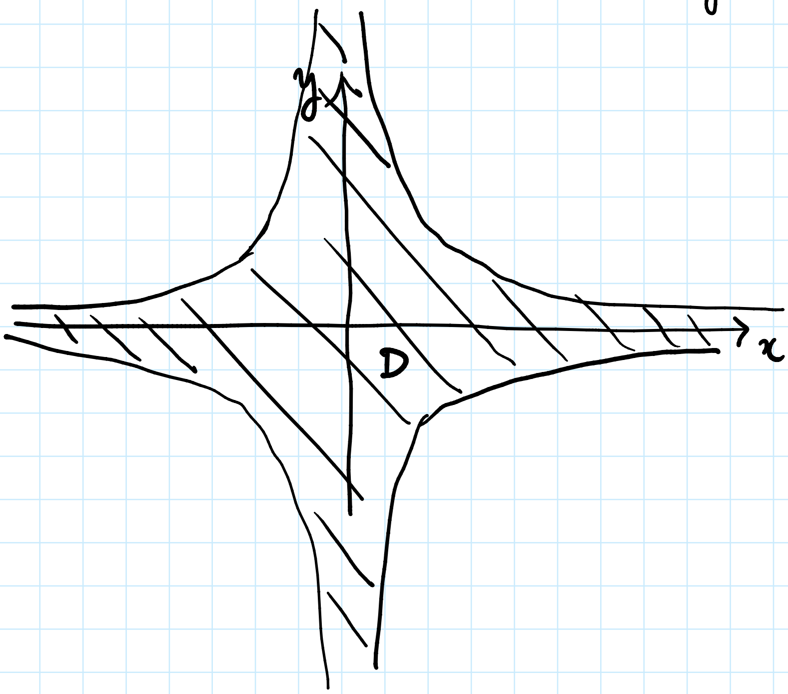
Ex 5.5.3 #2

domenica 14 giugno 2020 12:02

$$\int_D \frac{x^2 e^{-x^2}}{1+(xy)^2}\,dx\,dy \qquad D = \{ |xy| \leq 1 \}$$

Sol: Notice that $|xy| \leq 1 \underset{x \neq 0}{\Leftrightarrow} |y| \leq \frac{1}{|x|}$

$(x=0, \quad y \in \mathbb{R})$



Therefore

$$\text{Integral} \overset{RF}{=} \int_{-\infty}^{+\infty} \left(\int_{-1/|x|}^{1/|x|} \frac{x^2 e^{-x^2}}{1+(xy)^2}\,dy \right) dx$$

$$\overset{RF}{=} \int_{-\infty}^{+\infty} \left(\int_{-1/|y|}^{1/|y|} \frac{x^2 e^{-x^2}}{1+(xy)^2}\,dx \right) dy$$

Which is better ?

$$= \int_{-\infty}^{+\infty} x^{\cancel{2}} e^{-x^2} \int_{-1/|x|}^{1/|x|} \underbrace{\frac{x}{1+(xy)^2}}_{=\left(\operatorname{arctg}(xy)\right)'} dy \; dx$$

$$= \left[\operatorname{arctg}(xy)\right]_{y=-1/|x|}^{y=1/|x|}$$

(better to split $\int_{-\infty}^{+\infty} dx = \int_{-\infty}^{0} + \int_{0}^{+\infty}$)

$$= \int_{-\infty}^{0} x e^{-x^2} \left[\operatorname{arctg}(xy)\right]_{1/x}^{-1/x} dx$$

$$+ \int_{0}^{+\infty} x e^{-x^2} \left[\operatorname{arctg}(xy)\right]_{-1/x}^{1/x} dx$$

$$= \int_{-\infty}^{0} x e^{-x^2} \left[\underbrace{\operatorname{arctg}(-1)}_{-\pi/4} \underbrace{-}_{-} \underbrace{\operatorname{arctg} 1}_{\pi/4}\right] dx$$

