

If now $f = f(\vec{x})$ $\quad f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$, we see immediately that the prev. def cannot be given:

$$f'(\vec{x}) = \lim \quad f(\vec{x} + \vec{h}) - f(\vec{x})$$

given:

$$f'(\vec{x}) = \lim_{\vec{h}\to\vec{0}} \frac{f(\vec{x}+\vec{h}) - f(\vec{x})}{\vec{h}}$$

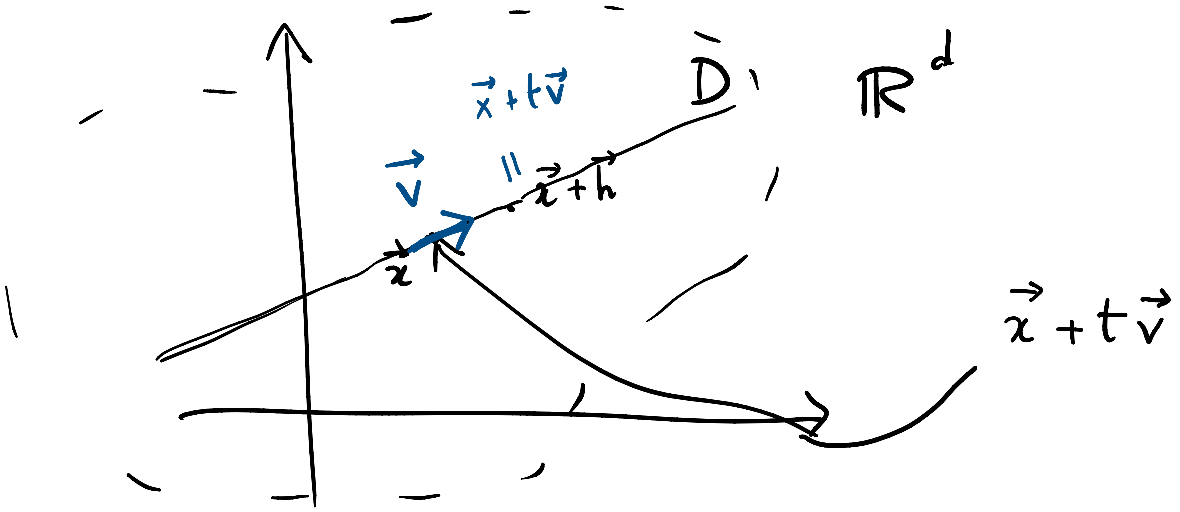
The pb is that we cannot divide vectors/scalars by vectors. There're two ways to overcome this pb:

1) Directional Derivative

2) Differential.

Directional Derivative

Idea: let's try to transform the pb of computing $f'(\vec{x})$ into the pb of computing a "traditional" (one real var) derivative.



$$\lim_{t\to 0} \frac{f(\vec{x}+t\vec{v}) - f(\vec{x})}{t} =: \partial_{\vec{v}} f(\vec{x})$$

↑ directional derivative

Rmk: $\boxed{\partial_{\vec{0}} f(\vec{x}) = 0.}$

directional derivative of f at point $\vec{x}$ along direction $\vec{v}$.

Example 3.1.2

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4+y^2} & (x,y) \neq (0,0) \\ 0 \leftarrow & (x,y) = (0,0) \end{cases}$$

(so $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$). Show that $\exists\, \partial_{\vec{v}} f(0,0)$ $\forall \vec{v} \in \mathbb{R}^2$ but f is not cont at $(0,0)$.

Sol: Let $\vec{v} \neq \vec{0}$ and let's compute $\partial_{\vec{v}} f(0,0)$

$$= \lim_{t\to 0} \frac{f((0,0)+t\vec{v}) - f(0,0)}{t}$$

$$= \lim_{t\to 0} \frac{f(t\vec{v})}{t} \qquad \vec{v} = (\alpha,\beta) \neq \vec{0}$$

$$= \lim_{t\to 0} \frac{f(t(\alpha,\beta))}{t} = \lim_{t\to 0} \frac{f(t\alpha, t\beta)}{t}$$

$$= \lim_{t\to 0} \frac{1}{t} \frac{(t\alpha)^2 t\beta}{(t\alpha)^4 + (t\beta)^2} = \lim_{t\to 0} \frac{1}{\cancel{t}} \frac{\cancel{t^3}^{\,2} \alpha^2\beta}{t^2\beta^2 + t^4\alpha^4}$$

$$= \lim_{t\to 0} \boxed{\frac{\alpha^2\beta}{\beta^2 + t^2\alpha^4}} = \frac{\alpha^2\beta}{\beta^2} = \frac{\alpha^2}{\beta} \qquad \text{if } \beta \neq 0$$

$$t \to 0 \quad \underbrace{\beta^2 + t^2\alpha^4}_{\to\, 0} \qquad \beta^2 \qquad \beta$$

$$(\text{if } \beta = 0 \qquad = \lim_{t\to 0} \frac{\alpha^2 \cdot 0}{0^2 + t^2\alpha^4} = 0$$

$$\partial_{(\alpha,\beta)} f(0,0) = \begin{cases} \alpha^2/\beta & \text{if } \beta \neq 0 \\ 0 & \text{if } \beta = 0 \end{cases}$$

Check that f is not cont at $(0,0)$ that is

$$\lim_{(x,y)\to\vec{0}} f(x,y) \not\to f(0,0) = 0$$

(hint: prove that lim doesn't $\exists$). ∎