Ex 3.4.2 #3

martedì 9 giugno 2020 14:02

Let

$$f(x,y) = \begin{cases} \dfrac{x^2y^3}{(x^2+y^2)^2} & (x,y) \neq 0_2 \\ 0 & (x,y) = 0_2 \end{cases}$$

1. Check f cont. at $(0,0) \equiv 0_2$.

f is cont at 0_2 iff

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$$

$$\Leftrightarrow \quad \lim_{(x,y)\to 0_2} \frac{x^2y^3}{(x^2+y^2)^2} = 0.$$

Clearly

$$f(x,0) \equiv 0 \longrightarrow 0 \qquad x \to 0$$

$$f(0,y) \equiv 0 \longrightarrow 0 \qquad y \to 0$$

Introducing polar coords

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \Rightarrow f(x,y) = \frac{\rho^5(\cos\theta)^2(\sin\theta)^3}{(\rho^2)^2}$$

$$= \rho(\cos\theta)^2(\sin\theta)^3$$

$$\Rightarrow \quad |f(x,y) - 0| = \rho(\cos\theta)^2|\sin\theta|^3 \leq \rho$$

$$= \|(x,y)\| \longrightarrow 0 \qquad (x,y) \longrightarrow 0_2$$

$$\Rightarrow \lim_{(x,y)\to 0_2} f(x,y) = 0 \Rightarrow f \text{ cont at } (0,0).$$

2. Check if f is differentiable at $(0,0)$.

We start trying to apply the total thm (warning: it's just a sufficient cond)

We have

$$\partial_x f(x,y) = \frac{2xy^3(x^2+y^2)^2 - x^2y^3\,2(x^2+y^2)2x}{(x^2+y^2)^4}$$

$$(x,y) \neq 0_2$$

To compute $\partial_x f(0,0)$ we use the def of partial derivative

$$\partial_x f(0,0) = \partial_{(1,0)} f(0,0) = \lim_{t\to 0} \frac{f((0,0)+t(1,0)) - f(0,0)}{t}$$

$$= \lim_{t\to 0} \frac{\overset{0}{\overset{\shortparallel}{f(t,0)}} - \overset{0}{\overset{\shortparallel}{f(0,0)}}}{t} = 0$$

Thus

$$\partial_x f(x,y) = \begin{cases} \dfrac{2xy^3(x^2+y^2) - 4x^3y^3}{(x^2+y^2)^3} & (x,y) \neq 0_2 \\ 0 & (x,y) = 0_2 \end{cases}$$

$$= \begin{cases} \dfrac{2xy^5 - 2x^3y^3}{(x^2+y^2)^3} & (x,y) \neq 0_2 \\ 0 & (x,y) = 0_2 \end{cases}$$

Tot diff thm ensures that IF $\partial_x f$, $\partial_y f$ are cont. on a neighbourhood of $(0,0)$ $\Rightarrow$ f is also diff in that neighb. Now, it is clear that $\partial_x f \in \mathcal{C}(\mathbb{R}^2 \setminus \{0_2\})$. Is also continuous at 0_2 ?? We should check

$$\lim_{(x,y)\to 0_2} \partial_x f(x,y) = \underset{\substack{\| \\ 0}}{\partial_x f(0,0)} \;?$$

$$\Updownarrow$$

$$\lim \frac{2xy^5 - 2x^3y^3}{\ldots} = 0$$

$$\lim_{(x,y)\to 0_2} \underbrace{\frac{\text{[illegible]}}{(x^2+y^2)^3}}_{g(x,y)} = 0$$

$$g(x,0) \equiv 0 \longrightarrow 0 \quad \checkmark$$

$$g(0,y) \equiv 0 \longrightarrow 0 \quad \checkmark$$

but $\quad g(2y,y) = \dfrac{4y^6 - 16y^6}{(5y^2)^3} = \dfrac{-12}{125} \downarrow -\dfrac{12}{125} \neq 0$

$\Rightarrow \partial_x f$ is <u>not</u> cont at $(0,0)$.

DOES THIS MEANS THAT f IS NOT DIFF. AT $(0,0)$? IN GENERAL, ANSWER IS NO

This because tot. diff thm is just a SUFFICIENT (but not NECESSARY) cond.

