Ex 4.5.1 v)

sabato 13 giugno 2020 07:59

$$f(x,y,z) = x - 2y + 2z$$

$$D = \{ (x,y,z) \in \mathbb{R}^2 : x^2 + y^2 + z^2 = 9$$

Find min/max on D.

Existence: $f \in \mathscr{C}(\mathbb{R}^3)$, D is closed (because def. through an eqn of a cont. funct) and bounded (because

$$x^2 + y^2 + z^2 = 9 \implies x^2, y^2, z^2 \leq 9$$

$$\iff |x|, |y|, |z| \leq 3$$

$\Rightarrow$ by Weierstrass thm $\exists \min_D / \max_D f$.

Determination: f is clearly differentiable. Let $g(x,y,z) = x^2 + y^2 + z^2 - 9$ in such a way that

$$D = \{ g = 0 \}$$

Let's check g is submersive on D. g is <u>not</u> submersive at $(x,y,z) \iff \nabla g = 0$

$$\iff (2x, 2y, 2z) = \vec{0}$$

$$\iff (x,y,z) = \vec{0} \notin D$$

$\Rightarrow \nabla g \neq \vec{0}$ on D $\Rightarrow$ g subm. on D

Therefore according to Lagrange multipliers thm

$\Rightarrow \nabla g \neq 0$ on D $\Rightarrow$ g subm. on D

Therefore, according to Lagrange multipliers thm,

IF $(x,y,z) \in D$ is min/max for f on D

$$\Downarrow$$

$$\exists \lambda \qquad \nabla f(x,y,z) = \lambda \nabla g(x,y,z)$$

$$\Updownarrow$$

$$\text{rk} \begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} < 2.$$

2x3 matrix

$$\Updownarrow$$

all 2x2 sub det vanish.

Now

$$\begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} = \begin{bmatrix} 1 & -2 & -2 \\ 2x & 2y & 2z \end{bmatrix}$$

thus

$$\text{rk} \begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} < 2 \iff \begin{cases} \det \begin{bmatrix} 1 & -2 \\ 2x & 2y \end{bmatrix} = 0 \\ \det \begin{bmatrix} -2 & -2 \\ 2y & 2z \end{bmatrix} = 0 \\ \det \begin{bmatrix} 1 & -2 \\ 2x & 2z \end{bmatrix} = 0 \end{cases}$$

$$\iff \begin{cases} y + 2x = 0 \\ -z + y = 0 \\ \ldots \end{cases} \iff \begin{cases} y = -2x \\ z = -2x \\ \ldots \end{cases}$$

$$\begin{cases} -z + y = 0 \\ z + 2x = 0 \end{cases} \quad \cdots \quad \Big|\; z = -2x$$

$$\Leftrightarrow \quad (x, -2x, 2x) \quad x \in \mathbb{R}$$

Now, we've to det which of these pts belong to D:

$$(x, -2x, +2x) \in D \Leftrightarrow \quad x^2 + 4x^2 + 4x^2 = 9$$

$$\Leftrightarrow \quad 9x^2 = 9$$

$$\Leftrightarrow \quad x^2 = 1 \Leftrightarrow \quad x = \pm 1$$

$$\Rightarrow \quad x = 1 \qquad (1, -2, 2)$$

$$x = -1 \qquad (-1, 2, -2)$$

Finally, because

$$f(1, -2, 2) = 1 - 2(-2) + 2 \cdot 2 = 9 \Rightarrow \text{max}$$

$$f(-1, 2, -2) = -1 - 2(2) + 2 \cdot (-2) = -9 \Rightarrow \text{min}$$

Ex 4.5.7

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$$\mathcal{M} = \{ (x,y,z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 1 \}$$

1. $\mathcal{M} \neq \emptyset$ and $\mathcal{M}$ is diff manifold

Let's take

$$(x,0,0) \in \mathcal{M} \iff x^2 = 1 \iff (\pm 1, 0, 0) \in \mathcal{M}$$

Also, for example

$$(0,y,z) \in \mathcal{M} \iff y^2 - z^2 = 1 \iff y^2 = z^2 + 1$$

$$\iff (0, \pm\sqrt{z^2+1}, z) \quad z \in \mathbb{R}.$$

In any case, $\mathcal{M} \neq \emptyset$

To say "$\mathcal{M}$ is differential manifold" means that $\mathcal{M} = \{g = 0\}$ where g is a submersion on $\mathcal{M}$. Now

$$\mathcal{M} = \{g = 0\} \quad \text{where} \quad g = x^2 + y^2 - z^2 - 1.$$

g is submersive on $\mathcal{M} \iff \nabla g \neq 0$ on $\mathcal{M}$.

Let's see where $\nabla g = \vec{0} \iff$

$$(2x, 2y, -2z) = \vec{0}$$

$$\iff (x,y,z) = \vec{0}$$

Is $\vec{0} \in M$? NO $\Rightarrow \nabla g \neq \vec{0}$ on M.

