Ex 1.5.4

martedì 9 giugno 2020 08:06

Consider

$$y' + (\sin t)\, y = \sin t \qquad t \in \mathbb{R}$$

1. General integral

The eqn is a linear first order non homog.

$$y' = a(t)\, y + b(t)$$

The gen sol is

$$y(t) = e^{\int a(t)\,dt} \left(\int e^{-\int a(t)\,dt}\, b(t)\,dt + C \right)$$

$$= e^{\int -\sin t\,dt} \left(\int e^{-\int -\sin t\,dt} \sin t\,dt + C \right)$$

$$= e^{\cos t} \left(\int e^{-\cos t} \underbrace{\sin t}_{=(-\cos t)'}\,dt + C \right)$$

$$= e^{\cos t} \left(e^{-\cos t} + C \right)$$

$$= 1 + C e^{\cos t}.$$

2. Are there sols s.t. $\exists \lim_{t \to +\infty} y(t) \in \mathbb{R}$?

Gen sol is

$$y(t) = 1 + C e^{\cos t}$$

If $t \to +\infty$, because $e^{\cos t}$ has not a limit, $y(t)$ has not a limit unless $C=0$

In this case

$$y(t) \equiv 1 \longrightarrow 1$$

Thus answer is: yes, there's just one solution that fulfills requirement.

3. Solve CP $y\left(\frac{\pi}{2}\right) = 1$

Gen int is

$$y(t) = 1 + C e^{\cos t}.$$

We have to det C in such a way

$$1 = y\left(\frac{\pi}{2}\right) = 1 + C e^{\cos \pi/2} = 1 + C$$

$\Rightarrow C = 0$ ☒

Ex 1.5.6 #3

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Consider the CP

$$\begin{cases} y' = \dfrac{\cos^2(2y)}{t(2-\log^2 t)} \\ y(1) = \pi/2 . \end{cases}$$

The eqn is a separable variable eqn

$$y' = a(t)\, f(y(t))$$

with

$$a(t) = \frac{1}{t(2-\log^2 t)} \qquad f(y) = \cos^2(2y).$$

Clearly $f \in \mathcal{C}^1$ and because

$$f(y(1)) = f\left(\frac{\pi}{2}\right) = \cos^2 \pi = 1 \neq 0$$

Thus, the solution of CP may be determined by separation of variables:

$$\frac{dy}{dt} = \frac{\cos^2(2y)}{t(2-\log^2 t)} \iff \frac{dy}{\cos^2(2y)} = \frac{dt}{t(2-\log^2 t)}$$

$$\iff \int \frac{dy}{\cos^2(2y)} = \int \frac{dt}{t(2-\log^2 t)} + C$$

$$\iff \int \frac{dy}{\cos^2(2y)} = \int \frac{dt}{t(2-\log^2 t)} + C$$

$$\int \frac{dy}{\cos^2(2y)} = \frac{1}{2} \int \underbrace{\frac{2}{\cos^2(2y)}}_{= (\mathrm{tg}\, 2y)'} dy = \frac{1}{2} \mathrm{tg}(2y)$$

$$\int \frac{dt}{t(2-\log^2 t)} \underset{\substack{u = \log t \\ du = \frac{dt}{t}}}{=} \int \frac{du}{2-u^2}$$

$$= -\int \frac{1}{(u-\sqrt{2})(u+\sqrt{2})} du$$

$$= -\frac{1}{2\sqrt{2}} \left[\int \frac{1}{u-\sqrt{2}} - \frac{1}{u+\sqrt{2}} du \right]$$

$$= -\frac{1}{2\sqrt{2}} \left[\log|u-\sqrt{2}| - \log|u+\sqrt{2}| \right]$$

$$= -\frac{1}{2\sqrt{2}} \log\left| \frac{u-\sqrt{2}}{u+\sqrt{2}} \right|$$

$$= -\frac{1}{2\sqrt{2}} \log\left| \frac{\log t - \sqrt{2}}{\log t + \sqrt{2}} \right|$$