[Thus, if verified OK
if not we don't know anything]

SO? TO ANSWER THERE'S ONLY A WAY:
USE THE TRUE DEF.

To this aim we first have to determine $\nabla f(0,0)$. We already proved $\partial_x f(0,0)=0$. In a similar way we have

$$\partial_y f(0,0) = 0$$

$\Rightarrow$ $\nabla f(0,0) = (0,0)$. Hence, according to def, f is diff. at $(0,0)$ $\Leftrightarrow$

$$f(\vec{0}+\vec{h}) - f(0) = \underbrace{\nabla f(\vec{0})\cdot\vec{h}}_{=\vec{0}} + o(\|\vec{h}\|)$$

$$= o(\|\vec{h}\|)$$

that is

$$\frac{f(\vec{0}+\vec{h}) - \overset{=0}{f(\vec{0})}}{\|\vec{h}\|} \longrightarrow 0 \quad \text{as} \quad \vec{h} \longrightarrow \vec{0}$$

Setting $\vec{h} = (u,v)$ we have to prove

$$\lim_{(u,v)\to(0,0)} \frac{f(u,v)}{\sqrt{u^2+v^2}} = 0$$

$$\Updownarrow$$

$$\lim_{(u,v)\to(0,0)} \frac{\frac{u^2v^3}{(u^2+v^2)^2}}{(u^2+v^2)^{1/2}} = \lim_{(u,v)\to(0,0)} \frac{u^2v^3}{(u^2+v^2)^{5/2}} = 0$$

$$\lim_{(u,v)\to(0,0)} \frac{(u^2+v^2)^2}{(u^2+v^2)^{1/2}} = \lim_{(u,v)\to(0,0)} \underbrace{\frac{\quad}{(u^2+v^2)^{5/2}}}_{g(u,v)} = 0$$

Clearly

$$g(u,0) \equiv 0 \longrightarrow 0$$

$$g(0,v) \equiv 0 \longrightarrow 0$$

however $g(u,u) = \dfrac{u^5}{(2u^2)^{5/2}} = \dfrac{u^5}{2^{5/2}|u|^5} = \dfrac{1}{2^{5/2}}\operatorname{sgn} u$

$$\longrightarrow \begin{cases} +\dfrac{1}{2^{5/2}} & u \to 0+ \\ -\dfrac{1}{2^{5/2}} & u \to 0- \end{cases}$$

$\Rightarrow \nexists \lim_{(u,v)\to 0_2} g(u,v) \Rightarrow g$ is not diff $\blacksquare$

Ex 3.4.11

martedì 9 giugno 2020 14:03

Let $f(x,y) := x^2\left(y^2 - (x-1)^2\right)$

1. $\lim\limits_{(x,y)\to\infty_2} f(x,y)$?

Notice

$f(0,y) \equiv 0 \longrightarrow 0 \qquad |y| \longrightarrow +\infty$

$f(x,0) = -x^2(x-1)^2 \longrightarrow -\infty \qquad |x| \longrightarrow +\infty$

$\Rightarrow$ lim does not exists.

2. Stationary points: find and classify

We've to det (x,y) : $\nabla f(x,y) = \vec{0}$

$$\Longleftrightarrow \begin{cases} \partial_x f = 0 \\ \partial_y f = 0 \end{cases} \Longleftrightarrow \begin{cases} 2x\left(y^2 - (x-1)^2\right) - x^2 2(x-1) = 0 \\ \boxed{2yx^2 = 0} \end{cases}$$

$y = 0 \quad \vee \quad x^2 = 0$

$\Updownarrow$

$x = 0$

$$\Longleftrightarrow \begin{cases} y = 0 \\ \cancel{-}\cancel{2}x(x-1)^2 \cancel{-}\cancel{2}x^2(x-1) = 0 \end{cases} \vee \begin{cases} x = 0 \\ 0 = 0 \end{cases}$$

$\Updownarrow \qquad \Updownarrow$

$$\Updownarrow$$

$$x(x-1)\left[(x-1)+x\right]=0 \qquad\qquad \Updownarrow \quad (0,y)\ \forall y$$

$$\Updownarrow$$

$$\begin{cases} y=0 \\ x(x-1)(2x-1)=0 \end{cases}$$

$$\Updownarrow$$

$$x=0 \ \vee\ x=1 \ \vee\ x=\tfrac{1}{2}$$

$$\Updownarrow$$

$$(0,0),\ (1,0),\ (\tfrac{1}{2},0)$$

<u>Conclusion:</u> st. pts $(1,0)$, $(\tfrac{1}{2},0)$, $(0,y)\ \forall y\in\mathbb{R}$