2. Is M compact?

M compact $\Leftrightarrow$ M is closed & bded.

M closed: true because $M = \{g = 0\}$ and $g \in \mathscr{C}(\mathbb{R}^3)$.

M bounded $:\Leftrightarrow$ coord are bounded.

However, in 1. we proved

$$(0, \pm\sqrt{z^2+1}, z) \in M \quad \forall z \in \mathbb{R}$$

and because

$$\|(0, \pm\sqrt{z^2+1}, z)\| = \sqrt{2z^2+1} \xrightarrow[|z| \to +\infty]{} +\infty$$

$\Rightarrow$ M is unbounded.

$\Rightarrow$ M is not compact.

3. Pts of M at min/max dist to $\vec{0}$.

We've to min/max $f(x,y,z) = \sqrt{x^2+y^2+z^2}$, or equivalently

$$f(x,y,z) = x^2+y^2+z^2.$$

Because M is not compact, we cannot invoke

Because $\mathcal{M}$ is not compact, we cannot invoke Weierstrass. thm. However, because clearly

i) $f \in \mathcal{C}(\mathbb{R}^3)$ and $\lim_{(x,y,z)\to\infty_3} f = +\infty$

ii) $\mathcal{M}$ is closed and unbounded

we can deduce that $\max f$ doesn't exist while $\min f$ exists.

Let $(x,y,z) \in \mathcal{M}$ be a min pt for f on $\mathcal{M}$.
According to Lagrange thm,

$$\exists \lambda : \nabla f = \lambda \nabla g \iff \operatorname{rk}\begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} < 2.$$

Now

$$\operatorname{rk}\begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} = \operatorname{rk}\begin{bmatrix} 2x & 2y & 2z \\ 2x & 2y & -2z \end{bmatrix} < 2$$

$$\iff \begin{cases} \det\begin{bmatrix} 2x & 2y \\ 2x & 2y \end{bmatrix} = 0 & \text{(true)} \\ \det\begin{bmatrix} \cancel{2}x & \cancel{2}z \\ \cancel{2}x & -\cancel{2}z \end{bmatrix} = 0 \iff xz = 0 \\ \det\begin{bmatrix} \cancel{2}y & \cancel{2}z \\ \cancel{2}y & -\cancel{2}z \end{bmatrix} = 0 \iff yz = 0 \end{cases}$$

$$\iff \begin{cases} xz = 0 \\ yz = 0 \end{cases} \iff \begin{cases} x = 0 \\ yz = 0 \end{cases} \vee \begin{cases} z = 0 \\ 0 = 0 \end{cases}$$

$\Updownarrow$ $\Updownarrow$

(x, y, z)

$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$$

$$\begin{cases} x=0 \\ y=0 \end{cases} \vee \begin{cases} x=0 \\ z=0 \end{cases} \qquad (x,y,0) \quad x,y\in\mathbb{R}$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

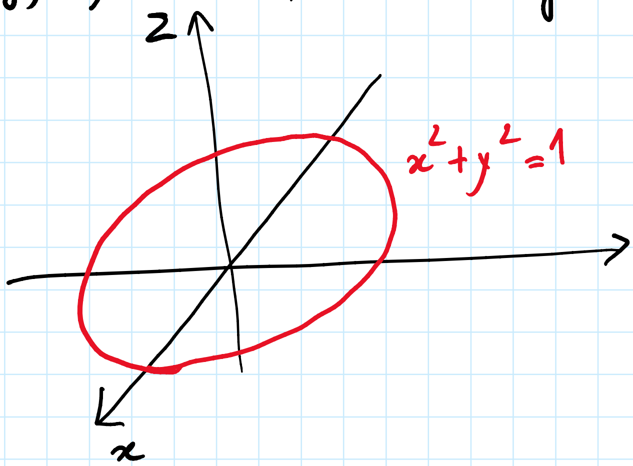
$$(0,0,z) \quad z\in\mathbb{R} \qquad (0,y,0) \quad y\in\mathbb{R}$$

Therefore:

$$\operatorname{rk}\begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} < 2 \iff (0,0,z) \ z\in\mathbb{R}, \quad (x,y,0) \ x,y\in\mathbb{R}.$$

To finish, we have to det. which of these pts are on $\mathcal{M}$:

- $(0,0,z)\in\mathcal{M} \iff 0^2+0^2-z^2=1$ impossible
- $(x,y,0)\in\mathcal{M} \iff x^2+y^2-0^2=1 \iff x^2+y^2=1.$



Now, because on all pts $\{(x,y,0) : x^2+y^2=1\}$ we have

$$f(x,y,0)=x^2+y^2=1$$

we conclude that

all pts $\{(x,y,0) : x^2+y^2=1\}$ are min pts for f on $\mathcal{M}$. ☒

Ex 4.5.14

sabato 13 giugno 2020 08:01

Let

$$f(x,y,z) = \frac{\sqrt{x^2 + y^2/4} - 3}{4} + z^2$$

1. $\lim\limits_{(x,y,z)\to\infty_3} f$.