Returning to eqn

$$\frac{1}{2}\,tg(2y) = -\frac{1}{2\sqrt{2}}\log\left|\frac{\log t-\sqrt{2}}{\log t+\sqrt{2}}\right| + C.$$

value of C is immediately determined by imposing initial cond:

$$0 = \frac{1}{2}\,tg\,\pi = -\frac{1}{2\sqrt{2}}\log\underbrace{\left|\frac{-\sqrt{2}}{\sqrt{2}}\right|}_{|-1|=1} + C$$

$$\Longleftrightarrow \quad \boxed{C=0}$$

We can finally extract y:

$$\cancel{2}y = \frac{1}{2}\,\mathrm{arctg}\left(-\frac{1}{\sqrt{2}}\log\left|\frac{\log t-\sqrt{2}}{\log t+\sqrt{2}}\right|\right) \qquad \blacksquare$$

Consider

$$y'' + y = \frac{1}{\cos t} \qquad t \in]-\pi/2, \pi/2[$$

1. General integral.

We have a second order linear non hom. eqn. The hom. eqn associated to this is

$$y'' + y = 0$$

whose characteristic eqn is

$$\lambda^2 + 1 = 0 \iff \lambda = \pm i$$

Therefore, a fund. syst. of sols for hom. eqn is

$$w_1 = \cos t \quad , \quad w_2 = \sin t$$

Gen sol of initial eqn has form

$$y(t) = c_1 w_1 + c_2 w_2 + U(t)$$
$$= c_1 \cos t + c_2 \sin t + U(t)$$

where U is a particular sol. We may find this applying the Lagrange formula

$$u(t) = -\left(\int \frac{w_2}{W} f \, dt\right) w_1 + \left(\int \frac{w_1}{W} f \, dt\right) w_2$$

where $$W = \det \begin{bmatrix} w_1 & w_2 \\ w_1' & w_2' \end{bmatrix} = \det \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} = 1$$

$$f = \frac{1}{\cos t}$$

Hence

$$u(t) = -\left(\int \frac{\sin t}{1} \cdot \frac{1}{\cos t} \, dt\right) \cos t + \left(\int \frac{\cos t}{1} \cdot \frac{1}{\cos t} \, dt\right) \sin t$$

$$= + \int \frac{-\sin t}{\cos t} \, dt \ \cos t + \int 1 \, dt \ \sin t$$

$$= (\log|\cos t|) \cos t + t \sin t.$$

In conclusion, the gen. int is

$$y(t) = (c_1 + \log|\cos t|) \cos t + (c_2 + t) \sin t.$$

2. True or false: $\lim_{t \to \pi/2^-} y(t) = +\infty$ for ever sol.

If $t \to \pi/2^-$ we have

$$c_1 \cos t \to 0$$

$$c_1 \cos t \longrightarrow 0$$

$$\Big(\underset{-\infty}{\underset{\downarrow}{\log}}\,|\underset{0}{\underset{\downarrow}{\cos t}}|\Big)\,\underset{0}{\underset{\downarrow}{\cos t}} \qquad \infty \cdot 0$$

$$\Big(\text{however: } \lim_{t\to \pi/2^-} \big(\log|\cos t|\big)\cos t \underset{u=\cos t}{=} \lim_{u\to 0^+} u\log u = 0\Big)$$

$$c_2 \sin t \longrightarrow c_2$$

$$t\sin t \longrightarrow \pi/2$$

$$\Rightarrow \quad y(t) \longrightarrow c_2 + \pi/2 .$$

3. Sols s.t. $y(0)=0$ and $y(t) \approx Ct^2$ $t \to 0$

Imposing $y(0)=0$ we have

$$0 = y(0) = c_1 \Rightarrow$$

$$y(t) = (c_2 + t)\sin t$$

Now

$$\lim_{t\to 0} \frac{y(t)}{t^2} \overset{0/0}{\underset{H}{=}} \lim_{t\to 0} \frac{\underset{\downarrow}{\overset{c_2}{\overset{\uparrow}{c_2\cos t}}} + \overset{0}{\overset{\uparrow}{\sin t}} + \overset{0}{\overset{\uparrow}{t\cos t}}}{2t \to 0}$$

thus if $c_2 \neq 0$ $\lim = \infty$

if $c_2 = 0$ $= \lim_{t \to 0} \frac{\sin t + t \cos t}{2t}$ $\frac{0}{0}$

$$\underset{H}{=} \lim_{t \to 0} \frac{\cos t + \cos t - t \sin t}{2}$$

$$= 1$$

Answer: the unique sol s.t $y(0) = 0$ and $y(t) \sim ct^2$ is $c_1 = c_2 = 0$

$$y(t) = U(t). \quad \blacksquare$$