# Fundamentals of Mathematical Analysis II

Paolo Guiotto

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CHAPTER 1

# Topology in $\mathbb{R}^d$

In a wide part of this course we will study Analysis in a multidimensional context,

$$\mathbb{R}^d := \{(x_1, \dots, x_d) \ : \ x_i \in \mathbb{R}, \ i = 1, \dots, d\}.$$

The basic tool of Mathematical Analysis is the concept of *limit* with its applications: continuous functions, differential calculus, integral calculus and more. The definition of limit passes through a suitable definition of *distance* between points. In $\mathbb{R}^d$ there's a natural notion of distance, namely the *Euclidean distance*

$$\mathrm{dist}(x, y) = \sqrt{(x_1 - y_1)^2 + \ldots + (x_d - y_d)^2}, \ \ x = (x_1, \dots, x_d), \ y = (y_1, \dots, y_d).$$

The euclidean distance is actually a function of the difference between the coordinates of the two points $x, y$ or, in other words, is translation invariant. The distance $d(x, 0)$ (where $0 = (0, \dots, 0)$ is the origin) can be interpreted as the length of a *vector* and plays the same role of the *modulus* for numbers. This is the fundamental concept of *norm* by wich we will begin.

## 1.1. Euclidean norm

We start recalling that $\mathbb{R}^d$ is a *vector space on* $\mathbb{R}$ with the operations of *sum* and *product*

$$(x_1, \dots, x_d) + (y_1, \dots, y_d) := (x_1 + y_1, \dots, x_d + y_d), \ \ \lambda(x_1, \dots, x_d) := (\lambda x_1, \dots, \lambda x_d).$$

Definition 1.1.1. *Given $x = (x_1, \dots, x_d) \in \mathbb{R}$ we call* **norm of** *$x$ the quantity*

$$\|x\| := \sqrt{\sum_{i=1}^{d} x_i^2}.$$

The norm plays the same role of the modulus in $\mathbb{R}$. Precisely

Proposition 1.1.2. *The norm fulfills the following properties:*

i) *positivity: $\|x\| \geqslant 0$, $\forall x \in \mathbb{R}^d$, and $\|x\| = 0$ iff $x = 0_d := (0, \dots, 0)$.*
ii) *homogeneity: $\|\lambda x\| = |\lambda| \|x\|$, $\forall \lambda \in \mathbb{R}$, $\forall x \in \mathbb{R}^d$.*
iii) *triangular inequality: $\|x + y\| \leqslant \|x\| + \|y\|$, $\forall \ x, y \in \mathbb{R}^d$.*

Proof — $\|x\| \geqslant 0$ for eny $x \in \mathbb{R}^d$ is evident. Let's check the vanishing:

$$\|x\| = 0, \iff \sum_{i=1}^{d} x_i^2 = 0, \iff x_i^2 = 0, \forall i, \iff x_i = 0, \forall i.$$

The homogeneity is very easy:

$$\|\lambda x\| = \sqrt{\sum_i (\lambda x_i)^2} = \sqrt{\lambda^2 \sum_i x_i^2} = |\lambda| \sqrt{\sum_i x_i^2} = |\lambda| \|x\|.$$

Finally the triangular inequality: for convenience let's square everything and notice that

$$\|x + y\|^2 = \sum_i (x_i + y_i)^2 = \sum_i (x_i^2 + y_i^2 + 2x_i y_i) = \|x\|^2 + \|y\|^2 + 2 \sum_i x_i y_i.$$

LEMMA 1.1.3 (CAUCHY–SCHWARZ INEQUALITY).

$$\sum_i x_i y_i \leqslant \left(\sum_i x_i^2\right)^{1/2} \left(\sum_i y_i^2\right)^{1/2}. \tag{1.1.1}$$

PROOF — (Lemma) Excluding the trivial cases when $\|x\| = 0$ or $\|y\| = 0$ we may assume $\|x\|, \|y\| \neq 0$ and prove

$$\sum_i \frac{x_i}{\|x\|} \frac{y_i}{\|y\|} \leqslant 1.$$

This simple trick make easy to conclude: recall the elementary inequality

$$ab \leqslant \frac{1}{2}(a^2 + b^2), \; \left( \Longleftrightarrow \; 2ab \leqslant a^2 + b^2, \; \Longleftrightarrow \; (a - b)^2 \geqslant 0 \right).$$

Then using $a = \frac{x_i}{\|x\|}$ and $b = \frac{y_i}{\|y\|}$ we have

$$\sum_i \frac{x_i}{\|x\|} \frac{y_i}{\|y\|} \leqslant \frac{1}{2} \sum_i \left( \frac{x_i^2}{\|x\|^2} + \frac{y_i^2}{\|y\|^2} \right) = \frac{1}{2} \left( \frac{\|x\|^2}{\|x\|^2} + \frac{\|y\|^2}{\|y\|^2} \right) = 1. \quad \blacksquare$$

By Cauchy–Schwarz,

$$\|x + y\|^2 \leqslant \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| = (\|x\| + \|y\|)^2, \; \Longleftrightarrow \; \|x + y\| \leqslant \|x\| + \|y\|. \quad \blacksquare$$

Through the norm we define the concept of *limit for a sequence* and, later, the limit of a function:

DEFINITION 1.1.4. *Let $(x_n) \subset \mathbb{R}^d$ be a sequence of vectors. We say that*

$$x_n \longrightarrow \xi \in \mathbb{R}^d, \; \Longleftrightarrow \; \|x_n - \xi\| \longrightarrow 0 \; (\textit{in } \mathbb{R}).$$

A little bit of care is needed for $x_n \longrightarrow \infty$ because, differently by $\mathbb{R}$, there's not a $+\infty$ and a $-\infty$:

DEFINITION 1.1.5. *Let $(x_n) \subset \mathbb{R}^d$ be a sequence of vectors. We say that*

$$x_n \longrightarrow \infty_d, \; \Longleftrightarrow \; \|x_n\| \longrightarrow +\infty.$$

A *neighborhood*, that is *a set of points closed to a given point $x$*:

DEFINITION 1.1.6. *Let $x \in \mathbb{R}^d$ and $r > 0$. We call*

**closed ball centered in** $x$ **of radius** $r$ *the set* $B(x, r] := \{y \in \mathbb{R}^d : \|y - x\| \leqslant r\}$,

**open ball centered in** $x$ **of radius** $r$ *the set* $B(x, r[ := \{y \in \mathbb{R}^d : \|y - x\| < r\}$.

*Every set $U_x$ containing a ball centered at $x \in \mathbb{R}^d$ its called* **neighborhood of** $x$. *We call also* **neighborhood of** $\infty_d$ *any set $U_\infty$ containing the exterior of a ball, that is $U_{\infty_d} \supset B(0, r]^c$.*

Remark 1.1.7. As $d = 1$ we have $B(x,r] = [x-r, x+r]$; as $d = 2$, $B(x,r]$ is the disk centered in $x$ with radius $r$. We used the square bracket in the notation $B(x,r]$ to recall that $B(x,r]$ contains all the points distant exactly $r$ from the center (that is the "skin" of the ball). ■

Definition 1.1.8 (open set). *A set $S \subset \mathbb{R}^d$ is said*

- **open**, *if $\forall\, x \in S\ \exists\, r(x) > 0\ :\ B(x, r(x)[\subset S$.*
- **closed**, *if its complementary $S^c = \mathbb{R}^d \backslash S$ is open.*

*By definition $\emptyset$ is assumed to be open.*

Remark 1.1.9. A common error consists in thinking that *every set $S$ is open or closed*. This a wrong idea probably due to the meaning of "closed" and "open" in the common language. For instance:

- in $\mathbb{R}$, $[a,b[$ is neither open or closed;
- $\emptyset$ is open by definition; according to the definition it is also closed: indeed $\emptyset^c = \mathbb{R}^d$ which is clearly open. Same for $\mathbb{R}^d$

There are no other simultaneously closed and open subsets in $\mathbb{R}^d$, but this is not easy to prove. ■

Proposition 1.1.10. *If $A, B \subset \mathbb{R}^d$ are open (closed) sets then $A \cup B, A \cap B$ are open (closed).*

Proof — Exercise. ■

An important characterization of closed sets is the following:

Theorem 1.1.11 (Cantor). *$S$ is closed iff it contains all the finite limits of all its sequences that is*

$$S \text{ closed} \iff \forall (x_n) \subset S,\ :\ x_n \longrightarrow \xi \in \mathbb{R}^d, \text{ then } \xi \in S. \tag{1.1.2}$$

Proof — $\Longrightarrow$ Assume that $S$ is closed and let's prove that if $(x_n) \subset S$ with $x_n \longrightarrow \xi \in \mathbb{R}^d$ then $\xi \in S$. Assume that this is false: then $\xi \in S^c$. But $S^c$ is open (being $S$ closed) and because $\xi \in S^c$,

$$\exists\, B(\xi, r[\subset S^c.$$

But $x_n \longrightarrow \xi$, that is $\|x_n - \xi\| \longrightarrow 0$ hence $\|x_n - \xi\| < r$ definitively: this means that $x_n \in B(\xi, r[\subset S^c$ definitively, and this is a contradiction being $(x_n) \subset S$.

$\Longleftarrow$ Assume the the property (1.1.2) is true and let's prove that $S$ is closed, that is $S^c$ is open. Take $\xi \in S^c$ and assume that, by contradiction,

$$\nexists\, B(\xi, r[\subset S^c.$$

Then

$$\forall r > 0,\ B(\xi, r[\not\subset S^c,\ \iff\ \forall r > 0,\ \exists x \in B(\xi, r[\ :\ x \in S.$$

Take $r = \frac{1}{n}$ and call $x_n \in B(\xi, \frac{1}{n}[$ such that $x_n \in S$. The sequence $(x_n) \subset S$ and because $\|x_n - \xi\| < \frac{1}{n} \longrightarrow 0$ we deduce $x_n \longrightarrow \xi$. But then, by (1.1.2), $\xi \in S$, and this is a contradiction. ■

Let's introduce two useful concepts:

Definition 1.1.12 (interior and boundary). *Let $S \subset \mathbb{R}^d$. We call*

- Int($S$) *(**interior of** $S$) the set of points of $x \in S$ such that there exists $U_x$ (neighborhood of $x$) such that $U_x \subset S$;*

- $\partial S$ (**boundary of** $S$) *the set of points* $x \in \mathbb{R}^d$ *such that every neighborhood* $U_x$ *of* $x$ *contains point of S and of* $S^c$*, that is* $U_x \cap S \neq \emptyset$, $U_x \cap S^c \neq \emptyset$.

In particular, $S$ is open if and only if $S = \text{Int}(S)$.

## 1.2. Limit

In this section we want to define the notion of limit

$$\lim_{x \to x_0} f(x) = \ell,$$

for a function $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$. As in one variable Calculus, to set this Defnition we need the concept *accumulation point*:

Definition 1.2.1. *Let* $S \subset \mathbb{R}^d$. *We say that*

- $\xi \in \mathbb{R}^d$ *is* **accumulation point for** $S$ *if* $\exists (x_n) \subset S \backslash \{\xi\}$ *such that* $x_n \longrightarrow \xi$;
- $\infty_d$ *is* **accumulation point for** $S$ *if* $\exists (x_n) \subset S$ *such that* $x_n \longrightarrow \infty_d$.

*The set of all accumulation points of S will be denoted by* $\text{Acc}(D)$.

By this and importing the same idea introduced for limits of one real variable functions we have the:

Definition 1.2.2. *Let* $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$ *and* $x_0 \in \text{Acc}(D)$. *We say that*

$$\lim_{x \to x_0} f(x) = \ell \in \mathbb{R}^m \cup \{\infty_m\}, \iff f(x_n) \longrightarrow \ell, \ \forall (x_n) \subset D \backslash \{x_0\}, \ x_n \longrightarrow x_0. \tag{1.2.1}$$

This Definition has the advantage to cover all the possibilities: limit at a finite point (when $x_0 \in \mathbb{R}^d$), at infinite (when $x_0 = \infty_d$) as well as finite limit (when $\ell \in \mathbb{R}^m$) or infinite limit ($\ell = \infty_m$). Despite this, the Definition is not helpful to compute practically a limit. Let's see some useful techniques.

### 1.2.1. Sections.

Let $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$ be a function such that $\lim_{x \to x_0} f(x) = \ell$. We may imagine that taking a "road" into $D$ going to $x_0$, $f$ will drive us just to $\ell$. With "road" we mean a line in the space.

Definition 1.2.3 (curve). *A function* $\gamma : [a, b] \longrightarrow \mathbb{R}^d$ *is called* **curve** *in* $D$ *if* $\gamma(t) \in D$ *for every* $t \in [a, b]$ *(notation* $\gamma \subset D$*). We call* **support** *of* $\gamma$ *the set* $\text{Supp}(\gamma) := \gamma([a, b])$. *The curve* $\gamma$ *is said to be* **continuous** *if* $\gamma \in \mathscr{C}([a, b])$.

Proposition 1.2.4. *Let* $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$, $x_0 \in \text{Acc}(D)$ *be such that* $\lim_{x \to x_0} f(x) = \ell$. *Let* $\gamma \subset D$ *be a curve such that* $\lim_{t \to t_0} \gamma(t) = x_0$ *and* $\gamma(t) \neq x_0$ *forall* $t$. *Then*

$$\lim_{t \to t_0} f(\gamma(t)) = \ell.$$

Proof — It's just an application of the definitions. Take $t_n \longrightarrow t_0$: then, because

$$\lim_{t \to t_0} \gamma(t) = x_0, \implies \gamma(t_n) \longrightarrow x_0.$$

By iii) we know that $x_n := \gamma(t_n) \neq x_0$ and we come to see that $x_n \longrightarrow x_0$. Therefore, by the Definition (1.2.1) $f(x_n) = f(\gamma(t_n)) \longrightarrow \ell$. So we proved that

$$\forall t_n \longrightarrow t_0, \implies f(\gamma(t_n)) \longrightarrow \ell,$$

and this is nothing but the conclusion. ∎

Corollary 1.2.5. *Let* $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$, $x_0 \in \mathrm{Acc}(D)$. *If there exists* $\gamma_1, \gamma_2$ *curves in D fulfilling ii) and iii) of the Proposition 1.2.4 and such that*

$$\lim_{t \to t_0} f(\gamma_1(t)) \neq \lim_{t \to t_0} f(\gamma_2(t))$$

*then* **the** $\lim_{x \to x_0} f(x)$ **doesn't exists**.

Example 1.2.6. *Show that*

$$\lim_{(x,y) \to 0_2} \frac{xy}{x^2 + y^2}$$

*doesn't exists.*

Sol. — Let

$$f(x,y) = \frac{xy}{x^2 + y^2},\ (x,y) \in D = \mathbb{R}^2 \backslash \{(0,0)\}.$$

Let's check what happens along the two sections along the axes. These are

$$f(t,0) = 0,\ \ f(0,t) = 0.$$

Here $\gamma_1(t) = (t,0) \longrightarrow (0,0)$ as $t \longrightarrow 0 =: t_0$ and clearly $\gamma_1(t) \neq (0,0)$ for all $t \neq t_0$. Hence

$$f(\gamma_1(t)) = f(t,0) = 0 \longrightarrow 0,\ \text{as } t \longrightarrow 0.$$

Similarly $f(0,t) \longrightarrow 0$. Is this enough to conclude that the limit exists? NO! Because we checked just two of the infinitely many sections. Let consider a new section, that is a point moving along a straight line $y = mx$. The curve describing this is simply

$$\gamma(t) := (t, mt),\ m \in \mathbb{R}.$$

Notice that the corresponding section of $f$ is

$$f(\gamma(t)) = f(t,mt) = \frac{mt^2}{t^2 + m^2 t^2} = \frac{m}{1+m^2} \longrightarrow \frac{m}{1+m^2},\ \text{as } t \longrightarrow 0.$$

We conclude that the behavior of $f$ along the axes is different to that one along straight lines through the origin with angular coefficient $m \neq 0$. The limit doesn't exists. ■

Example 1.2.7. *Show that*

$$\lim_{(x,y) \to (0,0)} \frac{xy^2}{x^2 + y^4}$$

*doesn't exists.*

Sol. — Let

$$f(x,y) = \frac{xy^2}{x^2 + y^4},\ (x,y) \in D = \mathbb{R}^2 \backslash \{(0,0)\}.$$

The sections along the axes are $f(t,0) \equiv 0$ and $f(0,t) \equiv 0$. Notice that this says, in particular, that **if the limit exists, it must be equal to** 0. Now if we take a section along the line $y = mx$,

$$f(t,mt) = \frac{m^2 t^3}{t^2 + m^2 t^4} = \frac{m^2 t}{1 + m^2 t^2} \longrightarrow 0,\ \text{as } t \longrightarrow 0.$$

So apparently again no contradictions! But if we consider the line $x = ay^2$ we have

$$f(at^2, t) = \frac{at^2 t^2}{a^2 t^4 + t^4} = \frac{a}{a^2+1} \longrightarrow \frac{a}{a^2+1},\ \text{as } t \longrightarrow 0.$$

This is different from 0 if $a \neq 0$: so we have found a family of curves on which the limit of $f$ exists but is different on any family: we deduce that the limit doesn't exists. ■

EXAMPLE 1.2.8. *Show that*

$$\lim_{(x,y)\to\infty_2} \left(x^2 + y^2 - 4xy\right)$$

*doesn't exists.*

SOL. — Let $f(x,y) := x^2 + y^2 - 4xy$. Sections along the axes are $f(t,0) = t^2$, $f(0,t) = t^2$. Clearly the points $(t,0),(0,t)$ go to $\infty_2$ iff $t \longrightarrow \pm\infty$. In any case $f(t,0), f(0,t) \longrightarrow +\infty$. So the candidate to be the eventual limit is $+\infty$. However, along the line $y = x$,

$$f(t,t) = t^2 + t^2 - 4t^2 = -2t^2,$$

and because $(t,t) \longrightarrow \infty_2$ iff $t \longrightarrow \pm\infty$ we have immediately that $f(t,t) \longrightarrow -\infty$. We conclude that the limit doesn't exists. ■

We have seen then that sections may be used to

- *guess the possible limit* (because **if** the limit exists **then** along any section the limit exists and it is the same);
- *exclude existence of the limit* (if there're two different sections along which the limits are different the global limit cannot exists).