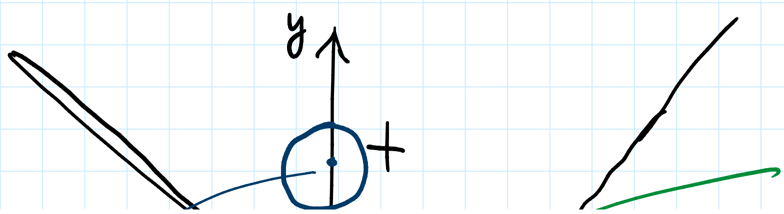
Let's pass to the classification.

We notice $f(1,0)=0$, $f(0,y)=0$

$$f(\tfrac{1}{2},0)=-\frac{1}{16}.$$

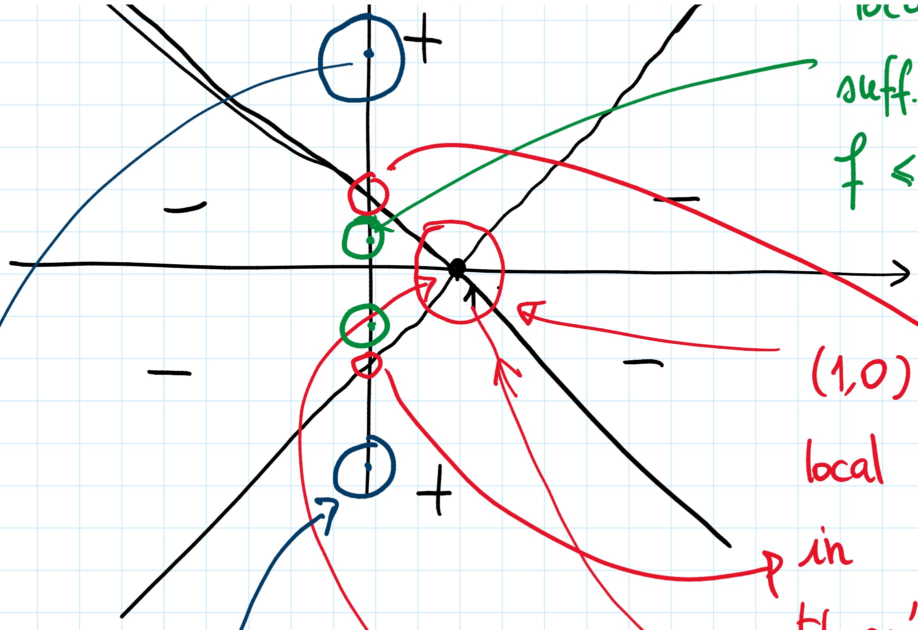
$$f\geq 0 \iff y^2-(x-1)^2\geq 0 \iff y^2\geq (x-1)^2$$

$$\iff y\geq |x-1| \ \vee\ y\leq -|x-1|$$



local max: in a suff. small neighb.

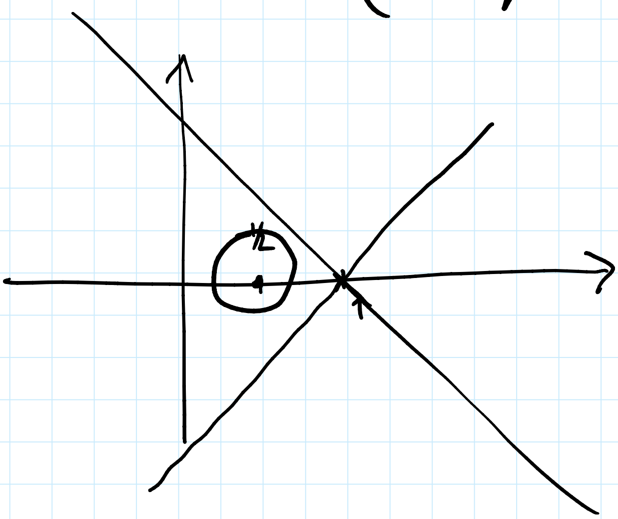
local max: in a suff. small neighb. $f \le 0 = f(0,y)$