$$+ \int_{0}^{+\infty} x e^{-x^2} \left[\underbrace{\operatorname{arctg} 1}_{\pi/4} \underbrace{-}_{-} \underbrace{\operatorname{arctg}(-1)}_{(-\pi/4)}\right] dx$$

$$= (\cancel{-2})\, \pi/4 \int_{-\infty}^{0} -2x e^{-x^2} dx \;\cancel{+}\; - \cancel{2}\, \frac{\pi}{4} \int_{0}^{+\infty} -2x e^{-x^2} dx$$

$$\Vert$$

$$\left(e^{-x^2}\right)'$$

$$= \frac{\pi}{4}\left[e^{-x^2}\right]_{-\infty}^{0} - \frac{\pi}{4}\left[e^{-x^2}\right]_{0}^{+\infty}$$

$$= \frac{\pi}{4}(1-0) - \frac{\pi}{4}(0-1) = \pi/2. \ \blacksquare$$

Ex 5.5.6 #7

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$$\mathrm{Vol}\ D = \left\{ (x,y,z) : z \geq \sqrt{x^2+y^2},\ x^2+y^2+z^2 \leq 1 \right\}$$

We have

$$\mathrm{Vol}\ D = \int_D 1 \, dx\, dy\, dz$$

$$= \int_{z \geq \sqrt{x^2+y^2},\ x^2+y^2+z^2 \leq 1} 1 \, dx\, dy\, dz$$

$$\underset{\text{cyl. coords}}{=} \int_{\substack{\rho \geq 0, \\ \theta \in [0,2\pi] \\ z \in \mathbb{R}}} \rho \, d\rho\, d\theta\, dz \qquad z \geq \rho,\ \rho^2+z^2 \leq 1$$

$x = \rho\cos\theta$

$y = \rho\sin\theta$

$z = z$

$$\overset{RF}{=} \int_0^{2\pi} \left(\int_{\rho \geq 0,\ z\in\mathbb{R} \quad z \geq \rho,\ \rho^2+z^2 \leq 1} \rho \, d\rho\, dz \right) d\theta$$

$$= 2\pi \int_{\rho \geq 0,\ z \in \mathbb{R},\ \underbrace{z \geq \rho,\ \rho^2+z^2 \leq 1}} \rho \, d\rho\, dz.$$

$\rho \leq z$ $\qquad z^2 \leq 1-\rho^2$

$\Downarrow \qquad\qquad \Downarrow$

$z \geq 0$

$$z \geq 0 \qquad\qquad 1-\rho^2 \geq 0 \Rightarrow \rho^2 \leq 1$$

$$0 \leq \rho \leq 1, \quad \underbrace{\rho \leq z \leq \sqrt{1-\rho^2}}_{\text{we need}}$$

$$\rho \leq \sqrt{1-\rho^2}$$
$$\Updownarrow$$
$$\rho^2 \leq 1-\rho^2$$
$$\Updownarrow$$
$$\rho^2 \leq 1/2$$
$$\Updownarrow$$
$$0 \leq \rho \leq 1/\sqrt{2} \quad (\leq 1)$$

stronger

$$\Rightarrow \operatorname{Vol} D = 2\pi \int_{0 \leq \rho \leq 1/\sqrt{2},\ \rho \leq z \leq \sqrt{1-\rho^2}} \rho \, d\rho \, dz$$

$$= 2\pi \int_0^{1/\sqrt{2}} \left(\int_\rho^{\sqrt{1-\rho^2}} \rho \, dz \right) d\rho$$

$$= 2\pi \int_0^{1/\sqrt{2}} \rho \underbrace{\int_\rho^{\sqrt{1-\rho^2}} dz}_{= \sqrt{1-\rho^2} - \rho} d\rho$$

$$= 2\pi \int^{1/\sqrt{2}} \rho (1-\rho^2)^{1/2} - \rho^2 \, d\rho$$

$$= 2\pi \int_0 \rho\,(1-\rho^2)^{1/2} - \rho^2 \; d\rho$$

$$\left[(1-\rho^2)^{3/2} \right]' = -\frac{3}{\cancel{2}} (1-\rho^2)^{1/2}\, \cancel{2}\rho$$