Of course to guess what the "right" sections are is not an easy business.

**1.2.2. Methods of calculus for scalar functions.** Sections are useful to find a candidate or to exclude existence of the limit, but are useless to prove that a function has a limit. In the following Example we will introduce an interesting method to answer to this problem.

EXAMPLE 1.2.9. *Compute*

$$\lim_{(x,y)\to 0_2} \frac{xy^2}{x^2 + y^2}.$$

SOL. — We have to begin with to guess a candidate. We remember that **if** the limit exists must coincide with the limit along any section. Now $f(x,0) = 0 \longrightarrow 0$, so if the limit exists must be 0. This is confirmed, by the way, by the $y-$axis section $f(0,y) = 0$ and by sections along $y = mx$, because

$$f(x,mx) = \frac{xm^2x^2}{x^2 + m^2x^2} = x\frac{m^2}{1+m^2} \longrightarrow 0, \quad x \longrightarrow 0.$$

Ok, if the limit exists it must be 0. How can we check that this is actually the case? Notice that by using polar coordinates

$$\begin{cases} x = \rho\cos\theta, \\ \\ y = \rho\sin\theta, \end{cases}$$

we have

$$f(\rho\cos\theta, \rho\sin\theta) = \frac{\rho^3(\cos\theta)(\sin\theta)^2}{\rho^2} = \rho(\cos\theta)(\sin\theta)^2,$$

so

$$|f(\rho\cos\theta, \rho\sin\theta)| \leqslant |\rho(\cos\theta)(\sin\theta)^2| \leqslant \rho.$$

Returning to euclidean coordinates this last says that

$$|f(x,y)| \leqslant \|(x,y)\|.$$

It seems now evident that if $(x,y) \longrightarrow 0_2$, that is $\|(x,y)\| \longrightarrow 0+$ then, by some argument similar to the two-policemen thm, we should have also $|f(x,y)| \longrightarrow 0+$, that is $f(x,y) \longrightarrow 0$. ■

What is the argument invoked at the end of the previous example? We found a numerical function $g = g(\rho) : [0,+\infty[\longrightarrow \mathbb{R}$ such that $g(\rho) \longrightarrow 0$ as $\rho \longrightarrow 0$ and

$$|f(x,y)| \leqslant g(\|(x,y)\|).$$

It is clear that, as $\|(x,y)\| \longrightarrow 0$ (that is $(x,y) \longrightarrow 0_2$) by the two policemen Lemma we have easily that $f(x,y) \longrightarrow 0$. We can extend this to a more general setting:

PROPOSITION 1.2.10. *Let $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$, $\xi \in \mathrm{Acc}(D)$. Suppose that there exist g such that*

i) $|f(x) - \ell| \leqslant g(\|x - \xi\|)$ *in some* $U_\xi \backslash \{\xi\}$*;*

ii) $\lim_{\rho \to 0+} g(\rho) = 0$.

*Then* $\exists \lim_{x \to \xi} f(x) = \ell$.

PROOF — Let $x_n \longrightarrow \xi$, that is $\|x_n - \xi\| \longrightarrow 0$. Then

$$|f(x_n) - \ell| \leqslant g(\|x_n - \xi\|) \longrightarrow 0, \implies f(x_n) \longrightarrow \ell. \quad ■$$

EXAMPLE 1.2.11. *Compute*

$$\lim_{(x,y,z) \to 0_3} \frac{\sin(xyz)}{x^2 + y^2 + z^2}.$$

SOL. — Let $f(x,y,z) := \frac{\sin(xyz)}{x^2+y^2+z^2}$ defined on its natural domain $D = \mathbb{R}^3 \backslash \{0_3\}$. The sections on the axes $f(x,0,0) = f(0,y,0) = f(0,0,z)$ vanish, so the eventual candidate to be the limit is 0. Using spherical coordinates

$$\begin{cases} x = \rho \cos\theta \sin\varphi, \\ y = \rho \sin\theta \sin\varphi, \\ z = \rho \cos\varphi, \end{cases}$$

we have

$$f(\rho\cos\theta\sin\varphi, \rho\sin\theta\sin\varphi, \rho\cos\varphi) = \frac{\sin\left(\rho^3(\cos\theta)(\sin\theta)(\sin\varphi)^2(\cos\varphi)\right)}{\rho^2}.$$

Recalling that $\sin(\xi) = \xi + o(\xi)$ we have

$$\begin{aligned} f(\rho\cos\theta\sin\varphi, \rho\sin\theta\sin\varphi, \rho\cos\varphi) &= \frac{\rho^3(\cos\theta)(\sin\theta)(\sin\varphi)^2(\cos\varphi)}{\rho^2} + \frac{o(\rho^3(\cos\theta)(\sin\theta)(\sin\varphi)^2(\cos\varphi))}{\rho^2} \\ &= \rho(\cos\theta)(\sin\theta)(\sin\varphi)^2(\cos\varphi) + o\left(\rho(\cos\theta)(\sin\theta)(\sin\varphi)^2(\cos\varphi)\right). \end{aligned}$$

Clearly,

$$|\rho(\cos\theta)(\sin\theta)(\sin\varphi)^2(\cos\varphi)| \leqslant \rho \longrightarrow 0, \text{ as } \rho \longrightarrow 0+,$$

hence $o(\ldots) \longrightarrow 0$. Therefore the limit exists and is 0. ■

We have a similar strategy in the case $\ell = +\infty$ (or $-\infty$):

Proposition 1.2.12. *Let* $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$, $\xi \in \mathrm{Acc}(D)$. *Suppose that there exist* $g$ *such that*

i) $|f(x)| \geqslant g(\|x-\xi\|)$ *in some* $U_\xi \backslash \{\xi\}$;

ii) $\lim_{\rho \to 0+} g(\rho) = +\infty$.

*Then* $\exists \lim_{x \to \xi} f(x) = +\infty$.

Proof — Exercise. ■

A final important case is when $x \longrightarrow \infty_d$. We just quote the following

Proposition 1.2.13. *Let* $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$, $\infty_d \in \mathrm{Acc}(D)$. *Suppose that there exist* $g$ *such that*

i) $f(x) \geqslant g(\|x\|)$ *in some* $U_{\infty_d}$;

ii) $\lim_{\rho \to +\infty} g(\rho) = +\infty$.

*Then* $\exists \lim_{x \to \infty_d} f(x) = +\infty$.

Proof — Exercise. ■

Example 1.2.14. *Compute*

$$\lim_{(x,y)\to\infty_2} \left(x^4 + y^4 - xy\right).$$

Sol. — Looking at the sections along the axes we have $f(x,0) = x^4 \longrightarrow +\infty$ and $f(0,y) = y^4 \longrightarrow +\infty$. So, if the limit exists must be $+\infty$. This seems reasonable because $x^4 + y^4$ should dominate $xy$. In this case we need just a "lower" policemen $g = g(\rho)$ such that

$$f(\rho\cos\theta, \rho\sin\theta) \geqslant g(\rho) \longrightarrow +\infty, \ \rho \longrightarrow +\infty.$$

We have

$$f(\rho\cos\theta,\rho\sin\theta) = \rho^4(\cos\theta)^4 + \rho^4(\sin\theta)^4 - \rho^2(\cos\theta)(\sin\theta) = \rho^4\left[(\cos\theta)^4 + (\sin\theta)^4\right] - \frac{1}{2}\rho^2\sin(2\theta).$$

Now: notice that the quantity $K(\theta) := (\cos\theta)^4 + (\sin\theta)^4$ is always positive and has a minimum as $\theta \in [0, 2\pi]$. Indeed: we don't need any computation because $K$ is clearly continuous, hence $K$ has a minimum by Weierstrass's theorem. Moreover $K(\theta) = 0$ iff $\cos\theta = \sin\theta = 0$, and this in impossible. We call $C$ the minimum value of $K$: $K(\theta) \geqslant C > 0$ for any $\theta \in [0, 2\pi]$. Recalling also that $|\sin(2\theta)| \leqslant 1$ we have

$$f(x,y) \geqslant C\rho^4 - \frac{1}{2}\rho^2\sin(2\theta) \geqslant C\rho^4 - \frac{1}{2}\rho^2 =: g(\rho) \longrightarrow +\infty.$$

By this the conclusion follows. ■

Example 1.2.15. *Compute*

$$\lim_{(x,y,z)\to\infty_3} \left[(x^2+y^2+z^2)^2 - xyz\right].$$

Sol. — A quick check on the sections along the axes show that they tend to $+\infty$. Again: it seems reasonable that the fourth order term $(x^2+y^2+z^2)^2$ dominates on $xyz$. Passing to spherical coordinates

$$f = (\rho^2)^2 - \rho^3(\cos\theta)(\sin\theta)(\sin\varphi)^2(\cos\varphi) = \rho^4 - \frac{1}{4}\rho^3(\sin(2\theta))(\sin(2\varphi))(\sin\varphi).$$

Now, because

$$|(\sin(2\theta))(\sin(2\varphi))(\sin\varphi)| \leqslant 1,$$

we have

$$f \geqslant \rho^4 - \frac{1}{4}\rho^3 =: g(\rho) \longrightarrow +\infty,$$

from which the conclusion follows. ■

EXAMPLE 1.2.16. *Compute*

$$\lim_{(x,y,z)\to\infty_3} \left[(x^2+y^2)^2 + z^2 - xy\right]$$

SOL. — Easily the sections are all convergent to $+\infty$ (e.g. $f(x,0,0) = x^4 \longrightarrow +\infty$ when $\|(x,0,0)\| = |x| \longrightarrow +\infty$). In this case it is convenient to introduce *cylindrical coordinates*

$$\begin{cases} x = \rho\cos\theta, \\ y = \rho\sin\theta, \\ z = z \end{cases}$$

because $x^2 + y^2 = \rho^2$. But be careful: $(x,y,z) \longrightarrow \infty_3$ means $\|(x,y,z)\| = \sqrt{x^2+y^2+z^2} = \sqrt{\rho^2+z^2} \longrightarrow +\infty$, and this doesn't mean necessarily that $\rho \longrightarrow +\infty$. However,

$$f_{cil} = (\rho^2)^2 + z^2 - \rho^2\cos\theta\sin\theta \geqslant \rho^4 + z^2 - \rho^2, \ \ (|\cos\theta\sin\theta| \leqslant 1).$$

Now: if we had $f(x,y,z) \geqslant \rho^2 + z^2 = \|(x,y,z)\|^2$ we would be done. To this aim we may hope that $\rho^4 - \rho^2 \geqslant \rho^2$ and indeed this is actually true if $\rho$ is big enough but not for every $\rho$. To get a lower bound true for any $\rho$ we may notice that

$$\exists K \ : \ \rho^4 - \rho^2 \geqslant \rho^2 + K, \ \forall \rho.$$

Indeed: this is equivalent to say that $\rho^4 - 2\rho^2 \geqslant K$, that is the function $\rho \longmapsto \rho^4 - 2\rho^2$ is bounded below. But a quick check shows that this function has a global minimum: so, if we call $K$ the minimum of the function $\rho \longmapsto \rho^4 - 2\rho^2$ we have the conclusion. ■

## 1.3. Continuity

One of the major application of the concept of limit is the definition of continuity:

DEFINITION 1.3.1 (CONTINUOUS FUNCTION). *Let $f : D \subset \mathbb{R}^d \to \mathbb{R}^m$, $x_0 \in D \cap \mathrm{Acc}(D)$. We say that*

$$f \textbf{ is continuous in } x_0 \textbf{ iff } \lim_{x\to x_0} f(x) = f(x_0).$$

*If $f$ is continuous in any point of $D$ we say that $f$* **is continuous over** *$D$ and we write $f \in \mathscr{C}(D)$.*

The usual properties of continuity for one variable functions remain true. For instance: sum, difference and products of continuous functions at some point (or in some domain) is a continuous function at that point (or in that domain). The same for the ratio with the extra requirement that the denominator is different from 0. It is quite easy to prove (we omit this) that

any polynomial in the $(x_1, \dots, x_d)$ variable is continuous on $\mathbb{R}^d$.

By polynomial we mean a finite sum of monomials of type

$$a x_1^{k_1} x_2^{k_2} \cdots x_d^{k_d}, \ \ k_1, \dots, k_d \in \mathbb{N}, \ a \in \mathbb{R}.$$

Quite useful is the chain rule:

*if $f$ is continuous in $x_0$, $g$ is continuous in $f(x_0)$ then $g \circ f$ is continuous in $x_0$.*

For instance: *any continuous scalar function of a polynomial is continuous where defined.*

Example 1.3.2. *Where is continuous the function $f(x,y) := \log(1 - x^2 - y^2)$?*

Sol. — The function is defined on

$$D = \left\{(x,y) \in \mathbb{R}^2 \ : \ 1 - x^2 - y^2 > 0\right\} = \left\{(x,y) \in \mathbb{R}^2 \ : \ x^2 + y^2 < 1\right\} = B(0,1[.$$

We may write $f = \log \circ p$ where $p$ is the polynomial $p(x,y) := 1 - x^2 - y^2$. Therefore $f \in \mathscr{C}(B(0,1[)$. ■

Example 1.3.3. *The euclidean norm is continuous on $\mathbb{R}^d$.*

Sol. — Remind that

$$\|x\| = \sqrt{x_1^2 + \ldots + x_d^2} \equiv \sqrt{\ } \circ p, \text{ where } p(x_1, \ldots, x_d) = x_1^2 + \ldots + x_d^2.$$

Now: $\sqrt{\ }$ is continuous where defined and $p \geqslant 0$. It follows $\| \cdot \| \in \mathscr{C}(\mathbb{R}^d)$. ■

It is easy to check that continuity *component wise*:

Proposition 1.3.4. *Let $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$, $f = (f_1, \ldots, f_m)$. Then $f$ is continuous in $x_0$ iff any $f_j$ is continuous in $x_0$, $j = 1, \ldots, m$.*

In particular, if $A$ is a linear transformation, that is

$$x \longmapsto Ax = \begin{bmatrix} a_{11} & \ldots & a_{1d} \\ \vdots & & \vdots \\ a_{m1} & \ldots & a_{md} \end{bmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + \ldots + a_{1d}x_d \\ \vdots \\ a_{m1}x_1 + \ldots + a_{md}x_d \end{pmatrix},$$

because every component is a first order polynomial, we get by the last proposition that $A$ is continuous:

Corollary 1.3.5. *Any linear transformation $T \in \mathscr{L}(\mathbb{R}^d; \mathbb{R}^m)$ is continuous.*

## 1.4. Properties of continuous functions

**1.4.1. Weierstrass Theorem.** We recall that any $f \in \mathscr{C}([a,b];\mathbb{R})$ has minimum/maximum over $[a,b]$. This is the well known Wierstrass Thm. The conclusion is false if the interval $[a,b]$ is not closed and bounded. These two properties are the key properties to extend the Thm to the case of functions of several variables. We need first to give the

Definition 1.4.1. *A set $S$ is said* **bounded** *if*

$$\exists M, \ : \ \|x\| \leqslant M, \ \forall x \in S.$$

Theorem 1.4.2 (Weierstrass). *Any continuous function $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$ on a domain $D$ closed and bounded has global minimum and global maximum on $D$, that is*

$$\exists \ x_{min}, x_{max} \in D, \ : \ f(x_{min}) \leqslant f(x) \leqslant f(x_{max}), \ \forall x \in D.$$

Weierstrass thm points out the importance of the class of closed and bounded subsets of $\mathbb{R}^d$:

Definition 1.4.3. *A set $S \subset \mathbb{R}^d$ is called* **compact** *if it is closed and bounded.*

So we may quickly say that *continuous functions on compact sets have min/max*. If we remove compactness we cannot assure the existence of global extreme points. There're however cases when the domain $D$ is still closed but *unbounded* (as for instance when $D = \mathbb{R}^d$) in which something can be said. We notice that

$$S \text{ unbounded} \iff \forall n\ \exists x_n \in S\ :\ \|x_n\| \geqslant n, \iff \exists (x_n) \subset S,\ x_n \longrightarrow \infty_d.$$

In particular, $S$ is unbounded iff $\infty_d \in \mathrm{Acc}(S)$.

Corollary 1.4.4. *Let $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$ be continuous on D, closed and unbounded, such that*

$$\lim_{x \to \infty_d} f(x) = +\infty\ (-\infty).$$

*Then $f$ has a global minimum (maximum).*

Proof — Fix a point $x_0 \in D$: by hypotheses, there exists $R$ such that

$$f(x) \geqslant f(x_0) + 1,\ \forall x \in D\ :\ \|x\| \geqslant R.$$

Indeed: if such $R$ wouldn't exists, for any $R = n \in \mathbb{N}$ then there should be a point $x_n \in D$ such that $f(x_n) \leqslant f(x_0)+1$ and $\|x_n\| \geqslant R = n$. But then $x_n \longrightarrow \infty_d$ hence by assumption $f(x_n) \longrightarrow +\infty$ which is impossible being $f(x_n) \leqslant f(x_0) + 1$ (that is bounded).