$(1,0)$ is not a local min/max: in every neighb there're pts where $f > 0 = f(1,0)$ and pts where $f < 0 = f(1,0)$

local min: in a sufficiently small neighb $f \geq 0 = f(0,y)$

It remains $(\frac{1}{2}, 0)$



Let's check on x-axis

$$f(x,0) = -x^2(x-1)^2 = g(x)$$

$$g' = -2x(x-1)^2 - 2x^2(x-1)$$

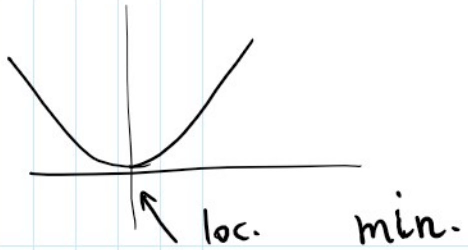
$$= -2x(x-1)(2x-1)$$

	0		1/2		1	
+		−		+		−
↗		↘		↗		↘

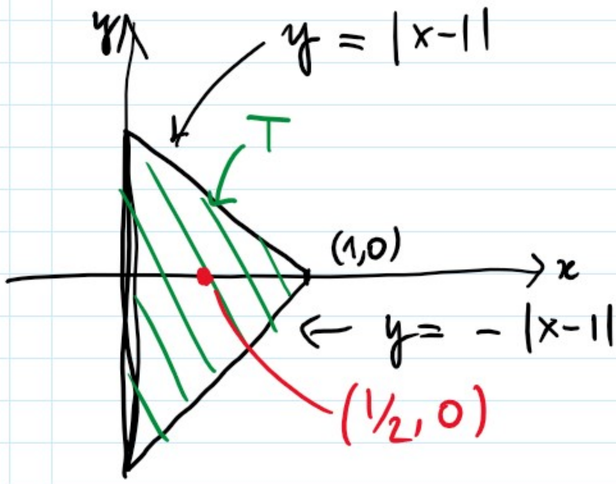
$x = \frac{1}{2}$ loc min.

$$f(\tfrac{1}{2}, y) = \frac{1}{4}\left(y^2 - \tfrac{1}{4}\right) = h(y)$$

$$h'(y) = \frac{1}{2}y \geq 0 \iff y \geq 0$$



Guess $(\frac{1}{2},0)$ is a min for f. Previous argument is not yet a proof (we only proved that f has min at $(1/2, 0)$ on two sects).
To deduce the conclusion, in this case we may argue in the following way. Consider the triangle



T is closed & bdd, that is compact. According to Weierstrass thm, f has min & max on T
If min/max belong to interior of T $\Rightarrow$

$$\nabla f = \vec{0} \qquad \text{(Fermat)}$$

But in Int T there's only $(1/2, 0)$ as st. pt
$\Rightarrow$ the unique candidate for min/max in Int T is $(1/2, 0)$. Here $\boxed{f(1/2, 0) = -1/16}$
If min/max belong to boundary of T $\Rightarrow$

$f(0,y) \Rightarrow \boxed{f(0,y) = 0}$

If min/max belong to boundary of T $\Rightarrow$

$$(x,y) = \begin{cases} (0,y) & \Rightarrow \quad \boxed{f(0,y) = 0} \\ y = |x-1| & \Rightarrow \quad y^2 = (x-1)^2 \Rightarrow \boxed{f(x,y) = 0} \\ y = -|x-1| & \Rightarrow \quad y^2 = (x-1)^2 \Rightarrow \boxed{f(\quad) = 0} \end{cases}$$