Sections: $f(x,0,0) = \frac{1}{4}\sqrt{x^2} = \frac{1}{4}|x| \xrightarrow[|x|\to+\infty]{} +\infty$

$$f(0,0,z^2) = z^2 \xrightarrow[|z|\to+\infty]{} +\infty$$

Thus, IF limit $\exists$ THEN it is $= +\infty$. To show this we may use adapted cyl. coords

$$\begin{cases} x = \rho\cos\theta \\ y = 2\rho\sin\theta \\ z = z \end{cases} \Rightarrow x^2 + \frac{y^2}{4} = \rho^2$$

$$\Rightarrow f(x,y,z) = \frac{\rho - 3}{4} + z^2$$

$$\geq \frac{1}{4}(\rho + z^2) - 3/4$$

$$\overset{?}{\geq} \varphi(\|(x,y,z)\|)$$

where $\varphi(r) \longrightarrow +\infty$ $r \to +\infty$.

Notice that

$$\|(x,y,z)\|^2 = \rho^2\left((\cos\theta)^2 + 4(\sin\theta)^2\right) + z^2$$
$$= \rho^2\left(1 + 3(\sin\theta)^2\right) + z^2$$
$$\leq \rho^2 \cdot 4 + z^2 .$$

Now

$$\rho + z^2 = \frac{1}{2}\left(2\rho + 2z^2\right)$$
$$= \frac{1}{2}\left(2\rho + |z| + \underbrace{2z^2 - |z|}_{\geq C \quad \forall z \in \mathbb{R}}\right)$$

(easy)

$$\geq \frac{1}{2}\left(2\rho + |z|\right) + \frac{C}{2}$$
$$\overset{?}{\geq} \frac{1}{2}\sqrt{(2\rho)^2 + |z|^2} + C/2$$

(yes: $\sqrt{(2\rho)^2 + |z|^2} \leq 2\rho + |z| \Leftrightarrow (2\rho)^2 + |z|^2 \leq (2\rho)^2 + |z|^2 + 2\cdot 2\rho|z|$)

$$= \frac{1}{2}\sqrt{4\rho^2 + z^2} + C/2$$
$$= 1/2\, \|(x,y,z)\| + C/2$$

<u>Conclusion:</u>

$$f(x,y,z) \geq \frac{1}{\,}\left(\rho + z^2\right) - \frac{3}{\,} \geq \frac{1}{\,}\|(x,y,z)\| + \frac{C}{\,} - \frac{3}{\,}$$

$$f(x,y,z) \geq \frac{1}{4}\left(\rho + z^2\right) - \frac{3}{4} \geq \underbrace{\frac{1}{8}\|(x,y,z)\| + \frac{c}{8} - \frac{3}{4}}_{\varphi(\|(x,y,z)\|)}$$

$$\frac{1}{4}\left(\rho + z^2\right) \geq \frac{1}{2}\|(x,y,z)\| + \frac{c}{2}$$

where $\varphi(r) = \frac{1}{8} r + K \xrightarrow[r \to +\infty]{} +\infty$.

2. St pts of f on $\mathbb{R}^3$. Min/max ?

We have

$$\nabla f = \vec{0} \iff \begin{cases} \partial_x f = 0 \\ \partial_y f = 0 \\ \partial_z f = 0 \end{cases} \iff \begin{cases} \dfrac{2x}{8\sqrt{x^2 + y^2/4}} = 0 \\ \dfrac{y/2}{8\sqrt{x^2 + y^2/4}} = 0 \\ 2z = 0 \end{cases}$$

$$\iff \begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases}$$

but f is <u>not diff</u> at $(0,0,0)$

$\Rightarrow$ no st. pts

Let's discuss min/max f on $\mathbb{R}^3$

<u>Existence:</u> $f \in \mathcal{C}(\mathbb{R}^3)$, $\mathbb{R}^3$ is closed and unbdd

and because

$$\lim f = +\infty$$

$(x,y,z) \to \infty_3$

we deduce:

- $\max_{\mathbb{R}^3} f$ doesn't exist
- $\min_{\mathbb{R}^3} f$ exists.

Determination: Recall that

i) min f exists

ii) f is differentiable on $\mathbb{R}^3 \setminus \{\vec{0}\}$

iii) f has no st. pts on $\mathbb{R}^3$

So, if $(x,y,z) \in \mathbb{R}^3$ is min pt for f on $\mathbb{R}^3$, then either

- $(x,y,z) \in \mathbb{R}^3 \setminus \{\vec{0}\} \Rightarrow$ Fermat $\nabla f = \vec{0}$ but there're no pts where $\nabla f = \vec{0} \Rightarrow (x,y,z) \notin \mathbb{R}^3 \setminus \{\vec{0}\}$
- $(x,y,z) = \vec{0}$.

Because first alternative doesn't produce any pts we conclude the unique possible min pt is $(0,0,0)$ $\blacksquare$