Now, with such $R$ we can notice that **if** the minimum exists it must belongs to $D \cap B(0, R]$ and also that $x_0 \in B(0, R]$ (otherwise $f(x_0)$ should be grater than $f(x_0) + 1$ which is impossible). But $D \cap B(0, R]$ is closed (intersection of two closed set) and bounded (because contained in $B(0, R]$). Therefore, by Weierstrass's theorem applied to $f$ on $D \cap B(0, R]$, there exists $x_{min} \in D \cap B(0, R]$ such that

$$f(x_{min}) \leqslant f(x),\ \forall x \in D \cap B(0, R].$$

In particular, also, $f(x_{min}) \leqslant f(x_0) < f(x_0) + 1 \leqslant f(x)$ for all $x \in D \cap B(0, R]^c$. By this follows that

$$f(x_{min}) \leqslant f(x),\ \forall x \in D. \quad \blacksquare$$

Remark 1.4.5. *Of course, because $\lim_{x \to \infty_d} f(x) = +\infty$ the function* **cannot have a maximum***!*

Example 1.4.6. *Show that the function $f(x, y) := x^4 + y^4 - xy$ has global minimum on $\mathbb{R}^2$. What about global maximum?*

Sol. — Of course $f \in \mathscr{C}(\mathbb{R}^2)$ (because it is a polynomial) and $\mathbb{R}^2$ is closed (its complementary is empty, so open by definition) and unbounded. We have also seen (see Example 1.2.14) that

$$\lim_{(x,y) \to \infty_2} f(x, y) = +\infty.$$

Therefore, by the Corollary of Weierstrass's thm we have that there exists a global minimum for $f$ on $\mathbb{R}^2$. On the other side, because $f$ is upper unbounded (by the limit at $\infty_2$) the global maximum doesn't exists. $\blacksquare$

**1.4.2. Domains defined by continuous functions.** Weiertrass' Thm shows how important is to say if a set $S$ is closed or less. A natural way to define subsets in $\mathbb{R}^d$ is through equalities or inequalities:
$S$ is the intersection of the closed unit ball $\left\{x^2 + y^2 + z^2 \leqslant 1\right\}$ with the cylinder $\left\{(x-1)^2 + y^2 \leqslant \frac{1}{4}\right\}$ with axis parallel to $z-$axis passing at the point $(1/2, 0, 0)$ with radius $\frac{1}{2}$. A quite general setting is to define a set as intersection of sets of type

$$\{f \leqslant 0\}, \text{ or } \{f = 0\}, \text{ or } \{f < 0\}.$$

An important question is to know if these sets are open or closed.

FIGURE 1. The set $S := \left\{(x, y, z) \in \mathbb{R}^3 \;:\; x^2 + y^2 + z^2 \leqslant 1,\ \left(x - \frac{1}{2}\right)^2 + y^2 \leqslant \frac{1}{4}\right\}$

THEOREM 1.4.7. *Let $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ be continuous on $R^d$. Then*

i) *$\{f < 0\}$ is open;*
ii) *$\{f \leqslant 0\}$ and $\{f = 0\}$ are closed.*

PROOF — i) Let's show that $\{f < 0\}^c = \{f \geqslant 0\}$ is closed. To this aim let's use the characterization (1.1.2): let $(x_n) \subset \{f \geqslant 0\}$ such that $x_n \longrightarrow \xi \in \mathbb{R}^d$. The goal is to show that $\xi \in \{f \geqslant 0\}$. We know that

$$(x_n) \subset \{f \geqslant 0\}, \implies f(x_n) \geqslant 0.$$

But $f$ is continuous at $\xi$ hence $f(x_n) \longrightarrow f(\xi)$, and because $f(x_n) \geqslant 0$ for any $n$, by permanence of sign it follows that $f(\xi) \geqslant 0$, that is $\xi \in \{f \geqslant 0\}$, and this concludes the proof.
ii) The proof is similar to that one of the previous point. ■

COROLLARY 1.4.8. *If $f_1, \ldots, f_k \in \mathscr{C}(\mathbb{R}^d; \mathbb{R})$ then $\{f_1 \geqslant 0, \ldots, f_k \geqslant 0\}$ and $\{f_1 = 0, \ldots, f_k = 0\}$ are closed, $\{f_1 > 0, \ldots, f_k > 0\}$ is open.*

PROOF — Just each of the sets is intersection of sets like $\{f_j \geqslant 0\}$, $\{f_j = 0\}$ and $\{f_j > 0\}$. ■

**1.4.3. Intermediate values theorem.** In the first year Analysis course we have seen the following important result: if $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ is continuous on $I$ interval takes two values $\alpha$ and $\beta$ then it takes any other value between $\alpha$ and $\beta$. Equivalently, continuous functions maps intervals to intervals. What is the form of this property for continuous functions $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$? The idea is: *continuous functions maps sets made by one piece into set made by one piece*. These sets are called *connected sets*. Here's a precise

DEFINITION 1.4.9. *A set $S \subset \mathbb{R}^d$ is said to be* **(arc-wise) connected** *iff any two of its points are connected by a continuous path that is:*

$$\forall x, y \in S,\ \exists\, \gamma \subset S,\ \gamma \in \mathscr{C},\ :\ \gamma(a) = x,\ \gamma(b) = y.$$

Here's the extension of the intermediate value thm:

THEOREM 1.4.10. *Let $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$ continue on $D$ connected. Then $f(D)$ is connected.*

Proof — Let $\xi, \eta \in f(D)$: we want to prove that they are connected by a continuous arc in $f(D)$. To this aim let $x, y \in D$ be such that

$$\xi = f(x),\ \eta = f(y).$$

Now, because $D$ is connected there exists a continuous path $\gamma$ connecting $x$ to $y$ in $D$. Mapping $\gamma$ in $f(D)$ through $f$, that is defining

$$\tilde{\gamma} : [a,b] \longrightarrow \mathbb{R}^m,\ \tilde{\gamma}(t) := f(\gamma(t)),\ t \in [a,b],$$

we have a continuous path (composition of continuous functions is continuous) connecting $\xi$ to $\eta$ in $f(D)$. ■

Corollary 1.4.11. *Let $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$ continue on $D$ connected. Then $f(D)$ is an interval.*

Proof — By previous thm $f(D)$ is connected. It is easy to recognize now that in $\mathbb{R}$ connected sets are nothing but intervals (exercise). ■

## 1.5. Exercises

Exercise 1.5.1. Discuss the following questions:

i) $B(\xi, \rho]$ is closed.
ii) $S \subset \mathbb{R}^d$ is closed iff $\partial S \subset S$.
iii) $S$ is open iff $S = \mathrm{Int}(S)$.
iv) $\mathrm{Int}(S)$ is open for every set $S \subset \mathbb{R}^d$.
v) A set $S$ is open iff $\partial S \subset S^c$.
vi) Are there cases of sets $S$ such that $\mathrm{Int}(S) = \emptyset$? And sets such that $\partial S = \emptyset$?
vii) Prove the proposition 1.1.10.

Exercise 1.5.2. Looking to suitable sections, prove that the following limits don't exist:

1. $\displaystyle\lim_{(x,y)\to 0_2} \frac{x^2 - y^2}{x^2 + y^2}$. 2. $\displaystyle\lim_{(x,y)\to 0_2} \frac{x^2 + y^3}{x^2 + y^2}$. 3. $\displaystyle\lim_{(x,y)\to 0_2} \frac{xy}{|x| + |y|}$.

4. $\displaystyle\lim_{(x,y)\to 0_2} \frac{y^2 - xy}{x^2 + y^2}$. 5. $\displaystyle\lim_{(x,y,z)\to 0_3} \frac{x + y^2 + z^3}{\sqrt{x^2 + y^2 + z^2}}$. 6. $\displaystyle\lim_{(x,y)\to 0_2} \frac{xy + \sqrt{y^2 + 1} - 1}{x^2 + y^2}$.

7. $\displaystyle\lim_{(x,y,z)\to 0_3} \frac{xyz}{x^4 + y^2 + z^2}$.

Exercise 1.5.3. Compute the following limits:

1. $\displaystyle\lim_{(x,y)\to 0_2} \frac{xy}{\sqrt{x^2 + y^2}}$. 2. $\displaystyle\lim_{(x,y)\to 0_2} \frac{x^2 y^3}{(x^2 + y^2)^2}$. 3. $\displaystyle\lim_{(x,y)\to 0_2} \frac{x^3 - y^3}{x^2 + y^2}$. 4. $\displaystyle\lim_{(x,y)\to 0_2} \frac{x\sqrt{|y|}}{\sqrt[3]{x^4 + y^4}}$. 5. $\displaystyle\lim_{(x,y)\to 0_2} \frac{xy}{|x| + |y|}$.

Exercise 1.5.4. *An open ball is an open set.*

Exercise 1.5.5. For each of the following limit, say if it exists (and in the case compute it) or less:

1. $\displaystyle\lim_{(x,y)\to 0_2} \frac{e^{4y^3} - \cos(x^2 + y^2)}{x^2 + y^2}$. 2. $\displaystyle\lim_{(x,y,z)\to 0_3} \frac{xyz}{x^2 + y^2 + z^2}$. 3. $\displaystyle\lim_{(x,y,z)\to 0_3} \frac{(x^2 + yz)^2}{\sqrt{(x^2 + y^2)^2 + z^4}}$.

4. $\displaystyle\lim_{(x,y)\to 0_2} \frac{\log(1 + 2x^3)}{\sinh(x^2 + y^2)}$. 5. $\displaystyle\lim_{(x,y)\to(0,1)} \frac{x^3 \sinh(y - 1)}{x^2 + y^2 - 2y + 1}$. 6. $\displaystyle\lim_{(x,y)\to(1,1)} \frac{(x-1)^2(y-1)^7}{\left((x-1)^2 + (y-1)^2\right)^{5/2}}$.

EXERCISE 1.5.6. For each of the following limit, say if it exists (and in the case compute it) or less:

1. $\lim_{(x,y)\to\infty_2} \left(x^3 + xy^2 - y^2\right)$.
2. $\lim_{(x,y)\to\infty_2} (x^4 - y^4 + y^2 - x^2)$.
3. $\lim_{(x,y)\to\infty_2} \left(x^2y^2 + x^2 + y^2 - xy\right)$.
4. $\lim_{(x,y,z)\to\infty_3} \left(x^4 + y^4 + z^4 - xyz\right)$.
5. $\lim_{(x,y,z)\to\infty_3} \left(x^2 + y^2 + z^4 - xz\right)$.
6. $\lim_{(x,y,z)\to\infty_3} \left(\sqrt{x^2 + y^2} + z^2 - z\right)$.
7. $\lim_{(x,y,z)\to\infty_3} \left(\sqrt{(x^2 + y^2)^2 + z^4} - xyz\right)$.

EXERCISE 1.5.7 (⋆). For wich values $\alpha > 0$ the function

$$f(x,y) := \frac{x^2|y|^\alpha}{x^4 + y^2},\ (x,y) \neq 0_2,$$

can be defined also in $0_2$ in such a way that the resulting function be continuous in $0_2$.

CHAPTER 2

# Differential Calculus

In this Chapter we extend the *Differential Calculus* to the general setting of *vector valued functions of several variables*,

$$f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m,$$

where $D$ is some domain in $\mathbb{R}^d$. The case $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$ is very important in view of the *optimization problems*, that is *to find min/max of $f$ on the set $D$*. This problem is one of the main reasons to introduce the Differential Calculus because the derivative should give informations useful to search extreme points.

The extension to the multi variable setting is not at all straightforward. Just beginning with the definition, in our setting we can't write

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

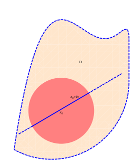
The problem is the domain: if the dimension $d > 1$, $x, h \in \mathbb{R}^d$ and we do not have an operation of "division" between vectors. As we will see, there're several possible definitions of derivative, but just precisely one is the more appropriate.

## 2.1. Directional derivative

In all the section we will assume that

$$\boxed{f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m,\ x \in \text{Int}(D),\ d > 1}$$

Fix a vector $v \neq 0_d$. The set $\{x + tv \ : \ t \in \mathbb{R}\}$ is the straight line passing by $x$ and with direction $v$. It is clear that being $x \in \text{Int}(D)$ at least for $t$ small $x + tv \in D$ (see the figure). Precisely: if $B(x, r] \subset D$ then $x + tv \in B(x, r]$ iff $\|tv\| \leqslant r$, iff $|t| \leqslant \frac{r}{\|v\|}$. Now, in a natural way we define



Definition 2.1.1. *Let $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$, $x \in \text{Int}(D)$. We call* **directional derivative of $f$ in the point $x$ along** $v \neq 0_d$ *the limit (if it exists finite)*

$$D_v f(x) := \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t}.$$

The directional derivative works as an ordinary derivative but, unfortunately, is not a good concept for derivative: it may happens that all the $D_v f$ exists but $f$ is not even continuous!

Example 2.1.2. *Let*

$$f(x,y) := \begin{cases} \dfrac{x^2 y}{x^4 + y^2}, & (x,y) \neq 0_2, \\ \\ 0, & (x,y) = 0_2. \end{cases}$$

*Then $f$ has all the directional derivatives in the point $0_2$ but it is not continuous in $0_2$.*

Sol. — Let's start by the continuity. Looking at the sections along axes we have $f(x,0) = f(0,y) \equiv 0 \longrightarrow 0$. But along the section $y = x^2$ we have

$$f(x,x^2) = \frac{x^2 x^2}{x^4 + x^4} = \frac{1}{2} \longrightarrow \frac{1}{2} \neq f(0,0) = 0.$$

Therefore $\nexists \lim_{(x,y)\to 0_2} f(x,y)$ and consequently the function cannot be continuous! Let's prove now that $\exists D_v f(0,0)$ for any $v$. Let $v = (a,b) \neq 0_2$. We have

$$D_v f(0,0) = \lim_{t\to 0} \frac{f\left((0,0) + t(a,b)\right) - f(0,0)}{t} = \lim_{t\to 0} \frac{f(ta,tb)}{t} = \lim_{t\to 0} \frac{\frac{t^3 a^2 b}{t^2(t^2 a^4 + b^2)}}{t} = \lim_{t\to 0} \frac{a^2 b}{t^2 a^4 + b^2},$$

that is

$$D_v f(0,0) = \begin{cases} 0, & \text{if } b = 0 \text{ (and of course } a \neq 0), \\ \\ \frac{a^2}{b^2}, & \text{if } b \neq 0. \end{cases} \qquad \blacksquare$$

Directional derivatives are just variations on one variable derivatives. A particularly important case is the following:

Definition 2.1.3 (partial derivative). *Let $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$, $x \in \mathrm{Int}(D)$ and let $e_1, \dots, e_d$ be the canonical base of $\mathbb{R}^d$, that is $e_j = (0, \dots, 0, 1, 0, \dots, 0)$ with $1$ in the $j-$th place. If it exists, we define* **partial derivative of $f$ with respect to the $j-$th variable in the point** *$x$ the*

$$\partial_j f(x) := D_{e_j} f(x).$$

Remark 2.1.4. *Partial derivative $\partial_j$ is nothing but an ordinary derivatives w.r.t $x_j$ considering all other variables $x_i$ $i \neq j$ as fixed parameters.* Indeed

$$\begin{aligned} \partial_j f(x) &= \lim_{t\to 0} \frac{f\left((x_1, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_d) + t(0, \dots, 0, 1, 0, \dots, 0)\right) - f(x_1, \dots, x_d)}{t} \\ &= \lim_{t\to 0} \frac{f\left(x_1, \dots, x_{j-1}, x_j + t, x_{j+1}, \dots, x_d\right) - f(x_1, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_d)}{t} \end{aligned}$$

So, for instance

$$\partial_x \left(y \sin x\right) = y \cos x, \quad \partial_y \left(y \sin x\right) = \sin x. \quad \blacksquare$$

## 2.2. Differential

Let's take the definition of derivative from another side. Notice that, for real functions of real variable,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}, \iff \lim_{h\to 0} \frac{f(x+h)-f(x)-f'(x)h}{h} = 0,$$

that is

$$f(x+h)-f(x)-f'(x)h = o(h).$$

If $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$, $f'(x)$ is a number and $f'(x)h$ is the algebraic product between $f'(x)$ and $h$. If now $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$ we expect that

i) $f'(x)h$ is of the same nature of $f(x+h)$ and $f(x)$, that is $f'(x)h\mathbb{R}^m$ (this because the quantity $f(x+h)-f(x)-f'(x)h$ should make sense);
ii) $f'(x)h$ depends linearly by $h$.

In other words, $f'(x)$ should be something that works linearly on vectors $h \in \mathbb{R}^d$ to vectors of $f'(x)h\mathbb{R}^m$. The natural object for $f'(x)$ is therefore an $m \times d$ *matrix*. This motivates the

Definition 2.2.1. *Let $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$, $x \in \mathrm{Int}(D)$. We say that $f$* **is differentiable in** *$x$ iff there exists a $m \times d$ matrix denoted by $f'(x)$ and called* **jacobian matrix** *such that*

$$f(x+h)-f(x)-f'(x)h = o(h), \tag{2.2.1}$$

*in the sense that*

$$\lim_{h\to 0} \frac{\|f(x+h)-f(x)-f'(x)h\|}{\|h\|} = 0. \tag{2.2.2}$$

The natural question is: *what are the entries of the jacobin matrix?*

Proposition 2.2.2. *If $f$ is differentiable in $x$ then there exists all the directional derivatives of $f$ in $x$ and*

$$D_v f(x) = f'(x)v, \ \forall v \in \mathbb{R}^d. \tag{2.2.3}$$

*In particular, if the components of $f$ are $f = (f_1, \dots, f_m)$ then*

$$f'(x) = \begin{bmatrix} \partial_1 f_1(x) & \partial_2 f_1(x) & \dots & \partial_d f_1(x) \\ \vdots & \vdots & & \vdots \\ \partial_1 f_m(x) & \partial_2 f_m(x) & \dots & \partial_d f_m(x) \end{bmatrix}. \tag{2.2.4}$$

Proof — Let's prove the (2.2.3). Fix $v \neq 0$. Then, by (2.2.2)

$$f(x+tv)-(f(x)+f'(x)(tv)) = o(tv), \implies \frac{f(x+tv)-f(x)}{t} = f'(x)v + \frac{o(tv)}{t}.$$

Now, because

$$\lim_{t\to 0}\left\|\frac{o(tv)}{t}\right\| = \lim_{t\to 0}\frac{\|o(tv)\|}{|t|} = \|v\| \lim_{t\to 0}\frac{\|o(tv)\|}{\|tv\|} = \|v\| \lim_{h\to 0}\frac{\|o(h)\|}{\|h\|} = 0,$$

we conclude that

$$\lim_{t\to 0}\frac{f(x_0+tv)-f(x_0)}{t} = f'(x_0)v.$$

Let's prove now the (2.2.4): if we call $f'(x_0) = [a_{ij}]$, it is well known by Linear Algebra that

$$f'(x_0)e_j$$

gives the $j-$th column of the matrix $f'(x_0)$. So the element $a_{ij}$ of $f'(x_0)$ is obtained by taking the $i-$th component of the vector $f'(x_0)e_j$. But: by (2.2.3) we have

$$f'(x_0)e_j = D_{e_j} f(x_0) = \partial_j f(x_0) = \left(\partial_j f_1(x_0), \partial_j f_2(x_0), \dots, \partial_j f_m(x_0)\right),$$

hence the $i-$th component is $\partial_j f_i(x_0)$, and this proves (2.2.4). ■

Remark 2.2.3. *In particular, differentiable $\Longrightarrow$ directionally derivable $\Longrightarrow$ partially derivable.* ■