Conclusion: $(1/2, 0)$ is a min for f on T

3. Estrema of f on $\mathbb{R}^2$.

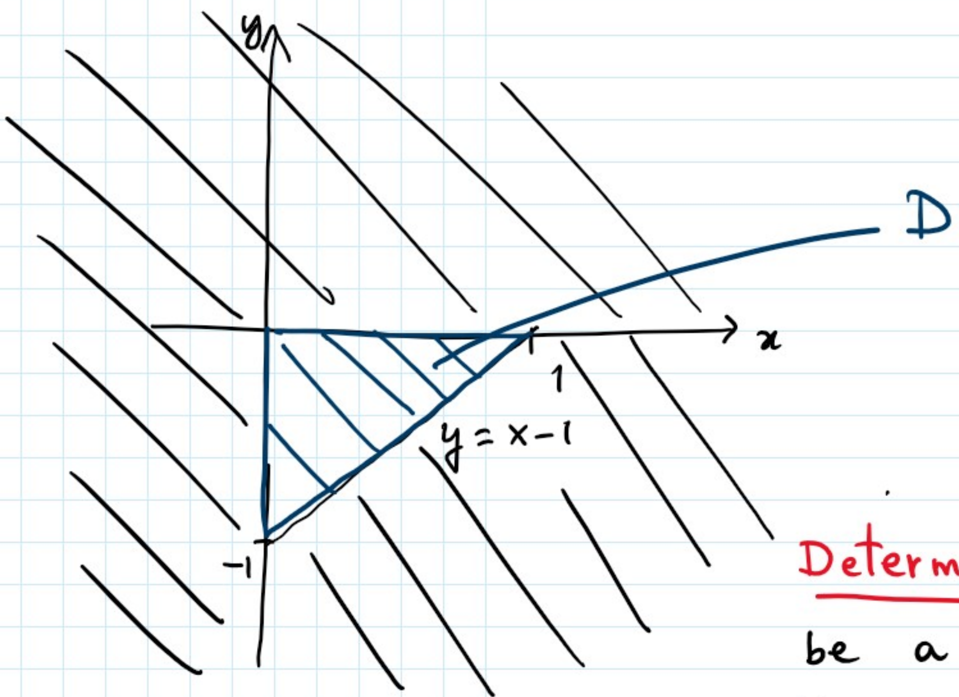
In 1. we already proved that

$$f(x,0) \to -\infty \quad |x| \to +\infty \Rightarrow \text{no min} f$$

On the other hand $f(1,y) = y^2 \underset{|y| \to +\infty}{\longrightarrow} +\infty$

$\Rightarrow$ no max f

4. min/max f on $D = \{y \leq 0,\ 0 \leq x \leq y+1\}$



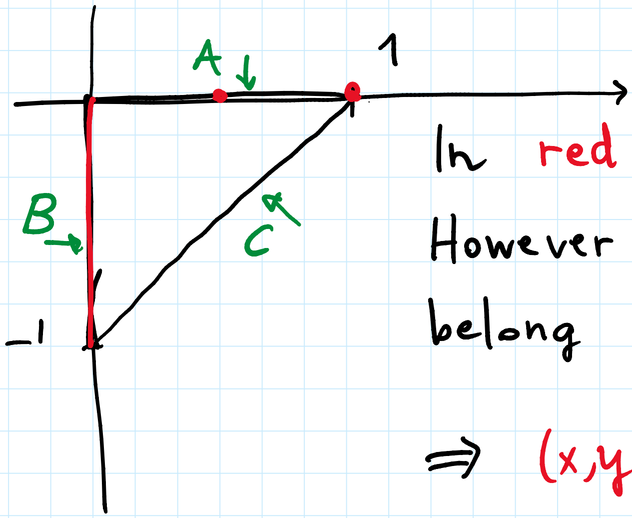
Existence: D is closed & bded, thus compact, $f \in \mathcal{C}(D)$

$\Downarrow$ (Weierstrass)

$\exists$ min/max f on D

Determination: Let $(x,y) \in D$ be a min/max pt. We've the following alternative the following alternative

- EITHER $(x,y) \in \operatorname{Int} D \Rightarrow \nabla f = 0$



In red, st pts for f.
However, none of these belong to Int D

$\Rightarrow (x,y) \notin \operatorname{Int} D$

- OR $(x,y) \in \partial D = \overset{A}{\{(x,0) ; 0 \leq x \leq 1\}} \cup$

$\overset{B}{\{(0,y) : -1 \leq y \leq 0\}} \cup$

$\overset{C}{\{(x, x-1) : 0 \leq x \leq 1\}}$

On A: $f(x,0) = -x^2 (x-1)^2 = g(x)$

$$g' = -2x(x-1)^2 - 2x^2(x-1)$$

$$= \underset{\ominus}{-2}\,\underset{\oplus}{x}\,\underset{\ominus}{(x-1)}(2x-1)$$