$$= 2\pi \left[-\frac{1}{3} \left[(1-\rho^2)^{3/2} \right]_0^{1/\sqrt{2}} - \frac{\rho^3}{3} \Big|_0^{1/\sqrt{2}} \right]$$

$$= \frac{2\pi}{3} \left[-\frac{1}{2^{3/2}} + 1 - \frac{1}{2^{3/2}} \right]$$

$$1 - \frac{2}{2^{3/2}} = 1 - \frac{1}{\sqrt{2}}$$

$$= \frac{2\pi}{3} \left(1 - 1/\sqrt{2} \right) \quad \blacksquare$$

$$\mathrm{Vol}\left(\left\{ z \geq x^2+y^2, \quad z \leq 18 - x^2 - y^2 \right\}\right)$$

Sol:

$$V = \int_{z \geq x^2+y^2, \quad z \leq 18-(x^2+y^2)} 1 \, dx\,dy\,dz$$

$$\underset{\text{cyl coords}}{=} \int_{\rho \geq 0,\ \theta \in [0,2\pi],\ z \in \mathbb{R} :\ z \geq \rho^2,\ z \leq 18-\rho^2} \rho \, d\rho\, d\theta\, dz$$

$$\overset{RF}{=} 2\pi \int_{\rho \geq 0,\ z \in \mathbb{R} :\ z \geq \rho^2,\ z \leq 18-\rho^2} \rho \, d\rho\, dz$$

$$\boxed{\rho^2 \leq z \leq 18-\rho^2}$$

$$\Downarrow$$

$$\rho^2 \leq 18 - \rho^2 \iff$$

$$2\rho^2 \leq 18 \iff \rho^2 \leq 9$$

$$\overset{\rho \geq 0}{\iff} 0 \leq \rho \leq 3$$

$$= 2\pi \int_{0 \leq \rho \leq 3,\ \rho^2 \leq z \leq 18-\rho^2} \rho \, d\rho\, dz$$

$$\overset{RF}{=} 2\pi \int^3 \left(\int^{18-\rho^2} \rho \, dz \right) d\rho$$

$$= 2\pi \int_0 \left(\int_{\rho^2} \rho \, dz \right) d\rho$$

$$= 2\pi \int_0^3 \rho \left(18 - \rho^2 - \rho^2 \right) d\rho$$

$$= 2\pi \int_0^3 18\rho - 2\rho^3 \, d\rho$$

$$= 2\pi \left[18 \frac{\rho^2}{2} \Big|_0^3 - 2 \frac{\rho^4}{4} \Big|_0^3 \right]$$

$$= \cancel{2}\pi \left[\frac{18 \cdot 9}{\cancel{2}} - \frac{1}{\cancel{2}} 81 \right]$$

$$= \pi \left[162 - 81 \right] = 81\pi. \quad \blacksquare$$

$$\int_{x^2+2y^2\leq 1} \frac{1}{1+x^2+2y^2}\,dx\,dy$$

Sol: we may start with polar coords

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$

Unfortunately, this is not a good choice. Indeed

$$x^2+2y^2 = x^2+y^2+y^2 = \rho^2+\rho^2(\cos\theta)^2$$
$$= \rho^2\left(1+(\cos\theta)^2\right)$$

$$\Rightarrow \text{Integral} = \int_{\substack{\rho\geq 0,\ \theta\in[0,2\pi] \\ \rho^2(1+(\cos\theta)^2)\leq 1}} \frac{1}{1+\rho^2(1+(\cos\theta)^2)}\,\rho\,d\rho\,d\theta$$

That does not sound better than initial integral.