Two important cases are

- $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$: in this case $f'(x)$ is a $1 \times d$ matrix, precisely
$$f'(x) = [\partial_1 f(x)\ \partial_2 f(x)\ \dots\ \partial_d f(x)] =: \nabla f(x),$$
called **gradient of** $f$ **in** $x$. In this case
$$f'(x)h = \nabla f(x) \cdot h,$$
where we denoted by $\cdot$ the scalar product of vectors in $\mathbb{R}^d$.
- $\gamma : [a,b] \subset \mathbb{R} \longrightarrow \mathbb{R}^d$: in this case $\gamma'(t)$ is a $d \times 1$ matrix, precisely
$$\gamma'(t) = \begin{bmatrix} \gamma_1'(t) \\ \vdots \\ \gamma_d'(t) \end{bmatrix}.$$

Example 2.2.4. *Discuss the differentiability at* $(0,0)$ *of*

$$f(x,y) := \begin{cases} \frac{x^2y^2}{x^2+y^2}, & \text{if } (x,y) \neq (0,0), \\ \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Sol. — We know that **the** candidate for $f'(0,0,) = \nabla f(0,0)$ if it exists. Notice that we cannot simply compute partial derivatives and evaluate in $(0,0)$ because, for instance,

$$\partial_x f(x,y) = \partial_x \frac{x^2y^2}{x^2+y^2} = \frac{2xy^2(x^2+y^2) - x^2y^2 2x}{(x^2+y^2)^2} = \frac{2xy^4}{(x^2+y^2)^2},$$

is of course not defined in $(0,0)$. In this case we have to proceed directly in the computation of $\partial_x f(0,0)$, that is

$$\partial_x f(0,0) = D_{(1,0)} f(0,0) = \lim_{t\to 0} \frac{f((0,0)+t(1,0)) - f(0,0)}{t} = \lim_{t\to 0} \frac{f(t,0)}{t} = \lim_{t\to 0} \frac{0}{t} = 0,$$

and similarly $\partial_y f(0,0) = 0$. We deduce $\nabla f(0,0) = (0,0)$. *What it remains to do?* We have to check that

$$f(0_2+h) - f(0_2) - \nabla f(0_2)\cdot h = o(h), \iff \lim_{h\to 0_2} \frac{\|f(0_2+h) - f(0_2) - \nabla f(0_2)\cdot h\|}{\|h\|} = 0.$$

Now: call $h = (u,v)$:

$$f(0_2+h) - f(0_2) - \nabla f(0_2)\cdot h = f(u,v) - 0 - (0,0)\cdot(u,v) = f(u,v).$$

We have therefore to prove that

$$\lim_{(u,v)\to(0,0)} \frac{f(u,v)}{\|(u,v)\|} = 0.$$

This is a limit in $\mathbb{R}^2$ that we will compute by using the methods of previous chapter. Notice that

$$\frac{f(u,v)}{\|(u,v)\|} = \frac{\frac{u^2v^2}{u^2+v^2}}{\sqrt{u^2+v^2}} = \frac{u^2v^2}{(u^2+v^2)^{3/2}} \overset{u=\rho\cos\theta,\ v=\rho\sin\theta}{=} \frac{\rho^4(\cos\theta)^2(\sin\theta)^2}{\rho^3} = \rho(\cos\theta)^2(\sin\theta)^2,$$

hence

$$\left|\frac{f(u,v)}{\|(u,v)\|}\right| \leqslant \rho \longrightarrow 0, \text{ as } \rho \longrightarrow 0.$$

This finishes the exercise and prove that $f$ is differentiable in $0_2$ and $f'(0_2) = (0,0)$. ■

To check differentiability by using the definition is long and complex. A useful test is the following

Theorem 2.2.5 (total differential theorem). *Let $f = (f_1, \ldots, f_m) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$, $D$ open. If*

$$\partial_j f_i \in \mathscr{C}(D),\ \forall i,j,\ \Longrightarrow\ \exists f \text{ is differentiable in any } x \in D$$

*A function $f$ fulfilling this hypothesis is called a $\mathscr{C}^1(D)$ function.*

Above we proved that to be differentiable implies that all the directional derivatives exists. Differentiability is actually a stronger concept. This follows as by product of the following

Proposition 2.2.6. *If $f$ is differentiable at $x$ it is therein continuous.*

Proof — By (2.2.2) we have $f(y) = f(x) + f'(x)(y-x) + o(y-x) \longrightarrow f(x)$, as $y \longrightarrow x$. ■

The rules of calculus of differentials basically are the same of those of ordinary calculus. For instance

$$(f+g)'(x) = f'(x) + g'(x).$$

if $f, g$ are differentiable at $x$. Similarly it holds the important

Theorem 2.2.7 (chain rule). *Let $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$, $g : E \subset \mathbb{R}^m \longrightarrow \mathbb{R}^k$, $x \in \operatorname{Int}(D)$ and $f(x) \in \operatorname{Int}(E)$ such that $\exists\, f'(x)$ and $\exists\, g'(f(x))$. Then*

$$\exists\ (g \circ f)'(x) = g'(f(x))f'(x). \tag{2.2.5}$$

A special important case of (2.2.5) is the following: suppose we want to compute

$$\frac{d}{dt} g(\gamma(t)), \text{ where } \gamma : I \subset \mathbb{R} \longrightarrow \mathbb{R}^m,\ \ g : E \subset \mathbb{R}^m \longrightarrow \mathbb{R}.$$

Assuming all the hypotheses fulfilled we have

$$\frac{d}{dt} g(\gamma(t)) = g'(\gamma(t))\gamma'(t) = [\partial_1 g(\gamma(t))\ \ldots\ \partial_m g(\gamma(t))] \begin{bmatrix} \gamma_1'(t) \\ \vdots \\ \gamma_m'(t) \end{bmatrix} = \nabla g(\gamma(t)) \cdot \gamma'(t). \tag{2.2.6}$$

This is also called *total derivative of $g$ along $\gamma$.*

## 2.3. Extrema

In this section we study the problem of finding min/max points of a function. Let's introduce first the important

Definition 2.3.1. *Let $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$. We say that a point $x_0 \in D$ is*

- **global maximum (minimum) of** $f$ **on** $D$ *if $f(x) \leqslant f(x_0)$ ($f(x) \geqslant f(x_0)$) for any $x \in D$;*
- **local maximum (minimum) of** $f$ **on** $D$ *if there exists a neighborhood $U_{x_0} \subset D$ of $x_0$ such that $f(x) \leqslant f(x_0)$ ($f(x) \geqslant f(x_0)$) for any $x \in U_{x_0}$.*

The goal of this section is to show how differential calculus can be used to determine local min/max points. Let's start by the extension of a well known property of Calculus in one variable:

Theorem 2.3.2 (Fermat). *Let $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$ and $x_0 \in \text{Int}(D)$ be a local min/max. If $f$ is differentiable at $x_0$ then $\nabla f(x_0) = 0$.*

Proof — The idea is quite easy: if $x_0$ is (for instance) a local maximum for $f$, it is also local maximum on any section. Let's translate formally this idea: assuming, for instance, that $x_0$ is a local maximum for $f$ on $D$,

$$\exists\, r > 0, : \ f(x) \leqslant f(x_0),\ \forall x \in B(x_0, r] \cap D.$$

Because $x_0 \in \text{Int}(D)$ we may assume directly that $B(x_0, r] \subset D$. Now, consider $f$ on a straight line passing by $x_0$ with direction $v$: $f(x_0 + tv)$. Once $x_0 + tv \in B(x_0, r]$ (and this happens iff $|t| \leqslant \frac{r}{\|v\|}$) we will have

$$f(x_0 + tv) \leqslant f(x_0),\ \forall t \in I_0 := \left[ -\frac{r}{\|v\|}, \frac{r}{\|v\|} \right].$$

This condition says that $t \longmapsto f(x_0 + tv)$ has a maximum at $t = 0$. By classical Fermat thm for one variable case, we deduce that

$$0 = \frac{d}{dr} f(x_0 + tv)\Big|_{t=0} \overset{(2.2.6)}{=} \nabla f(x_0) \cdot v.$$

This must be true for any $v \in \mathbb{R}^d$, that is $\nabla f(x_0) \cdot v = 0$, $\forall v \in \mathbb{R}^d$. This means that $\nabla f(x_0)$ is orthogonal to any vector $v$ of $\mathbb{R}^d$. Hence $\nabla f(x_0) = 0$. ■

**Be careful!** First: the condition $\nabla f(x_0) = 0$ works only if $x_0 \in \text{Int}(D)$.

Example 2.3.3. *Let $f(x, y) = x^2 + y^2$ on $D = \{x^2 + y^2 \leqslant 1\}$.* Clearly $\min_D f = 0$ and the minimum is attained at $(0, 0)$, $\max_D f = 1$ and every point on $\{x^2 + y^2 = 1\}$ is a maximum point for $f$ on $D$. Of course these points are not in $\text{Int}(D) = \{x^2 + y^2 < 1\}$ and being $\nabla f(x, y) = (2x, 2y)$ we see that $\nabla f(x, y) = 0_2$ iff $(x, y) = (0, 0)$, so $\nabla f \neq 0$ at every $(x, y) \in \{x^2 + y^2 = 1\}$. ■

Second: the condition $\nabla f = 0$ not necessarily identifies min/max.

Example 2.3.4. *Let $f(x, y) = x^2 - y^2$ on $D = \mathbb{R}^2$.* Clearly, $\nabla f(x, y) = (2x, -2y)$, therefore $\nabla f(0, 0) = (0, 0)$. However $(0, 0)$ is not an extremum because if we section $f$ along the $x$ axis we get $f(x, 0) = x^2$, and this says that $(0, 0)$ is a minimum (even global) for this section, whereas when we take the $y$ axis section $f(0, y) = -y^2$ we have that $(0, 0)$ is a global maximum for the section. Again, taking $f(x, x) = 0$ we get that the function is constant! ■

There's no surprise with the previous example because the same happens for functions of real variable. For instance, $f(x) = x^2$ on $[-1,1]$ has max at $x = \pm 1$ but $f'(\pm 1) \neq 0$ and $f(x) = x^3$ on $\mathbb{R}$ has no min/max but $f'(0) = 0$. In particular, points where $\nabla f = 0$ are not necessarily min/max, but we will give to them a name:

DEFINITION 2.3.5. *If* $\nabla f(x_0) = 0$ *we say that* $x_0$ **is a stationary point for** $f$.

In other words: *local min/max are stationary points* but not viceversa. However, stationary points are the natural candidates for local min/max in the interior of the domain of $f$. Notice that, around such points

$$f(x) - f(x_0) = \nabla f(x_0) \cdot (x - x_0) + o(x - x_0) = o(x - x_0).$$

To discern between a min or a max we need to determine the sign of $f(x) - f(x_0)$, roughly positive in the case of a minimum, negative for a maximum. However, the previous formula doesn't allow to decide. We would need to "explore" a bit more deeply the behavior of $o(x - x_0)$. The idea how to proceed here comes from functions of one real variable. Recall indeed that, under suitable hypotheses, the Taylor's formula says

$$f(x) - f(x_0) = f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + o((x - x_0)^2) \overset{x_0 \ stat.}{\approx} \frac{f''(x_0)}{2}(x - x_0)^2.$$

By this we could easily deduce that

- if $f''(x_0) > 0$ then $x_0$ is a local minimum;
- if $f''(x_0) < 0$ then $x_0$ is a local maximum.

Our strategy will be now to extend this argument to the case when $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$.

**2.3.1. Second derivative and Hessian matrix.** The first step of the program outlined here above is to define the *second derivative* of $f$. Let $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$ be derivable on $D$ open set. Then $\nabla f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a well defined function.

DEFINITION 2.3.6. *We say that* $f$ **is twice differentiable in** $x \in D$ *if exists* $(\nabla f)'(x) =: f''(x)$*. In particular, by (2.2.2) applied to* $\nabla f$ *we have*

$$f''(x) = \begin{bmatrix} \partial_1(\partial_1 f) & \partial_2(\partial_1 f) & \dots & \partial_d(\partial_1 f) \\ \partial_1(\partial_2 f) & \partial_2(\partial_2 f) & \dots & \partial_d(\partial_2 f) \\ \vdots & \vdots & & \vdots \\ \partial_1(\partial_d f) & \partial_2(\partial_d f) & \dots & \partial_d(\partial_d f) \end{bmatrix}(x) \in M_{d\times d}(\mathbb{R}),$$

*is called* **hessian matrix of** $f$ **in** $x_0$.

As consequence of the *total differential thm* 2.2.5 we have

COROLLARY 2.3.7. *If* $f, \nabla f \in \mathscr{C}^1(D)$ *then* $f$ *is twice differentiable on D. We say that* $f \in \mathscr{C}^2(D)$.

EXAMPLE 2.3.8. Let $f(x,y) = x^2 - y^2$ on $D = \mathbb{R}^2$. Then

$$\partial_x f(x,y) = 2x, \quad \partial_y f(x,y) = 2y,$$

exists and are continuous (because polynomials) on $\mathbb{R}^2$. Moreover

$$\partial_{xx} f(x,y) = 2, \ \ \partial_{yx} f(x,y) = 0, \ \ \partial_{yy} f(x,y) = 2, \ \ \partial_{xy} f(x,y) = 0,$$

and, in particular, $f \in \mathscr{C}^2(\mathbb{R}^2)$. Then

$$f''(x,y) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad \blacksquare$$

Notice that $\partial_i(\partial_j f)$ means that you do first the partial derivative of $f$ wrt $x_j$ hence you derive partially what you get wrt $x_i$ while $\partial_j(\partial_i f)$ means that you "switch" the two derivations. This is a general fact

Theorem 2.3.9 (Schwarz). *Let $f \in \mathscr{C}^2(D)$ then $\partial_i(\partial_j f) \equiv \partial_j(\partial_i f)$. In this case we pose $\partial_{ij} f := \partial_i(\partial_j f) \equiv \partial_j(\partial_i f)$. In particular: if $f \in \mathscr{C}^2(D)$ the hessian matrix is symmetric.*

However there're cases (where $f \notin \mathscr{C}^2$ of course) where the switch doesn't work!

Example 2.3.10. *Let*

$$f(x,y) := \begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & \text{if } (x,y) \neq 0_2, \\ 0, & \text{if } (x,y) = 0_2. \end{cases}$$

*Then: $f \in \mathscr{C}^1(\mathbb{R}^2)$, $\exists \partial_{xy} f(0,0), \partial_{yx} f(0,0)$ but $\partial_{xy} f(0,0) \neq \partial_{yx} f(0,0)$.*

Sol. — Exercise. $\blacksquare$

**2.3.2. Taylor's formula and application to extrema.** We have now all the ingredients to state the Taylor's formula (at the second order):

Theorem 2.3.11. *Let $f \in \mathscr{C}^2(D)$, $x_0 \in D \subset \mathbb{R}^d$ open set. Then*

$$f(x_0 + h) = f(x_0) + \nabla f(x_0) \cdot h + \frac{1}{2} f''(x_0)h \cdot h + o(\|h\|^2). \tag{2.3.1}$$

*where $f''(x_0)h$ is the vector obtained multiplying line by column $f''(x_0)$ with the vector $h$.*

Proof — Let's consider the auxiliary function $g(t) := f(x_0 + tv)$ where $v := \frac{h}{\|h\|}$ is the unitary vector with same direction of $h$ (of course $h \neq 0$ otherwise everything is trivial). In our hypotheses it is easy to check that $f, g \in \mathscr{C}^2$. Let's write down the Taylor formula for $g$:

$$g(t) = g(0) + g'(0)t + \frac{1}{2} g''(0)t^2 + o(t^2). \tag{2.3.2}$$

Now: $g(0) = f(x_0)$, $g'(t) = \nabla f(x_0 + tv) \cdot v$ so $g(0) = \nabla f(x_0) \cdot v$. Similarly

$$g''(t) = \left( \sum_j \partial_j f(x_0 + tv) v_j \right)' = \sum_j \left( \sum_i \partial_i(\partial_j f)(x_0 + tv) v_i \right) v_j = \sum_{i,j} \partial_{ij} f(x_0 + tv) v_i v_j = f''(x_0 + tv) v \cdot v,$$

so $g''(0) = f''(x_0)v \cdot v$. Therefore, the (2.3.2) becomes

$$f(x_0 + tv) = f(x_0) + (\nabla f(x_0) \cdot v)\, t + \frac{1}{2} \left( f''(x_0 + tv) v \cdot v \right) t^2 + o(t^2).$$

Setting finally $t = \|h\|$ and noticing that $tv = \|h\| \frac{h}{\|h\|} = h$ we get the thesis. $\blacksquare$

Therefore, by Taylor formula, if $x_0$ is a stationary point for $f$ we have

$$f(x) - f(x_0) \approx \frac{1}{2} f''(x_0)(x - x_0) \cdot (x - x_0).$$

Let's introduce now the

Definition 2.3.12. *Let $M \in M_{d\times d}(\mathbb{R})$. We say that*

i) *$M$ is* **positive** *(notation $M \geqslant 0$) if $Mv \cdot v \geqslant 0$ for any $v \in \mathbb{R}^d$;*
ii) *$M$ is* **strictly positive** *(notation $M > 0$) if $Mv \cdot v > 0$ for any $v \in \mathbb{R}^d \backslash \{0_d\}$.*

Example 2.3.13. *Of course: a strictly positive definite matrix is also positive definite but not viceversa.* For instance the identity $\mathbb{I}$ on $\mathbb{R}^d$ is strictly positive definite, the null matrix is positive definite but not strictly positive.