On $[0,1]$: $g' \geq 0 \iff 2x - 1 \geq 0$

$\iff x \geq 1/2$

	0		1/2		1
g'		$-$		$+$	

g'	$-$	$+$
g	$\searrow$	$\nearrow$

Potential min $\left(\tfrac{1}{2}, 0\right) \to f = -\tfrac{1}{16}$

max $(0,0)$, $(1,0) \to f = 0$

On B: $f(0,y) \equiv 0$

On C: $f(x, x-1) \equiv 0$

Conclusion:

$\left(\tfrac{1}{2}, 0\right)$ is min pt for f on D

$(0,y)$, $y \in [-1,0]$, $(x, x-1)$, $x \in [0,1]$ are max pts for f on D

$\blacksquare$

Ex 3.4.12

martedì 9 giugno 2020 14:03

Let $f(x,y) = x^4 + y^4 - 8(x^2+y^2)$

1. $\lim\limits_{(x,y)\to\infty_2} f$

Sections: $f(x,0) = x^4 - 8x^2 \xrightarrow[|x|\to+\infty]{} +\infty$

$f(0,y) = y^4 - 8y^2 \xrightarrow[|y|\to+\infty]{} +\infty$

thus if limit exists it is equal to $+\infty$.

To prove that this is the case, let's use polar coords:

$$f(x,y) = \rho^4 \underbrace{\left((\cos\theta)^4 + (\sin\theta)^4 \right)}_{g(\theta)} - 8\rho^2$$

Clearly, $g \in \mathscr{C}([0,2\pi]) \Rightarrow$ it has a minimum (Weierstrass),

$$g(\theta) \geq g(\theta^*) \quad \forall\, \theta \in [0,2\pi].$$

<u>Claim:</u> $g(\theta^*) > 0$. Indeed, $g(\theta^*) \geq 0$ (obvious)

and if $g(\theta^*) = 0 \Rightarrow$

$\underset{\oplus}{(\cos\theta^*)^4} + \underset{\oplus}{(\sin\theta^*)^4} = 0$

$\Leftrightarrow \begin{cases} \cos\theta^* = 0 \\ \sin\theta^* = 0 \end{cases}$ impossible!

$$\Rightarrow \quad g(\theta) \geq C > 0 \quad \forall \theta \in [0, 2\pi]$$

Therefore

$$f(x,y) \geq C\rho^4 - 8\rho^2 \longrightarrow +\infty$$

$$\rho = \|(x,y)\| \longrightarrow +\infty$$

Conclusion: $\lim_{(x,y)\to\infty_2} f = +\infty.$

2. St pts.

(x,y) is st pt for $f \iff \nabla f = \vec{0}$

$$\iff \begin{cases} \partial_x f = 0 \\ \partial_y f = 0 \end{cases}$$

$$\iff \begin{cases} 4x^3 - 16x = 0 \\ 4y^3 - 16y = 0 \end{cases}$$

$$\iff \begin{cases} x(x^2 - 4) = 0 \\ y(y^2 - 4) = 0 \end{cases} \iff x = 0 \ \vee \ x^2 - 4 = 0$$

$$\iff \begin{cases} x = 0 \\ y(y^2 - 4) = 0 \end{cases} \vee \begin{cases} x^2 - 4 = 0 \\ y(y^2 - 4) = 0 \end{cases}$$

$$\Updownarrow$$

$$y = 0 \ \vee \ y^2 - 4 = 0$$

$$\iff$$

$$\begin{cases} x = 0 \\ \ldots \end{cases} \vee \begin{cases} x = 0 \\ \ldots \end{cases} \vee \begin{cases} x^2 - 4 = 0 \\ y = 0 \end{cases} \vee \begin{cases} x^2 - 4 = 0 \\ y^2 - 4 = 0 \end{cases}$$

$$\begin{cases} x=0 \\ y=0 \end{cases} \vee \begin{cases} x=0 \\ y^2-4=0 \end{cases} \vee \begin{cases} \\ y=0 \end{cases} \vee \begin{cases} \\ y^2-4=0 \end{cases}$$

$\Updownarrow$

$(0,0)$ $\quad$ $(0,\pm 2)$ $\quad$ $(\pm 2, 0)$ $\quad$ $(\pm 2, \pm 2)$ (four points)

3. Min/max f on $\mathbb{R}^2$

Existence: $f \in \mathscr{C}(\mathbb{R}^2)$, $\lim_{\infty_2} f = +\infty \Rightarrow f$ has not global max

but there's a global min.