The solution passes through adapted polar coords
We recognize both domain and function depends on $x^2+2y^2 \equiv x^2+(\sqrt{2}y)^2$. Thus natural change of vars is

$$\phi^{-1}: \begin{cases} x = \rho\cos\theta \\ \sqrt{2}y = \rho\sin\theta \end{cases} \qquad \left(\text{or} \quad y = \frac{\rho}{\sqrt{2}}\sin\theta\right)$$

In this way

$$x^2 + 2y^2 = x^2 + (\sqrt{2}y)^2 = \rho^2$$

$$\Rightarrow \text{Integral} = \int_{\rho^2 \leq 1,\ \theta\in[0,2\pi]} \frac{1}{1+\rho^2} \left|\det(\phi^{-1})'(\rho,\theta)\right| d\rho\, d\theta$$

We compute the volume element:

$$\left|\det(\phi^{-1})'(\rho,\theta)\right| = \left|\det \begin{array}{c} \begin{array}{cc} \partial_\rho & \partial_\theta \end{array} \\ \begin{bmatrix} \cos\theta & -\rho\sin\theta \\ \frac{1}{\sqrt{2}}\sin\theta & \frac{\rho}{\sqrt{2}}\cos\theta \end{bmatrix} \end{array}\right|$$

$$= \left|\frac{\rho}{\sqrt{2}}(\cos\theta)^2 + \frac{\rho}{\sqrt{2}}(\sin\theta)^2\right| = \frac{\rho}{\sqrt{2}}$$

$$\Rightarrow \boxed{\left|\det(\phi^{-1})'\right| = \frac{\rho}{\sqrt{2}}}$$

Therefore

$$\text{Integral} = \int_{\rho^2\leq 1,\ \theta\in[0,2\pi]} \frac{1}{1+\rho^2}\frac{\rho}{\sqrt{2}}\, d\rho\, d\theta$$

$$\overset{RF}{=} \frac{1}{\sqrt{2}} \int_0^{2\pi} \left(\int_0^1 \frac{\rho}{1+\rho^2} \, d\rho \right) d\vartheta$$

$$= \frac{\cancel{2}\pi}{\sqrt{2}} \int_0^1 \underset{\substack{\parallel \\ (\log(1+\rho^2))'}}{\frac{2\rho}{1+\rho^2}} \, d\rho$$

$$= \frac{\pi}{\sqrt{2}} \log(1+\rho^2) \Big|_0^1$$

$$= \frac{\pi}{\sqrt{2}} \log 2 . \qquad \blacksquare$$

Compute $\displaystyle\int_{\mathbb{R}^3} \sqrt{x^2+y^2+z^2}\; e^{-(x^2+y^2+z^2)}\, dx\,dy\,dz$

<u>Sol:</u> We use spherical coords (thus $x^2+y^2+z^2=\rho^2$)

We have

$$\text{integral} = \int_{\rho\geq 0,\ \theta\in[0,2\pi],\ \varphi\in[0,\pi]} \sqrt{\rho^2}\, e^{-\rho^2} \cdot \rho^2 \sin\varphi \, d\rho\, d\theta\, d\varphi$$

$$\overset{RF}{=} \int_0^{2\pi} \left(\int_0^{\pi} \left(\int_0^{+\infty} \rho^3 e^{-\rho^2} d\rho \right) \sin\varphi \, d\varphi \right) d\theta$$

$$= 2\pi \left(\int_0^{\pi} \sin\varphi \, d\varphi \right) \left(-\frac{1}{2} \int_0^{+\infty} \rho^2 \underbrace{(-2)\rho\, e^{-\rho^2}}_{= (e^{-\rho^2})'} d\rho \right)$$

$$= 2\pi \cdot \cancel{2} \cdot \left(-\frac{1}{\cancel{2}} \right) \left[\rho^2 e^{-\rho^2} \Big|_0^{+\infty} + \int_0^{+\infty} -2\rho\, e^{-\rho^2} d\rho \right]$$

$$= -2\pi \left[0 + e^{-\rho^2} \Big|_0^{+\infty} \right] = 2\pi \quad \blacksquare$$