For the future development we will need the following

Lemma 2.3.14. *A matrix $M \in M_{d\times d}(\mathbb{R})$ is strictly positive iff it is* **coercive** *that is iff*

$$\exists \alpha > 0 \text{ such that } Mv \cdot v \geqslant \alpha \|v\|^2, \ \forall v \in \mathbb{R}^d. \tag{2.3.3}$$

Proof — Clearly coerciveness is stronger than strictly positiveness. The viceversa it is the non trivial part. To this aim we first notice that

$$\mathbb{S}^{d-1} := \left\{ (v_1, \ldots, v_d) \in \mathbb{R}^d \ : \ v_1^2 + \ldots + v_d^2 = 1 \right\},$$

it is clearly bounded and it is also closed, hence compact. By Weierstrass's Thm the function

$$v \longmapsto Mv \cdot v$$

has a global minimum on $\mathbb{S}^{d-1}$. Call $\alpha$ the minimum value. In particular

$$Mv \cdot v \geqslant \alpha, \ \forall v \ : \ \|v\| = 1, \text{ and } \exists v_0 \ : \ Mv_0 \cdot v_0 = \alpha.$$

By the second fact and strict positivity it follows that $\alpha > 0$. Now: if $v \in \mathbb{R}^d$, $v \neq 0$, the vector $\frac{v}{\|v\|}$ has norm 1, so it belongs to $\mathbb{S}^{d-1}$. Hence by previous argument

$$\alpha \leqslant M \frac{v}{\|v\|} \cdot \frac{v}{\|v\|} = \frac{1}{\|v\|^2} Mv \cdot v, \iff Mv \cdot v \geqslant \alpha \|v\|^2. \quad \blacksquare$$

We are now ready to prove the connection with our problem to look for extremes points:

Theorem 2.3.15. *Let $f \in \mathscr{C}^2(D)$, $x_0 \in \mathrm{Int}(D)$ a stationary point of $f$. Then:*

i) *if $x_0$ is a local minimum then $f''(x_0) \geqslant 0$;*
ii) *if $f''(x_0) > 0$ then $x_0$ is a local minimum for $f$.*

*Similarly conclusions hold for maximum points switching from "positive" definite to "negative" definite.*

Proof — First notice that, combining Taylor's and Fermat's theorems, we have

$$f(x) - f(x_0) = \frac{1}{2} f''(x_0)(x - x_0) \cdot (x - x_0) + o(\|x - x_0\|^2).$$

i) Let $B(x_0, r]$ such that $f(x) \geqslant f(x_0)$ for all $x \in B(x_0, r]$. Then, renaming $u = x - x_0$,

$$f''(x_0)u \cdot u + o(\|u\|^2) \geqslant 0, \ \ \forall u \in B(0, r].$$

Fix $v \in \mathbb{R}^d$: then $\varepsilon v \longrightarrow 0$ sa $\varepsilon \longrightarrow 0$, hence will belong to $B(0, r]$ for $\varepsilon$ small enough. Therefore, setting $u = \varepsilon v$ we have

$$0 \leqslant \varepsilon^2 f''(x_0)v \cdot v + o(\varepsilon^2 \|v\|^2), \iff f''(x_0)v \cdot v + \frac{o(\varepsilon^2\|v\|^2)}{\varepsilon^2} \geqslant 0.$$

Letting $\varepsilon \longrightarrow 0$ we have $\frac{o(\varepsilon^2\|v\|^2)}{\varepsilon^2} \longrightarrow 0$, then $f''(x_0)v \cdot v \geqslant 0$.

ii) Because $f''(x_0)$ is strictly positive it is also coercive by Lemma 2.3.14. Hence

$$f(x) - f(x_0) \geqslant \frac{\alpha}{2}\|x - x_0\|^2 + o(\|x - x_0\|^2) = \frac{\alpha}{2}\|x - x_0\|^2 \left(1 + \frac{o(\|x - x_0\|^2)}{\|x - x_0\|^2}\right).$$

By definition

$$\frac{o(\|x - x_0\|^2)}{\|x - x_0\|^2} \longrightarrow 0, \ x \longrightarrow x_0, \implies \exists r > 0, : \frac{o(\|x - x_0\|^2)}{\|x - x_0\|^2} \geqslant -\frac{1}{2}, \ \forall x \in B(x_0, r]\backslash\{x_0\}.$$

Therefore

$$f(x) - f(x_0) \geqslant \frac{\alpha}{4}\|x - x_0\|^2 \geqslant 0, \ \forall x \in B(x_0, r]\backslash\{x_0\}. \quad \blacksquare$$

In the applications it is just the statement ii) to be the important one. Indeed we will proceed as follows: i) we will look for stationary points; ii) we will check the "sign" of $f''$ at the stationary points. Of course, if the sign of $f''$ is not strictly positive or negative we cannot say anything on the nature of the point, as the following example shows:

Example 2.3.16 (Important!). *Find and classify all the critical points of*

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}, \ f(x, y) = x^4 - y^4.$$

Sol. — Of course $f \in \mathscr{C}^2(\mathbb{R}^2)$. Starting with stationary points,

$$\nabla f(x, y) = (4x^3, -4y^3),$$

hence $\nabla f(x, y) = 0_2$ iff $(x, y) = 0_2$ and this is the unique stationary point. The hessian matrix is

$$f''(x, y) = \begin{bmatrix} 12x^2 & 0 \\ 0 & -12y^2 \end{bmatrix}, \ f''(0, 0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

so $f''(0, 0)$ is positive definite. But taking the $x$ section $f(x, 0) = x^4$ you see that $(0, 0)$ is a strict minimum for this section, whereas taking the $y$ section, $f(0, y) = -y^4$ we see that $(0, 0)$ is a strict maximum for this section. Hence $(0, 0)$ is not an extremum. $\blacksquare$

Remark 2.3.17. *The strict positiveness is just a* **sufficient condition** *but is not necessary!* For instance: $f(x, y) = x^4 + y^4$ has hessian matrix null in $(0, 0)$, hence not strictly positive definite, but clearly $(0, 0)$ is a strict global minimum! $\blacksquare$

**2.3.3. Aside on positive definite matrices.** How can be quickly checked the positive definiteness of an Hessian matrix? We review here some facts known from Linear Algebra. First: because $f''(x) =: H = [h_{ij}]$ is symmetric, it is *diagonalizable*, that is

$$\exists T \in M_{d\times d}(\mathbb{R}) \ : \ T^{-1}HT = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \lambda_d \end{bmatrix} \equiv \mathrm{diag}(\lambda_1, \dots, \lambda_d) =: D$$

The numbers $\lambda_1, \dots, \lambda_d$ are the *eigenvalues*. Because $H$ is symmetric easily it follows that $T$ is *orthogonal*, that is $T^{-1} = T^{\star}$. Knowing the eigenvalues it is easy to check positiveness: *$H$ is positive (strictly) iff $\lambda_j \geqslant 0$ ($\lambda_j > 0$) for all $j$.* Indeed

$$Hv \cdot v = TDT^{-1}v \cdot v = DT^{-1}v \cdot T^{\star}v \overset{T^{\star}=T^{-1}}{=} DT^{-1}v \cdot T^{-1}v.$$

If $u = T^{-1}v$ then

$$Hv \cdot v = Du \cdot u = \sum_j \lambda_j u_j^2.$$

We see therefore that

- $\mathrm{diag}(\lambda_1, \dots, \lambda_n) \geqslant 0$ ($> 0$) iff $\lambda_1, \dots, \lambda_d \geqslant 0$ ($> 0$);
- $\mathrm{diag}(\lambda_1, \dots, \lambda_d) \leqslant 0$ ($< 0$) iff $\lambda_1, \dots, \lambda_d \leqslant 0$ ($< 0$).

In general it is not necessary to find the eigenvalues to see if $H$ is (strictly) positive. If $m_k$ denotes the determinant of the sub-matrix of $D$ taking the first $k$ rows and $k$ columns we have

- if $D > 0$ then $\mathrm{sgn}(m_k) = 1$ for any $k$;
- if $D < 0$ then $\mathrm{sgn}(m_k) = (-1)^k$ for any $k$.

Notice also that if $v_j \neq 0$ is an eigenvector for the eigenvalue $\lambda_j$, that is $Hv_j = \lambda_j v_j$ then

$$f(x_0 + tv_j) - f(x_0) = \frac{1}{2}H(tv_j) \cdot (tv_j) + o(t^2) = \lambda_j t^2 \|v_j\|^2 + o(t^2),$$

by which we see that *$f$ has a strict min/max along the direction $v_j$ according $\lambda_j > 0$ or $\lambda_j < 0$.*

DEFINITION 2.3.18. *A stationary point is called* **saddle point** *if there at least one direction along which $f$ has a strict minimum and another directions along which $f$ has a strict maximum.*

EXAMPLE 2.3.19. A typical example of saddle point is $(0, 0)$ for $f(x, y) = x^2 - y^2$. Notice that in this case

$$f''(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}.$$

To determine the eigenvalues would be nice, but it is in general very difficult (if not possible) and basically useless to classify the nature of a stationary point. Indeed, if $m_k := \det[\mathrm{diag}(\lambda_1, \dots, \lambda_k)] = \lambda_1 \cdots \lambda_k$ we see that

- $D$ is positive definite iff $\mathrm{sgn}(m_k) = +1$ for every $k$;
- $D$ is negative definite iff $\mathrm{sgn}(m_k) = (-1)^k$ for every $k$;

Now, it is possible to prove that the same holds by replacing the matrix $D$ with $H$:

THEOREM 2.3.20. *Let* $m_k := \det[h_{ij}]_{i,j=1,\dots,k}$. *Then*

- $H > 0$ *iff* $\text{sgn}(m_k) = 1$ *for any* $k$*;*
- $H < 0$ *iff* $\text{sgn}(m_k) = (-1)^k$ *for any* $k$.

In certain cases it might happens that $\text{sgn}(m_k) = \pm 1$ but not according to the previous cases. In these situations the stationary point is a saddle point.

EXAMPLE 2.3.21. *Let*

$$f(x, y) := x(x^4 + y^2) - y^2 x^3,\ (x, y) \in \mathbb{R}^2.$$

*Find and classify the stationary points of* $f$ *and discuss the existence of local/global extrema. Find* $f(\mathbb{R}^2)$.

SOL. — It is immediate to see that $f \in \mathscr{C}^1(\mathbb{R}^2)$ because clearly

$$\partial_x f(x, y) = 5x^4 + y^2 - 3x^2y^2,\ \ \partial_y f(x, y) = 2xy - 2yx^3,$$

are continuous functions on $\mathbb{R}^2$ (polynomials). We look at stationary points: to this aim we have to solve the equation $\nabla f(x, y) = (0, 0)$, that is

$$\begin{cases} 5x^4 + y^2 - 3x^2y^2 = 0, \\ \\ 2xy - 2yx^3 = 0, \end{cases} \iff \begin{cases} 5x^4 + y^2 - 3x^2y^2 = 0, \\ \\ xy(1 - x^2) = 0, \end{cases}$$

The second equation poses three cases: $x = 0$, $y = 0$ and $x^2 = 1$.

$$\begin{cases} y^2 = 0, \\ \\ x = 0, \end{cases} \iff (x, y) = (0, 0),\ \begin{cases} 5x^4 = 0, \\ \\ y = 0, \end{cases} \iff (x, y) = (0, 0),$$

and

$$\begin{cases} 5 + y^2 - 3y^2 = 0, \\ \\ x^2 = 1, \end{cases} \iff \begin{cases} y^2 = \frac{5}{2}, \\ \\ x^2 = 1, \end{cases} \iff (x, y) = \left(\pm 1, \pm\sqrt{\frac{5}{2}}\right),$$

with all possible combinations on sings. Therefore, there are five stationary points. To classify them we look at the Hessian matrix. Notice first that $f \in \mathscr{C}^2(\mathbb{R}^2)$ and

$$f''(x, y) = \begin{bmatrix} 20x^3 - 6xy^2 & 2y - 6x^2y \\ \\ 2y - 6x^2y & 2x - 2x^3 \end{bmatrix}.$$

$(0, 0)$: here $f''$ is null, so we need to check directly what happens. Notice first that the $x$ section $f(x, 0) = x^5$ says immediately that $(0, 0)$ is not an extremum. To see if it is a saddle point is more difficult. By looking at easy sections like $x = y^2$ we have

$$f(y^2, y) = y^2(x^8 + y^2) - y^2 y^6 = y^{10} + y^4 - y^8 \sim y^4,\ \text{on } I_0,$$

that is, along $x = y^2$ $f$ has a minimum. Taking $x = -y^2$ we have the reversed situation

$$f(-y^2, y) = -y^2(y^8 + y^2) + y^2 y^6 = -y^{10} - y^4 + y^8 \sim -y^4,\ \text{on } I_0$$

that is, along $x = -y^2$ $f$ has a maximum. We deduce that $(0,0)$ is a saddle point. The other points are easier and the criterium applies:

$$f''\left(1, \sqrt{\frac{5}{2}}\right) = \begin{bmatrix} 20 - 6\frac{5}{2} & \sqrt{\frac{5}{2}}(2-6) \\ \sqrt{\frac{5}{2}}(2-6) & 2-2 \end{bmatrix} = \begin{bmatrix} 15 & -4\sqrt{\frac{5}{2}} \\ -4\sqrt{\frac{5}{2}} & 0 \end{bmatrix}.$$

The sub-determinants are $15 > 0$ and $-16\frac{5}{2} = -40 < 0$, therefore $\left(1, \sqrt{\frac{5}{2}}\right)$ turns out to be a saddle point. The same happens for all the other points. Therefore, $f$ has not local extrema points, hence even global extrema. About $f(\mathbb{R}^2)$, finally, because $\mathbb{R}^2$ is connected $f(\mathbb{R}^2)$ is an interval (see Cor 1.4.11). Moreover $f$ is unbounded, just check the $x$ section $f(x,0) = x^5$. Therefore $f(\mathbb{R}^2) = ]-\infty, +\infty[$. ■

Example 2.3.22. *Let*

$$f(x,y) = xye^{x^2+y^2}, \quad (x,y) \in D := \{(x,y) \in \mathbb{R}^2 : x \geqslant 0,\ y \geqslant 0,\ x^2+y^2 \leqslant 1\}.$$

*Draw D. What can you say about D: is open? closed? compact? connected? Show that f admits global extrema on D and find these points. Finally, determine f(D).*

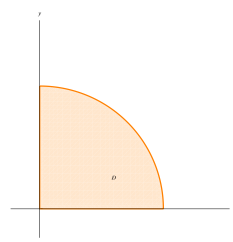


Figure 1. The boundary of $D$ is colored in thick orange.

Sol. — Being $D$ defined by large inequalities it is closed. It is not open because the points on the boundary belongs to $D$. Moreover, $D$ is a subset of unit disk, therefore is bounded, hence is compact. A picture of $D$ is easy.

*Existence.* Because $f$ is continuous and $D$ compact, Weierstrass's thm says that $f$ has global min and max.

*Determination.* Because $D$ is not open the argument (which is, by the way, a standard argument) requires some care. Assume $(x,y)$ be a min/max. We have two cases: either $(x,y) \in \mathrm{Int}(D)$ (hence it is necessarily a stationary point according to Fermat Theorem) or $(x,y) \in \partial D$ (then it is not necessarily a stationary point and we have to discuss directly).

**If** $(x,y) \in \mathrm{Int}(D)$ then, by Fermat thm, $\nabla f(x,y) = 0$. We have

$$\nabla f(x,y) = \left(e^{x^2+y^2}(y+2x^2y), e^{x^2+y^2}(x+2y^2x)\right) = 0, \iff \begin{cases} y(1+2x^2) = 0, \\ x(1+2y^2) = 0, \end{cases} \iff x = y = 0.$$

But this means that there aren't stationary points of $f$ in $\mathrm{Int}(D)$. In particular, the extrema belongs to $\partial D$.

**Hence,** $(x,y) \in \partial D$. We have to proceed by direct inspection. First of all

$$\partial D = \{(x,0) : 0 \leqslant x \leqslant 1\} \cup \{(0,y) : 0 \leqslant y \leqslant 1\} \cup \left\{(x,y) : x^2+y^2 = 1,\ x \geqslant 0,\ y \geqslant 0\right\} =: A \cup B \cup C.$$

On $A$ we have $f(x,0) = 0$, so $f$ is constant; on $B$ we have the same $f(0,y) = 0$. Let see what happens on $C$. It's better to describe $C$ in the way

$$C = \left\{(\cos\theta, \sin\theta) \ : \ \theta \in \left[0, \frac{\pi}{2}\right]\right\}.$$

In this way

$$f(\cos\theta, \sin\theta) = (\cos\theta)(\sin\theta)e^1 = \frac{e}{2}\sin(2\theta),$$

and clearly this quantity is maximum as $2\theta = \frac{\pi}{2}$, that is $\theta = \frac{\pi}{4}$, that is in the point $\frac{1}{\sqrt{2}}(1,1)$, whereas the minimum is 0 (as $\theta = 0, \frac{\pi}{2}$, corresponding to the points $(1,0)$ and $(0,1)$). In conclusion: on $A$ and $B$ $f$ is constant 0 whereas on $C$ the minimum is 0 (taken on $(1,0) \in A$ and $(0,1) \in B$) whereas the maximum is $\frac{e}{2}$ reached in the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. The moral is:

$$\min_D f = 0, \text{ minimum points: } A \cup B, \quad \max_D f = \frac{e}{2}, \text{ maximum points: } \left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\}.$$

Finally: because $D$ is connected, $f(D)$ is connected in $\mathbb{R}$, hence interval. Of course, by what we have said before, $f(D) = [0, \frac{e}{2}]$. ■

Example 2.3.23. *Find the eventual global extrema of the function*

$$f : D = B(0,1] \subset \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad f(x,y) := \sqrt{|x+y|}e^{-(x^2+y^2)}.$$

Sol. — *Existence.* Notice first that $f \in \mathscr{C}(D)$ and clearly $D$ is compact. By Weierstrass's Thm $f$ admits global min/max.

*Determination.* Let $(x,y) \in D$ be an extreme point. We have the following alternative: either $(x,y) \in \mathrm{Int}(D)$ or $(x,y) \in \partial D$.