Determination: Let (x,y) be a global min $\Rightarrow$ (Fermat) $\nabla f = \vec{0}$ (because $(x,y) \in \operatorname{Int} \mathbb{R}^2$)

$\Rightarrow (x,y)$ is one among

$(0,0)$, $(0, \pm 2)$, $(\pm 2, 0)$, $(\pm 2, \pm 2)$.

To decide:

$f(0,0) = 0$

$f(0,\pm 2) = f(\pm 2, 0) = 16 - 8 \cdot 4 = -16$

$f(\pm 2, \pm 2) = 16 + 16 - 8(4+4) = 32 - 64 = -32$

$\Rightarrow$ $(\pm 2, \pm 2)$ are all global min pts for f on $\mathbb{R}^2$.

4. min/max f on $D = \{x^2+y^2 \leq 9\}$

Existence: Clearly D is closed & bdded, thus compact, $f \in \mathscr{C}(D) \Rightarrow f$ has global compact, $f \in \mathcal{C}(D) \Rightarrow f$ has global min/max on D.

Determination: Let $(x,y) \in D$ be a global min/max. Then:

- IF $(x,y) \in \operatorname{Int} D \Rightarrow$ (Fermat) $\nabla f = 0 \Rightarrow$ (x,y) is one among $(0,0)$, $(\pm 2, 0)$, $(0, \pm 2)$, $(\pm 2, \pm 2)$

  Among these

  $$\underset{\text{max on Int } D}{\underset{\uparrow}{f(0,0) = 0}}, \quad f(0,\pm 2) = f(\pm 2, 0) = -16$$
  $$\underset{\text{min on Int } D}{\underset{\uparrow}{f(\pm 2, \pm 2)}} = -32.$$

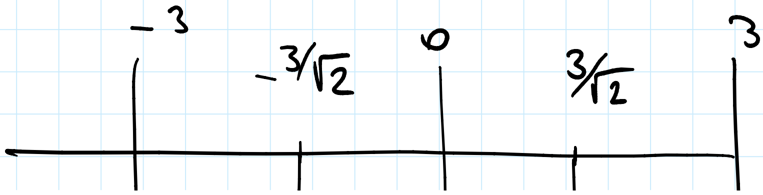
- IF $(x,y) \in$ Boundary D $\Rightarrow x^2 + y^2 = 9 \Rightarrow$

$$\begin{aligned} f(x,y) &= x^4 + y^4 - 8 \cdot 9 \\ &= x^4 + (9 - x^2)^2 - 72 \\ &= x^4 + 81 - 18x^2 + x^4 - 72 \\ &= 2x^4 - 18x^2 + 9 \quad = g(x) \end{aligned}$$

$x \in [-3, 3]$

Let's study g on $[-3,3]$:

$$g' = 8x^3 - 36x = 4x(2x^2 - 9)$$



x	$-$	$-$	$+$	$+$
$2x^2-9$	$+$	$-$	$-$	$+$
g'	$-$	$+$	$-$	$+$
g	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$

$x = \pm 3, 0$ are max pts. for g on $[-3,3]$

$x = \pm 3/\sqrt{2}$ " min " " " " "

points $(\pm 3, 0)$, $(0, \pm 3) \longrightarrow f = 9$ (max on ∂D)

points $\left(\pm \frac{3}{\sqrt{2}}, \pm \frac{3}{\sqrt{2}}\right) \longrightarrow f = \cancel{2}\,\frac{81}{\cancel{4}2} - 72 = -\frac{63}{2}$

(min on ∂D)

<u>Conclusion:</u> Candiates for max

$(0,0) \in \operatorname{Int} D \qquad f = 0$

$(\pm 3, 0), (0, \pm 3) \in \partial D \qquad f = 9 \Leftarrow$ global max D

Candidates for min

$(\pm 2, \pm 2) \in \operatorname{Int} D \qquad f = -32 \Leftarrow$ global min D

$\left(\pm \frac{3}{\sqrt{2}}, \pm \frac{3}{\sqrt{2}}\right) \in \partial D \qquad f = -\frac{63}{2} = -31.5$

$\blacksquare$