**Case** $(x,y) \in \mathrm{Int}(D)$. If $f$ is differentiable at $(x,y)$ then $\nabla f = 0$. We have to specify *if* $f$ is differentiable because $f$ is not differentiable where $x + y = 0$ (because of the term $|x+y|$). On that points however $f(x,y) \equiv 0$. If $x + y \neq 0$ we need $\nabla f = 0$. Let's compute the gradient.

$$\partial_x f = \frac{\mathrm{sgn}(x+y)}{2\sqrt{|x+y|}}e^{-(x^2+y^2)} + \sqrt{|x+y|}e^{-(x^2+y^2)}(-2x) = \frac{e^{-(x^2+y^2)}}{2\sqrt{|x+y|}}\left(\mathrm{sgn}(x+y) - 4x|x+y|\right),$$

$$\partial_y f = \frac{e^{-(x^2+y^2)}}{2\sqrt{|x+y|}}\left(\mathrm{sgn}(x+y) - 4y|x+y|\right).$$

Then $\nabla f(x,y) = 0_2$ iff

$$\begin{cases} \mathrm{sgn}(x+y) - 4x|x+y| = 0, \\ \\ \mathrm{sgn}(x+y) - 4y|x+y| = 0, \end{cases} \iff \begin{cases} \mathrm{sgn}(x+y)(1 - 4x(x+y)) = 0, \\ \\ \mathrm{sgn}(x+y)(1 - 4y(x+y)) = 0, \end{cases} \iff \begin{cases} x(x+y) = \frac{1}{4}, \\ \\ y(x+y) = \frac{1}{4}. \end{cases}$$

By this, easily $x, y \neq 0$ and $\frac{x}{y} = 1$, that is $x = y$, and plugging again into the system be get $x(2x) = \frac{1}{4}$, that is $x^2 = \frac{1}{2}$, $x = \pm\frac{1}{\sqrt{2}}$. Therefore we find

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \in \mathrm{Int}(D) \backslash \{y = -x\}.$$

Now, we could also compute the hessian matrix, but look at the question posed: we have to find the global extrema. The two points founded are of course candidates. But we won't need to study precisely their nature, because we need just to know what is the value of $f$ in such points. In this sense, notice that Notice that

$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \sqrt{\frac{2}{\sqrt{2}}}e^{-\frac{1}{2}} = \frac{\sqrt[4]{2}}{\sqrt{e}}.$$

**Case** $(x, y) \in \partial D = \{x^2 + y^2 = 1\} \cup \{y = -x\}$**.** On $\{y = -x\}$ we have $f(x, -x) = 0$, and because $f \geqslant 0$ clearly for every $(x, y) \in \mathbb{R}^2$, the points $\{y = -x\}$ are surely global minimum points; on $\{x^2 + y^2 = 1\}$ it is convenient to use the standard parametrization: we have

$$f(\cos\theta, \sin\theta) = \sqrt{|\cos\theta + \sin\theta|}e^{-1}.$$

It is easy to check that $\cos\theta + \sin\theta$ gets its maximum value as $\theta = \frac{\pi}{4}, \frac{3}{4}\pi$. Therefore

$$\max_{x^2+y^2=1} f(x, y) = \sqrt{\frac{2}{\sqrt{2}}}e^{-1} = \sqrt[4]{2}e^{-1} < \sqrt[4]{2}e^{-1/2} = f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right).$$

By this follows that $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ e $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ are global maximum. ■

## 2.4. Exercises

Exercise 2.4.1. Compute the following directional derivatives (if they exists):

1. $D_{(\sqrt{3},1)} \log(1 + x^2y^2)$, at $(1, 1)$. 2. $D_{(2,2)} \arctan(x + y)$, at $(1, 0)$. 3. $D_{(1,1)} \dfrac{x^2y}{|x| + y^2}$, at $(0, 0)$.

4. $D_{(-1,1)} \dfrac{xy}{x^2 + y^4}$, at $(0, 0)$. 5. $D_{(-1,-2)} \dfrac{y(e^x - 1)}{\sqrt{x^2 + y^2}}$, at $(0, 0)$.

Exercise 2.4.2. For each of the following functions say if a) is continuous at point $(0, 0)$; ii) there exist $\partial_x f(0, 0)$, $\partial_y f(0, 0)$; iii) is differentiable in $(0, 0)$.

1. $f(x, y) := \begin{cases} \frac{x^3}{x^2+y^2}, & (x, y) \neq 0_2, \\ 0, & (x, y) = 0_2. \end{cases}$ 2. $f(x, y) := \begin{cases} \frac{x^2+y^2}{|x|+|y|}, & (x, y) \neq 0_2, \\ 0, & (x, y) = 0_2. \end{cases}$

3. $f(x, y) := \begin{cases} \frac{x^2y^3}{(x^2+y^2)^2}, & (x, y) \neq 0_2, \\ 0, & (x, y) = 0_2. \end{cases}$ 4. $f(x, y) := \begin{cases} \frac{x^2y}{x^2+y^2} + x - y, & (x, y) \neq 0_2, \\ 0, & (x, y) = 0_2. \end{cases}$

Exercise 2.4.3. Show that the function $f(x, y) = x\sqrt{x^2 + y^2}$, $(x, y) \in \mathbb{R}^2$ is differentiable on $\mathbb{R}^2$.

Exercise 2.4.4. Determine the stationary points of each of the following functions:

1. $f(x, y) = xy(x + 1)$. 2. $f(x, y) = x^2 + y^2 + xy$. 3. $f(x, y) = x^3 + y^3 + 2x^2 + 2y^2 + x + y$.

4. $f(x, y) = xe^y + ye^x$. 5. $f(x, y, z) = (x^3 - 3x - y^2)z^2 + z^3$.

Exercise 2.4.5. For each of the following functions a) find the stationary points, b) find eventual min/max on the domain, c) find the image of the domain.

(1) $f(x,y) = x^4 + y^4 - xy$, on $D = \mathbb{R}^2$.
(2) $f(x,y) = x\left((\log x)^2 + y^2\right)$ on $D = ]0,+\infty[\times\mathbb{R}$.
(3) $f(x,y) = xy(x+y)$, on $D = \mathbb{R}^2$.
(4) $f(x,y,z) = x^2 + 3y^2 + 2z^2 - 2xy + 2xz$ on $D = \mathbb{R}^3$.
(5) $f(x,y,z) = x^4 + y^4 + z^4 - xyz$, on $D = \mathbb{R}^3$.

Exercise 2.4.6. Let
$$f(x,y) = (x^2+y^2)^2 - 3x^2 y,\ (x,y) \in \mathbb{R}^2.$$
i) Compute (if it exists) $\lim_{(x,y)\to\infty_2} f(x,y)$. ii) Find stationary points of $f$. iii) Find eventual global min/max of $f$ on $\mathbb{R}^2$ and find $f(\mathbb{R}^2)$.

Exercise 2.4.7. Let
$$f(x,y,z) := (x^2+y^2+z^2)^2 - xyz,\ (x,y,z) \in \mathbb{R}^3.$$
i) Show that $\lim_{(x,y,z)\to\infty_3} f(x,y,z) = +\infty$. ii) Find stationary points of $f$. iii) Show that $f$ has global minimum on $\mathbb{R}^3$ and find $f(\mathbb{R}^3)$.

Exercise 2.4.8. Let$f(x,y) := x^2(1-y)$ on $D := \{(x,y) \in \mathbb{R}^2\ :\ x^2 + |y| \leqslant 4\}$. Study the sign of $f$, determine its eventual stationary points on $D$ and find min/max of $f$ on $D$.

Exercise 2.4.9. Find the extrema of $f(x,y) := xye^{-xy}$ on $D = \{(x,y) \in \mathbb{R}^2\ :\ 1 \leqslant x \leqslant 4,\ y \geqslant 0,\ |xy| \leqslant 1\}$.

Exercise 2.4.10. Find the value of the parameter $\lambda \in \mathbb{R}$ such that the function $f(x,y) := x^2 + \lambda y^2 - 4x + 2y$ has a stationary point in $(2,-1)$. What kind of point is this?

Exercise 2.4.11. Let $f(x,y) := x^2\left(y^2 - (x-1)^2\right)$, $(x,y) \in \mathbb{R}^2$. i) Does it exists $\lim_{(x,y)\to\infty_2} f(x,y)$? If yes, compute it. ii) Find and classify the stationary points of $f$ on $\mathbb{R}^2$. iii) What about extrema of $f$ on $\mathbb{R}^2$? Determine $f(\mathbb{R}^2)$. iv) Show that $f$ has min/max on $D := \{(x,y) \in \mathbb{R}^2\ :\ y \leqslant 0,\ 0 \leqslant x \leqslant y+1\}$ and find them. What is $f(D)$?

Exercise 2.4.12. Consider the function $f(x,y) := x^4 + y^4 - 8(x^2+y^2)$ on $\mathbb{R}^2$. i) Compute $\lim_{(x,y)\to\infty_2} f(x,y)$. ii) Find and classify the stationary points of $f$. What can you say about global extreme points of $f$? What about $f(\mathbb{R}^2)$? iii) Find the extreme points of $f$ on the domain $D := \{(x,y) \in \mathbb{R}^2\ :\ x^2 + y^2 \leqslant 9\}$.

Exercise 2.4.13. Consider the function
$$f(x,y) := \begin{cases} \frac{x^5 y^2}{(x^4+y^2)^2}, & (x,y) \neq 0_2, \\ \\ 0, & (x,y) = 0_2. \end{cases}$$
i) Say if $f$ is continuos, differentiable in $0_2$ (and in this case compute $\nabla f(0_2)$). ii) Find the eventual stationary points of $f$ on $\mathbb{R}^2$ and discuss their nature. Does $f$ has extreme points on $\mathbb{R}^2$? iii) Show that $f$ has min/max on the domain $D = \{(x,y) \in \mathbb{R}^2\ :\ |x| \leqslant 1,\ |y| \leqslant 1\}$ and find them.

Exercise 2.4.14. Let $f$ be the function defined as
$$f(x,y) := \begin{cases} xye^{\frac{xy}{x^2+y^2}}, & (x,y) \in \mathbb{R}^2 \backslash \{0_2\}, \\ \\ 0, & (x,y) = 0_2. \end{cases}$$
i) Say if $f$ is continuous and differentiable at $0_2$. ii) Does it exists $\lim_{(x,y)\to\infty_2} f(x,y)$? (in the case affirmative, what is the value?). Is $f$ bounded on $\mathbb{R}^2$? iii) Show that $f$ has min/max on $\{(x,y) \in \mathbb{R}^2\ :\ x^2 + y^2 \leqslant 1\}$ and find them.

CHAPTER 3

# Constrained Optimization

Many applied questions can be formalized as the maximization/minimization of a certain quantity (function) $f$ of several variables over certain *constraints* on the variables. For instance: consider the problem to find the parallelepiped with maximum volume among those with fixed surface $S$. This means to determine

$$\max_{x,y,z>0\ :\ 2(xy+yz+xz)=S} xyz.$$

A general form for this problem is

$$\text{find}\max/\min f(x_1,\dots,x_d) \text{ subject } g_1(x_1,\dots,x_d)=0,\dots,g_k(x_1,\dots,x_d)=0.$$

The method developed in the previous Chapter, based on the individuation of the stationary points, doesn't work in this context. Indeed, the set of points

$$\mathscr{M} := \{(x_1,\dots,x_d)\in\mathbb{R}^d\ :\ g_1(x_1,\dots,x_d)=0,\dots,g_k(x_1,\dots,x_d)=0\},$$

has no interior points in general. This means that conditions of Fermat's Thm are never fulfilled here, hence min/max are not necessarily stationary points for $f$. To tackle this problem, let's consider a simplified version of the initial isoperimetric problem, that is let's consider the problem to find, among all the rectangles with fixed perimeter $S$, those with maximum area. Formally, we want to determine

$$\max_{(x,y)\in]0,+\infty[\ :\ 2(x+y)=S} xy.$$

In this case, of course, we can reduce the problem to a well known one just by noticing that, by the constraint $2(x+y)=S$ we have, in particular, $y=\frac{S}{2}-x$. Hence we have to maximize

$$xy = x\left(\frac{S}{2}-x\right).$$

This is now a function of one variable that we've to maximize for $0\leqslant x\leqslant \frac{S}{2}$. To do this we can use the tools of one variable calculus. Trying to catch the moral, we could think that if we have to solve

$$\max_{g(x,y)=0} f(x,y),$$

by the equation $g(x,y)=0$ we can "extract" for instance $y=\varphi(x)$. Then the problem can be reduced to

$$\max_x f(x,\varphi(x)).$$

This last, differently from the former, looks like an "unconstrained maximization" to which we can apply the usual tools of calculus. Of course, to make real this idea, there're two main points to solve:

- first, is it always possible say that $g(x,y)=0$ iff $y=\varphi(x)$?

- second, admitting $\varphi$ exists, how this can be used to maximize $f(x, \varphi(x))$?

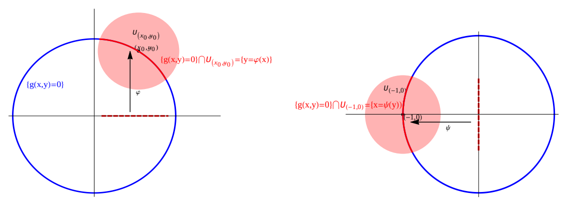
The answer to the first question is given by the *Dini's theorem*, the second is the so called *Lagrange's multipliers theorem*. For pedagogical reasons, we will first see these results in the simplest (but non trivial) case when $f = f(x, y)$ and $g = g(x, y)$, then we will extend to the general case.

### 3.1. Implicit functions: scalar case

Let's consider a set of the form

$$\mathscr{M} := \{(x, y) \in \mathbb{R}^2 \ : \ g(x, y) = 0\}$$

where $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$. Excluding degenerate cases, we expect that $\mathscr{M}$ be a curve in $\mathbb{R}^2$. This curve should be actually a graph of a function because, at least intuitively, through the equation we should able to express one of the two coordinates as function of the other. For instance, if the equation is $x^2 + y^2 - 1 = 0$ (that is $x^2 + y^2 = 1$) we can see that $x = \pm\sqrt{1 - y^2}$ as well as $y = \pm\sqrt{1 - x^2}$.




If we fix a point $(x_0, y_0) \in \mathscr{M}$, then *in a neighborhood of* $(x_0, y_0)$ *the set* $\mathscr{M}$ *is precisely the graph of just one function of type* $y = \varphi(x)$ *or* $x = \psi(y)$. Notice that for all points except $(\pm 1, 0)$ this graph could be in the form $y = \varphi(x)$ but for points $(\pm 1, 0)$ it is impossible to represents the set of the solution (namely the circumference) as a graph of this type. It is, however, possible to express the set as a graph of type $x = \psi(y)$.

Suppose that *the set* $\mathscr{M} = \{g(x, y) = 0\}$ *be, locally, the graph of a function of type* $y = \varphi(x)$, that is

$$\{g(x, y) = 0\} \cap U_{(x_0, y_0)} = \{y = \varphi(x)\}.$$

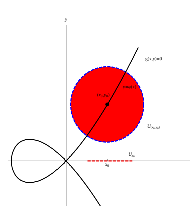
Assuming that $\varphi$ be also a regular function,

$$g(x, \varphi(x)) \equiv 0, \quad \Longrightarrow \quad 0 \equiv \frac{d}{dx} g(x, \varphi(x)) = \partial_x g(x, \varphi(x)) + \partial_y g(x, \varphi(x)) \varphi'(x). \tag{3.1.1}$$

If $\partial_y g(x, \varphi(x)) \neq 0$ to get

$$\varphi'(x) = -\frac{\partial_x g(x, \varphi(x))}{\partial_y g(x, \varphi(x))}.$$

In particular, setting $x = x_0$, being $y_0 = \varphi(x_0)$ we would have $\varphi'(x_0) = -\frac{\partial_x g(x_0, y_0)}{\partial_y g(x_0, y_0)}$.



The condition $\partial_y g(x_0, y_0) \neq 0$ turns out to be the right condition for the existence of the function $\varphi$:

THEOREM 3.1.1 (DINI). *Let $g : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$, $D$ open and $g \in \mathscr{C}^1(D)$. Let $(x_0, y_0) \in D$ such that $g(x_0, y_0) = 0$ and suppose that*

$$\partial_y g(x_0, y_0) \neq 0. \tag{3.1.2}$$

*There exists then $U_{(x_0,y_0)}$ neighborhood of $(x_0, y_0)$ in D, $U_{x_0}$ neighborhood of $x_0$ in $\mathbb{R}$, $\varphi : U_{x_0} \longrightarrow \mathbb{R}$ such that*

$$\{g = 0\} \cap U_{(x_0,y_0)} = \left\{(x, \varphi(x)) \ : \ x \in U_{x_0}\right\}.$$

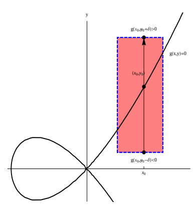
*Moreover $\varphi \in C^1$ and*

$$\varphi'(x) = -\frac{\partial_x g(x, \varphi(x))}{\partial_y g(x, \varphi(x))}, \ \forall x \in U_{x_0}. \tag{3.1.3}$$

*The function $\varphi$ is called* **implicit function** *defined by $g$.*

PROOF — We will limit to construction of $\varphi$. Assume for instance that $\partial_y g(x_0, y_0) > 0$ (similar argument for the case $< 0$). By continuity (recall $g \in \mathscr{C}^1(D)$), there's a neighborhood $U_{(x_0,y_0)}$

$$\partial_y g(x, y) > 0, \ \forall (x, y) \in U_{(x_0,y_0)}.$$




Without loosing any generality, we may assume that $U_{(x_0,y_0)} = [x_0 - \varepsilon, x_0 + \varepsilon] \times [y_0 - \delta, y_0 + \delta]$. Now pick the section $y \longmapsto g(x_0, y)$. As $y$ is such that $(x_0, y) \in U_{(x_0,y_0)}$ this function is *strictly increasing* (its derivative is just $\partial_y g(x_0, y) > 0$) and $g(x_0, y_0) = 0$. Then

$$g(x_0, y_0 - \delta) < 0 < g(x_0, y_0 + \delta).$$

Eventually shrinking the rectangle we may assume that $g < 0$ on the bottom and $g > 0$ on the top. Again: because each vertical section has strictly positive derivative (its our hypothesis) this mean that

$$\forall x \in [x_0 - \varepsilon, x_0 + \varepsilon], \ \exists \text{ a unique } \varphi(x) \in [y_0 - \delta, y_0 + \delta] \text{ such that } g(x, \varphi(x)) = 0.$$

This defines $\varphi : [x_0 - \varepsilon, x_0 + \varepsilon] \longrightarrow \mathbb{R}$ such that $\{g = 0\} \cap U_{(x_0,y_0)} = \{(x, y) \ : \ y = \varphi(x)\}$. ■

REMARK 3.1.2. *A similar statement holds to explicit $x = \psi(y)$. The key hypothesis that replaces (3.1.2) is*

$$\partial_x g(x_0, y_0) \neq 0. \tag{3.1.4}$$

*Then exists $\psi : U_{y_0} \subset \mathbb{R} \longrightarrow \mathbb{R}$, such that $\{g = 0\} \cap U_{(x_0,y_0)} = \{(\psi(y), y) \ : \ y \in U_{y_0}\}$.*

REMARK 3.1.3. *In particular:* **if $\nabla g(x_0, y_0) \neq (0, 0)$ we have that $\{g = 0\}$ is locally the graph of some function $y = \varphi(x)$ or $x = \psi(y)$ in a neighborhood of $(x_0, y_0)$.** ■

Remark 3.1.4 (Warning!). **Dini Thm gives a sufficient condition to explicit one variable in term of the other. A frequent error is to think that the hypothesis (3.1.2) (or (3.1.4)) is sufficient: this is false!** In other words, a common error is to believe that *"if one of (3.1.2) or (3.1.4) is not fulfilled, then is not possible to explicit one variable in term of the other"*. Look at the following "stupid" example: let

$$g(x,y) = (x-y)^2.$$

Of course $\{g(x,y)=0\} = \{y = x\}$ is a global graph of the function $y = x$ or $x = y$. But

$$\partial_y g(x,y) = -2(x-y) \equiv 0,\ \forall (x,y) \in \{g=0\},$$

as well $\partial_x g(x,y) \equiv 0$ for every $(x,y) \in \{g=0\}$. Therefore, hypotheses (3.1.2) and (3.1.4) are **never** fulfilled! ■

Example 3.1.5. *Consider the equation*

$$x^3 + y^3 - 3xy - 3 = 0.$$

*Show that if $(x,y)$ is a solution of this equation is always possible to explicit at least one between $x$ and $y$ as function of the other variable.*

Sol. — Define $g(x,y) := x^3+y^3-3xy-3$. If we prove that when $(x,y)$ is a solution then **at least one** between $\partial_x g(x,y)$ or $\partial_y g(x,y)$ is different from 0 then at least one of (3.1.2) and (3.1.4) is fulfilled: therefore, Dini Thm applies at least in one of the two cases and we are done. To this aim let's see if there are points on $\{g=0\}$ where both (3.1.2) and (3.1.4) fail: we have to find solution of

$$\begin{cases} (x,y) \in \{g=0\}, \\ \\ \partial_x g(x,y) = 0, \\ \\ \partial_y g(x,y) = 0. \end{cases} \iff \begin{cases} x^3+y^3-3xy-3=0 \\ \\ 3x^2-3y=0, \\ \\ 3y^2-3x=0. \end{cases} \iff \begin{cases} x^3+y^3-3xy-3=0, \\ \\ x^2=y, \\ \\ y^2=x. \end{cases}$$

By two last equations $y^4 = y$, that is $y(y^3-1)=0$ whose solutions are $y = 0, 1$. As $y = 0$ we have $x = 0$, and for $y = 1$ we have $x = 1$, that is the points $(0,0)$ and $(1,1)$. Now the question is: do they satisfy also the first condition? It is easy to check that the answer is no! ■

## 3.2. Lagrange multipliers: scalar case

Let's now consider the problem

$$\min_{\mathscr{M}}/\max_{\mathscr{M}} f(x,y), \quad \text{where } \mathscr{M} := \{g(x,y)=0\}.$$

Our goal here is to find an analogous of condition $\nabla f = 0$ for constrained min/max points. Assume that $(x_0,y_0)$ be a min/max for $f$ on $\mathscr{M}$ and assume that, in a neighborhood of $(x_0,y_0)$, $\mathscr{M}$ be the graph of some regular function. According to Dini's Theorem and, in particular, to the Remark 3.1.3, a sufficient condition for this is

$$\nabla g(x_0,y_0) := (\partial_x g(x_0,y_0), \partial_y g(x_0,y_0)) \neq 0_2.$$

Definition 3.2.1. *We say that $g$ is* **submersive** *at $(x_0,y_0)$ if $\nabla g(x_0,y_0) \neq 0_2$.*

In this setting we have

THEOREM 3.2.2 (LAGRANGE). *Assume that* $f \in \mathscr{C}^1(D)$, $D \supset \mathscr{M} := \{g = 0\}$, $g \in \mathscr{C}^1(\mathbb{R}^2)$, *g submersive at* $(x_0, y_0)$. *If* $(x_0, y_0) \in \mathscr{M}$ *is a local min/max for* $f$ *on* $\mathscr{M}$ *then*

$$\exists \lambda \in \mathbb{R} \ : \ \nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0). \tag{3.2.1}$$

*Points* $(x_0, y_0)$ *where (3.2.1) holds are called* **constrained stationary points**.

PROOF — Suppose that, for instance, $\partial_y g(x_0, y_0) \neq 0$ (the other case is treated similarly). Then, by Dini's theorem, there exists a neighborhood $U$ of $(x_0, y_0)$ and a function $y = \varphi(x)$ such that

$$\mathscr{M} \cap U = \{(x, \varphi(x)) \ : \ x \in I_{x_0}\}.$$

Now, because $(x_0, y_0)$ is a local minimum for $f$ on $\mathscr{M}$ (same argument in the case of a maximum),

$$f(x, \varphi(x)) \geqslant f(x_0, \varphi(x_0)), \ \forall x \in I_{x_0},$$

that is the auxiliary function $h(x) := f(x, \varphi(x))$ has a minimum at $x_0$. Therefore, by the one variable Fermat theorem,

$$h'(x_0) = 0.$$

But

$$h'(x) = \frac{d}{dx} f(x, \varphi(x)) = \partial_x f(x, \varphi(x)) + \partial_y f(x, \varphi(x))\varphi'(x),$$

hence

$$0 = \partial_x f(x_0, y_0) + \partial_y f(x_0, y_0)\varphi'(x_0).$$

Now, recalling that for the implicit function $\varphi$ we have the (3.1.3) we deduce

$$0 = \partial_x f(x_0, y_0) - \partial_y f(x_0, y_0)\frac{\partial_x g(x_0, y_0)}{\partial_y g(x_0, y_0)},$$

that is

$$\partial_x f(x_0, y_0)\partial_y g(x_0, y_0) - \partial_y f(x_0, y_0)\partial_x g(x_0, y_0) = 0, \iff \nabla f(x_0, y_0) \perp \left(\partial_y g(x_0, y_0), -\partial_x g(x_0, y_0)\right).$$

But then

$$\nabla f(x_0, y_0) \propto (\partial_x g(x_0, y_0), \partial_y g(x_0, y_0)) = \nabla g(x_0, y_0). \quad \blacksquare$$

In practice, to determine constrained min/max we can proceed as follows. First we make sure if they exists (by a Weierstrass like argument). Then we look for constrained stationary points of $f$ on $\mathscr{M}$. The min/max is among them. Finally, we could just evaluate $f$ on these points: those where $f$ is minimum are the mins, those where $f$ is maximum are the maxs.

EXAMPLE 3.2.3. *Find points of the ellipse* $x^2 + 2y^2 - xy = 9$ *at min/max distance to the origin.*

SOL. — Let $\mathscr{M} := \{x^2 + 2y^2 - xy = 9\} = \{x^2 + 2y^2 - xy - 9 = 0\} =: \{g = 0\}$. We have to minimize/maximize the distance to the origin, that is the function

$$f(x, y) = \sqrt{x^2 + y^2}.$$

Because of the properties of the root, to minimize this function or just $x^2 + y^2$ is the same (it produces the same points but of course not the same values!) being $\sqrt{x^2 + y^2}$ min/max iff $x^2 + y^2$ it is, we replace the previous $f$ with

$$f(x, y) = x^2 + y^2,$$

which is easier to be managed.

**Existence**: $f \in \mathscr{C}(\mathbb{R}^2)$ and $\mathscr{M}$ is clearly closed. If we don't recognize an ellipse immediately (hence we can conclude that $\mathscr{M}$ is also bounded) we can easily show easily this: recalling that

$$xy \leqslant \frac{1}{2}(x^2 + y^2),$$

if $(x, y) \in \mathscr{M}$ then

$$x^2 + 2y^2 = 9 + xy \leqslant 9 + \frac{1}{2}(x^2 + y^2), \implies \frac{1}{2}x^3 + \frac{3}{2}y^2 \leqslant 9, \implies \frac{1}{2}x^2, \frac{3}{2}y^2 \leqslant 9,$$

by which $x^2 \leqslant 18$ (hence $|x| \leq \sqrt{18}$) and $y^2 \leqslant 6$ (that is $|y| \leqslant \sqrt{6}$). In any case both $x, y$ are bounded hence $\mathscr{M}$ is bounded. The conclusion is that, according to Weierstrass theorem, $\mathscr{M}$ is compact, hence $f$ admits both min/max.

**Determination**: by the previous argument we know that min/max points for $f$ exist. Let's see if we can apply the previous theorem. We need to check if $g$ is submersive on $\mathscr{M}$. To this aim let's see where $g$ is not submersive. This happens iff

$$\nabla g = 0, \iff (2x - y, 4y - x) = (0, 0), \iff \begin{cases} 2x - y = 0, \\ \\ 4y - x = 0, \end{cases} \iff x = y = 0.$$

Therefore $g$ is not submersive at point $(0, 0) \notin \mathscr{M}$, hence $g$ is submersive on $\mathscr{M}$.

According to the previous theorem in a min/max point we must have $\nabla f \propto \nabla g$. Being $\nabla f = (2x, 2y)$ this means

$$(2x, 2y) = \lambda(2x - y, 4y - x), \iff \begin{cases} 2x = \lambda(2x - y), \\ \\ 2y = \lambda(4y - x) \end{cases}$$

Of course $(0, 0)$ is a solution of the system, but because $(0, 0) \notin \mathscr{M}$ it cannot be considered as candidate to be en extrema for $f$. Are there non trivial solutions? Notice that if $x = 0$ (or $y = 0$) then necessarily $y = 0$ ($x = 0$). Hence we can assume $x, y \neq 0$: in such case by dividing the two equations we get

$$\frac{2x}{2y} = \frac{2x - y}{4y - x}, \iff x(4y - x) - (2x - y)y = 0, \iff 2xy + y^2 - x^2 = 0.$$

This can be rewritten as

$$(x + y)^2 - 2x^2 = 0, \iff (x + y)^2 = 2x^2, \iff x + y = \pm\sqrt{2}x, \iff y = (\pm\sqrt{2} - 1)x.$$

Therefore we have points $(x, (\pm\sqrt{2} - 1)x)$. Of course we have to look at those of them that belongs to $\mathscr{M}$:

$$(x, (\sqrt{2} - 1)x) \in \mathscr{M}, \iff x^2 + 2(\sqrt{2} - 1)^2 x^2 - (\sqrt{2} - 1)x^2 = 9, \iff (8 - 5\sqrt{2})x^2 = 9, \iff x = \pm\frac{3}{\sqrt{8 - 5\sqrt{2}}}.$$

This produces points $\left(\pm\frac{3}{\sqrt{8-5\sqrt{2}}}, \pm\frac{3(\sqrt{2}-1)}{\sqrt{8-5\sqrt{2}}}\right)$ (same sign, 2 points). Similarly

$$(x, (-\sqrt{2} - 1)x) \in \mathscr{M}, \iff x^2 + 2(-\sqrt{2} - 1)^2 x^2 - (-\sqrt{2} - 1)x^2 = 9, \iff (8 + 5\sqrt{2})x^2 = 9, \iff x = \pm\frac{3}{\sqrt{8 + 5\sqrt{2}}}.$$

This produces points $\left(\pm\frac{3}{\sqrt{8+5\sqrt{2}}}, \pm\frac{3(-\sqrt{2}-1)}{\sqrt{8+5\sqrt{2}}}\right)$ (same sign, two points). Now, being

$$f\left(\pm\frac{3}{\sqrt{8 - 5\sqrt{2}}}, \pm\frac{3(\sqrt{2} - 1)}{\sqrt{8 - 5\sqrt{2}}}\right) = \frac{36 - 18\sqrt{2}}{8 - 5\sqrt{2}} > f\left(\pm\frac{3}{\sqrt{8 + 5\sqrt{2}}}, \pm\frac{3(-\sqrt{2} - 1)}{\sqrt{8 + 5\sqrt{2}}}\right) = \frac{36 + 18\sqrt{2}}{8 + 5\sqrt{2}}$$

we have that the first points are max for $f$ (hence point of $\mathscr{M}$ at max distance from $0_2$), the latter are min. ■

### 3.3. Dini Theorem: case of systems

A natural extension of the classical Dini's theorem is the following. Consider a set defined as

$$\mathscr{M} := \{(x_1, \dots, x_d) \in \mathbb{R}^d \ : \ g_1(x_1, \dots, x_d) = 0, \dots, g_n(x_1, \dots, x_d) = 0\}.$$

The intuitive idea is that, under good conditions, it should be possible to represent points $(x_1, \dots, x_d) \in \mathscr{M}$ in a neighborhood of a certain $(\xi_1, \dots, \xi_d) \in \mathscr{M}$ by expressing $n$ coordinates in function of the remaining $d - n$, as for instance

$$(x_1, \dots, x_d) \in \mathscr{M}, \iff \begin{cases} x_1 = \psi_1(x_{n+1}, \dots, x_d), \\ \vdots \\ x_n = \psi_n(x_{n+1}, \dots, x_d). \end{cases}$$

We can see this under the same form of the Dini's Thm. Call $g = (g_1, \dots, g_n) : \mathbb{R}^d \longrightarrow \mathbb{R}^n$, and write

$$x = (x_1, \dots, x_n), \quad y = (x_{n+1}, \dots, x_d),$$

in such a way that $(x_1, \dots, x_d) \in \mathscr{M}$ iff $g(x, y) = 0$. We set also $x_0 := (\xi_1, \dots, \xi_n)$ and $y_0 := (\xi_{n+1}, \dots, \xi_d)$. We look then for a function $x = \psi(y)$ such that

$$\{g = 0\} \cap U_{(x_0, y_0)} = \{(\psi(y), y)\}.$$

Such $\psi$ would be a function of $y \in \mathbb{R}^{d-n}$ into $\mathbb{R}^n$ and should fulfill

$$g(\psi(y), y) \equiv 0.$$

Deriving this identity with the chain rule

$$\partial_x g(\psi(y), y)\psi'(y) + \partial_y g(\psi(y), y) = 0.$$

Here $\partial_x g$ denotes the Jacobian matrix of $g$ respect to $x = (x_1, \dots, x_n)$ while $\partial_y g$ is the same respect to $y = (x_{n+1}, \dots, x_d)$,

$$\partial_x g = \begin{bmatrix} \partial_1 g_1 & \partial_2 g_1 & \dots & \partial_n g_1 \\ \partial_1 g_2 & \partial_2 g_2 & \dots & \partial_n g_2 \\ \vdots & & & \vdots \\ \partial_1 g_n & \partial_2 g_n & \dots & \partial_n g_n \end{bmatrix}, \quad \partial_y g = \begin{bmatrix} \partial_{n+1} g_1 & \partial_{n+2} g_1 & \dots & \partial_d g_1 \\ \partial_{n+1} g_2 & \partial_{n+2} g_2 & \dots & \partial_d g_2 \\ \vdots & & & \vdots \\ \partial_{n+1} g_n & \partial_{n+2} g_n & \dots & \partial_d g_n \end{bmatrix}.$$

Therefore

$$\psi'(y) = -[\partial_x g(\psi(y), y)]^{-1} \partial_y g(\psi(y), y),$$

**provided** $\partial_x g(\psi(y), y)$ **be invertible**. As $y = y_0$, $\psi(y_0) = x_0$ and we have $\partial_x g(x_0, y_0)$ needs to be invertible. As for the Dini's Theorem this turns out to be the appropriate condition to ask:

Theorem 3.3.1. *Let $g = (g_1, \ldots, g_n) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^n$ be a $\mathscr{C}^1$ function and $(x_0, y_0) \in \mathbb{R}^d$ be such that $g(x_0, y_0) = 0$. Suppose that*

$$\det \partial_x g(x_0, y_0) \neq 0,$$

*There exists then an implicit function $\psi \in \mathscr{C}^1$ and a neighborhood $U_{(x_0,y_0)}$ of $(x_0, y_0)$ such that*

$$\{g = 0\} \cap U_{(x_0,y_0)} = \{(\psi(y), y)\}.$$

*Moreover*

$$\psi'(y) = -\left[\partial_x g(\psi(y), y)\right]^{-1} \partial_y g(\psi(y), y). \tag{3.3.1}$$

Example 3.3.2. *Show that the system*

$$\begin{cases} x^3 - 3xy^2 + z^3 + 1 = 0, \\ x - 2y^2 - 3z^2 + 4 = 0, \end{cases}$$

*is a graph of $(y, z)$ as function of $x$ in a neighborhood of the point $(x, y, z) = (1, 1, 1)$. Compute $y'(1)$.*

Sol. — Easily we see that $(x, y, z) = (1, 1, 1)$ is a solution. The problem asks to express $(y, z)$ as function of $x$ in a neighborhood of $(1, 1, 1)$: this is possible, according to Dini's thm, if the jacobian $\partial_{(y,z)} g(1, 1, 1)$ is invertible, where of course $g(x, y, z) = (x^3 - 3xy^2 + z^3 + 1, x - 2y^2 - 3z^2 + 4)$. We have

$$\partial_{(y,z)} g(x, y, z) = \begin{bmatrix} -6xy & 3z^2 \\ -4y & -6z \end{bmatrix}, \implies \partial_{(y,z)} g(1, 1, 1) = \begin{bmatrix} -6 & 3 \\ -4 & -6 \end{bmatrix},$$

clearly invertible (its determinant is $36 + 12 = 48$). So the first requirement is fulfilled. For the second, instead to use the (3.3.1) we proceed directly: *let's derive the two equations considering $y = y(x)$ and $z = z(x)$*. We get

$$\begin{cases} (x^3 - 3xy^2 + z^3 + 1)' = 0, \\ (x - 2y^2 - 3z^2 + 4)' = 0, \end{cases} t, \iff \begin{cases} 3x^2 - 3(y^2 + 2xyy') + 3z^2 z' = 0, \\ 1 - 4yy' - 6zz' = 0. \end{cases}$$

Now, replacing $(x, y, z) = (1, 1, 1)$ we obtain

$$\begin{cases} 3 - 3(1 + 2y'(1)) + 3z'(1) = 0, \\ 1 - 4y'(1) - 6z'(1) = 0, \end{cases} \iff \begin{cases} -2y'(1) + z'(1) = 0, \\ 4y'(1) + 6z'(1) = 1, \end{cases} \iff y'(1) = \frac{1}{16}, \ z'(1) = \frac{1}{8}. \ \blacksquare$$

## 3.4. Lagrange Multipliers: general case

Let's now consider the problem

$$\min_{\mathscr{M}}/\max_{\mathscr{M}} f(x_1, \ldots, x_d), \ \text{ on } \mathscr{M} = \{(x_1, \ldots, x_d) \in \mathbb{R}^d \ : \ g_1(x_1, \ldots, x_d) = 0, \ldots, g_n(x_1, \ldots, x_d) = 0\}.$$

The general argument is similar but technically much more involved than that one presented as special case above. We will limit to sketch it.

The first step is to discuss when $\mathscr{M} = \{g = 0\}$ is, locally (that is in a neighborhood of each of its points), a graph of some function. According to Dini's Thm 3.3.1, to express locally $\mathscr{M}$ as graph where $(x_{i_1}, x_{i_2}, \dots, x_{i_n})$ are functions of the remaining $d - n$ we need that

$$\det[\partial_{(x_{i_1},x_{i_2},\dots,x_{i_n})} g] = \det \begin{bmatrix} \partial_{x_{i_1}} g_1 & \partial_{x_{i_2}} g_1 & \dots & \partial_{x_{i_n}} g_1 \\ \partial_{x_{i_1}} g_2 & \partial_{x_{i_2}} g_2 & \dots & \partial_{x_{i_n}} g_2 \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{x_{i_1}} g_n & \partial_{x_{i_2}} g_n & \dots & \partial_{x_{i_n}} g_n \end{bmatrix} \neq 0.$$

The matrix $\partial_{(x_{i_1},x_{i_2},\dots,x_{i_n})} g$ is the sub-matrix of the Jacobian matrix of $g$ by which we select columns $i_1, i_2, \dots, i_n$. Now, because it is indifferent which are the $i_1, \dots, i_n$, we wish just

$$\exists 1 \leqslant i_1 < i_2 < \dots < i_n \leqslant d \ : \ \det[\partial_{(x_{i_1},x_{i_2},\dots,x_{i_n})} g] \neq 0,$$

that is *at least one of the $n \times n$ sub-determinants of the Jacobian of $g$ be $\neq 0$.* It is well known that this is equivalent to say that

$$\operatorname{rank}[\partial g] = n.$$

Definition 3.4.1. *We say that $g : \mathbb{R}^d \longrightarrow \mathbb{R}^n$ is* **submersive at** *$x$ if* $\operatorname{rank}[\partial g(x)] = n$. *If $g$ is submersive at every point of a set $S$, we say that $g$ is a* **submersion on** *$S$. In particular, if $g$ is submersion on $\mathscr{M} := \{x \in \mathbb{R}^d \ : \ g(x) = 0\}$ we say that $\mathscr{M}$ is a* **differential manifold**.

We notice that $\partial g$ is a $n \times d$ matrix (with $d > n$ in our setting), hence to say that $\operatorname{rank}[\partial g] = n$ means also that the rank is maximum. With a proof similar to that one seen above it is possible now to prove

Theorem 3.4.2 (Lagrange multipliers theorem). *Assume that $f \in \mathscr{C}^1(D; \mathbb{R})$, $D \supset \mathscr{M} := \{g = 0\}$, be a differential manifold. Then, if $\xi \in \mathscr{M}$ is a local min/max for $f$ on $\mathscr{M}$ we necessarily have*

$$\exists \lambda_1, \dots, \lambda_n \in \mathbb{R} \ : \nabla f(\xi) = \sum_{i=1}^{n} \lambda_i \nabla g_i(\xi). \tag{3.4.1}$$

*Points $\xi$ where (3.4.1) holds are called* **constrained stationary points**.

The (3.4.1) says that $\nabla f(\xi)$ is linearly dependent by $\nabla g_1(\xi), \dots, \nabla g_n(\xi)$ or, equivalently,

$$\operatorname{rank}[\nabla f(\xi) \ \nabla g_1(\xi) \ \dots \nabla g_n(\xi)] = n. \tag{3.4.2}$$

Because $g$ is assumed to be submersive on $\mathscr{M}$, $\operatorname{rank}[\nabla g_1(\xi) \ \dots \nabla g_n(\xi)] = \operatorname{rank}[\partial g] = n$, Therefore, to check the (3.4.2) it is sufficient to check that *all the $(n+1) \times (n+1)$ sub-determinants of the matrix $[\nabla f \ \nabla g_1 \ \cdots \ \nabla g_n]$ vanish.*

Example 3.4.3. *Let $\mathscr{M} := \left\{(x, y, z) \in \mathbb{R}^3 \ : \ xy + z^2 = 1, \ x^2 + y^2 = 1\right\}$. i) Show that $\mathscr{M}$ is a non empty differential manifold. ii) Say if $\mathscr{M}$ is compact or less. iii) Find points of $\mathscr{M}$ at minimum/maximum distance to the origin.*

Sol. — **i)** Let's check that $\mathscr{M} \neq \emptyset$. To this aim let's look to points of type $(x, x, z) \in \mathscr{M}$. Imposing this we get $2x^2 = 1$, that is $x = \pm\frac{1}{\sqrt{2}}$. By the first, then, $x^2 + z^2 = 1$, that is $z^2 = 1 - x^2 = 1 - \frac{1}{2} = \frac{1}{2}$, i.e. $z = \pm\frac{1}{\sqrt{2}}$. Therefore $(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}) \in \mathscr{M}$ (all combinations of sign provided sign of the first two coordinates are equal).

Let $g:\mathbb{R}^3\longrightarrow\mathbb{R}^2$, $g(x,y,z):=(xy+z^2-1,x^2+y^2-1)$. Clearly $g\in\mathscr{C}^1$ and $\mathscr{M}=Z(g)$. Let's find points where $g$ **is not** submersive. This means

$$\operatorname{rank} g'(x,y,z)<2,\iff \operatorname{rank}\begin{bmatrix} y & x & 2z \\ 2x & 2y & 0\end{bmatrix}<2,\iff \begin{cases} 2(y^2-x^2)=0,\\ -4xz=0,\\ -4yz=0.\end{cases}$$

Now, this produces the two cases

$$\begin{cases} x=0,\\ y^2=0,\iff y=0,\\ z\in\mathbb{R}\end{cases}\quad\text{or}\quad \begin{cases} z=0,\\ x^2-y^2=0,\iff y=x,\ \vee\ y=-x.\end{cases}$$

Therefore, $g$ is not submersive at points $(0,0,z)$, $z\in\mathbb{R}$ and $(x,x,0),(x,-x,0)$, $x\in\mathbb{R}$. Which of them belongs to $\mathscr{M}$? Clearly $(0,0,z)\notin\mathscr{M}$ for any $z\in\mathbb{R}$; moreover,

$$(x,x,0)\in\mathscr{M},\iff \begin{cases} x^2=1,\\ 2x^2=1,\end{cases}\text{(impossible)},\ (x,-x,0)\in\mathscr{M},\iff \begin{cases} -x^2=1,\\ 2x^2=1,\end{cases}\text{(impossible)}.$$

Therefore $g$ is submersive on $\mathscr{M}$ and consequently $\mathscr{M}$ is a differential manifold of dimension 1 in $\mathbb{R}^3$.

**ii)** Because $\mathscr{M}=Z(g)$ and $g\in\mathscr{C}$ it follows that $\mathscr{M}$ is closed. It is also bounded because, by second constraint, $x^2+y^2=1$ we deduce $|x|,|y|\leqslant 1$, and by the first

$$z^2=1-xy\leqslant 2,\implies |z|\leqslant\sqrt{2}.$$

**iii)** We should minimize/maximize the function $f(x,y,z)=\sqrt{x^2+y^2+z^2}$. Because this is min/max exactly when the same happens for $f(x,y,z)=x^2+y^2+z^2$, we use this last function to find extrema. In the previous point we have seen that $\mathscr{M}$ is compact, hence min/max exist by Weierstrass Thm. Because $f\in\mathscr{C}^1(\mathbb{R}^3)$ we have that extrema points are stationary points. To find them we use the Lagrange Thm (that can be used by point i)): we have

$$(x,y,z)\in\mathscr{M}\text{ stationary point}\iff \operatorname{rank}\begin{bmatrix} F'(x,y,z)\\ \nabla f(x,y,z)\end{bmatrix}=2,\iff \det\begin{bmatrix} y & x & 2z\\ 2x & 2y & 0\\ 2x & 2y & 2z\end{bmatrix}=0.$$

Computing the determinant by third column,

$$2z(2y^2-2x^2)=0,\iff z(y-x)(y+x)=0.$$

Candidates are therefore the points $(x,y,0)$, $x,y\in\mathbb{R}$, $(x,x,z),(x,-x,z)$, with $x,z\in\mathbb{R}$. Now

$$(x,y,0)\in\mathscr{M},\iff \begin{cases} x^2=1,\\ x^2+y^2=1,\end{cases}\iff (x,y,0)=(\pm1,0,0).$$

Similarly

$$(x,x,z)\in\mathscr{M},\iff \begin{cases} x^2+z^2=1,\\ 2x^2=1,\end{cases}\iff \left(\pm\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}}\right);$$

$$(x,-x,z)\in\mathscr{M},\iff \begin{cases} -x^2+z^2=1,\\ 2x^2=1,\end{cases}\iff \left(\pm\frac{1}{\sqrt{2}},\mp\frac{1}{\sqrt{2}},\pm\frac{\sqrt{3}}{\sqrt{2}}\right);$$

It is easy to conclude: $(\pm1,0,0)$ are the points at min distance, $\left(\pm\frac{1}{\sqrt{2}},\mp\frac{1}{\sqrt{2}},\pm\frac{\sqrt{3}}{\sqrt{2}}\right)$ are at max distance. ∎

Example 3.4.4. *Let* $\mathscr{M} := \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^4 = 1,\ x^2-yz=0\}$. *i) Prove that* $\mathscr{M}$ *is a differential manifold. ii) Prove that* $\mathscr{M}$ *is compact. iii) Find points of* $\mathscr{M}$ *with maximum quote.*

Sol. — **i)** $\mathscr{M}$ is defined by constraints $g_1(x,y,z) := x^2+y^2+z^4-1$, $g_2(x,y,z) := x^2-yz$, clearly $\mathscr{C}^1(\mathbb{R}^3)$. Setting $g(x,y,z) := (g_1(x,y,z), g_2(x,y,z))$, we have to check that $g$ is submersive on $\mathscr{M}$. Now

$$g \text{ is not submersive on } (x,y,z) \iff \operatorname{rank} g'(x,y,z) = \operatorname{rank}\begin{bmatrix} 2x & 2y & 4z^3 \\ \\ 2x & -z & -y \end{bmatrix} < 2,$$

and this happens iff all the $2\times 2$ sub-determinants of $g'(x,y,z)$ vanish. This means

$$\begin{cases} -2xz-4xy=0, \\ -2xy-8xz^3=0, \\ -2y^2+4z^4=0 \end{cases} \iff \begin{cases} x(z+2y)=0, \\ x(y+4z^3)=0, \\ y^2=2z^4, \end{cases}$$

which produces the alternatives

$$(1)\begin{cases} x=0, \\ y^2=2z^4, \end{cases} \implies (x,y,z)=(0,\pm\sqrt{2}z^2,z)\in\mathscr{M} \iff \begin{cases} 2z^2+z^4=1, \\ \\ \pm\sqrt{2}z^3=0, \end{cases} \implies \text{impossible}$$

$$(2)\begin{cases} z=-2y, \\ y=-4z^3, \\ y^2=2z^4. \end{cases} \iff \begin{cases} z=-2y, \\ y=8y^3, \\ y^2=32y^4. \end{cases} \implies \begin{matrix} (2a)\begin{cases} y=0, \\ z=0, \end{cases} \implies (x,0,0)\notin\mathscr{M} \\ \\ (2b)\begin{cases} z=-2y, \\ 1=8y^2, \\ 1=32y^2, \end{cases} \implies \text{impossible.} \end{matrix}$$

All this means that there aren't points on $\mathscr{M}$ on which $g$ is not submersive.
**ii)** Notice that

$$\mathscr{M} \subset \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^4=1\},$$

(which isn't the unit ball!), and because

$$x^2+y^2+z^4=1, \implies x^2\leqslant 1,\ y^2\leqslant 1, z^4\leqslant 1, \implies |x|\leqslant 1,\ |y|\leqslant 1,\ |z|\leqslant 1.$$

Therefore $\mathscr{M}$ is bounded. Moreover $\mathscr{M}$ is closed because is the zero set of $g$ continuous, and by this follows that $\mathscr{M}$ is compact.
**iii)** We have to maximize the function $f(x,y,z) := z$ on $\mathscr{M}$. Because $f$ is continuous and $\mathscr{M}$ compact, the existence of global max is assured by Weierstrass Thm. Moreover $f\in\mathscr{C}^1$ and $\mathscr{M}$ is a differential manifold, therefore maximum points have to be stationary for $f$ on $\mathscr{M}$. By Lagrange multipliers Thm, these are necessarily such that

$$(x,y,z)\in\mathscr{M},\ \text{s.t.}\ \operatorname{rank}\begin{bmatrix} \nabla f(x,y,z) \\ \nabla g_1(x,y,z) \\ \nabla g_2(x,y,z) \end{bmatrix} = 2,$$

and this happens iff

$$0=\det\begin{bmatrix} \nabla f(x,y,z) \\ \nabla g_1(x,y,z) \\ \nabla g_2(x,y,z) \end{bmatrix} = \det\begin{bmatrix} 0 & 0 & 1 \\ 2x & 2y & 4z^3 \\ 2x & -z & -y \end{bmatrix} = -2xz-4yz = -2z(x+2y).$$

This produces the following alternatives:

$$z=0, \iff (x,y,z)=(x,y,0), \ \text{ or }\ x+2y=0, \iff x=-2y, \iff (x,y,z)=(-2y,y,z).$$

We have to check now which of these points belong to $\mathscr{M}$:

$$(x,y,0)\in\mathscr{M},\iff\begin{cases} x^2+y^2=1,\\ x^2=0,\end{cases}\iff (x,y,z)=(0,\pm1,0).$$

In the second case

$$(-2y,y,z)\in\mathscr{M}\iff\begin{cases}4y^2+y^2+z^4=1,\\ 4y^2-yz=0,\end{cases}\iff\begin{cases}5y^2+z^4=1,\\ \\ y(4y-z)=0.\end{cases}$$

By this we have the alternatives $y=0$ (hence $z^4=1$ and we find points $(0,0,\pm1)$) or $z=4y$ which produces

$$\begin{cases}5y^2+4^4y^4=1,\\ z=4y.\end{cases}$$

Solving the first equation we get

$$y^2=\frac{-5\pm\sqrt{25+4^5}}{2\cdot4^4},\iff y=\sqrt{\frac{-5+\sqrt{25+4^5}}{2\cdot4^4}}=:\hat{y}.$$

Therefore we have a further candidate, the point $(-2\hat{y},\hat{y},4\hat{y})$. The maximum point is therefore between $(0,\pm1,0)$, $(0,0,\pm1)$ and $(-2\hat{y},\hat{y},4\hat{y})$, and is simply that one with the maximum $z$. Because $4\hat{y}<1$ we have easily that $(0,0,1)$ is the maximum. ■

Example 3.4.5. *A segment of length L is divided into n parts $x_1,\ldots,x_n$. Find the maximum of $x_1\cdots x_n$. Deduce by this the classical inequality*

$$\sqrt[n]{x_1\cdots x_n}\leqslant\frac{x_1+\ldots+x_n}{n},\ \ \forall x_1,\ldots,x_n\geqslant0.$$

Sol. — We have to find

$$\max_{x_1+\ldots+x_n=L,\ x_1,\ldots,x_n>0} x_1\cdots x_n.$$

First: let's prove that the maximum exists. Indeed, let

$$\mathscr{M}:=\{x_1+\ldots+x_n=L\ :\ x_1,\ldots,x_n>0\}.$$

Clearly $\mathscr{M}$ is an $n-1$ differential manifold defined by a unique constraint $g(x_1,\ldots,x_n)=x_1+\ldots+x_n-L$, clearly submersive on all $\mathbb{R}^n$ ($\nabla g\equiv(1,1,\ldots,1)$). In particular $\mathscr{M}$ is closed as zero set of a continuous function ($g$). Moreover is bounded. Indeed, because

$$x_1,\ldots,x_n>0,\ x_1+x_2+\ldots+x_n=L,\implies 0<x_j<L,\ \forall j=1,\ldots,n.$$

Therefore $\mathscr{M}$ is compact and because $f(x_1,\ldots,x_n)=x_1\cdots x_n$ is of course continuous we have the existence by Weierstrass Thm. Now, let's find the stationary points of $f$ on $\mathscr{M}$. We have $(x_1,\ldots,x_n)\in\mathscr{M}$ is stationary iff

$$1=\operatorname{rank}\begin{bmatrix}\nabla f(x_1,\ldots,x_n)\\ \\ \nabla g(x_1,\ldots,x_n)\end{bmatrix}=\operatorname{rank}\begin{bmatrix}x_2\cdots x_n & x_1x_3\cdots x_n & \cdots & x_1\cdots x_{n-1}\\ \\ 1 & 1 & \cdots & 1\end{bmatrix},$$

that is iff all the $2\times2$ sub determinants vanish. Choosing column $i$ and $j$ respectively we have

$$\det\begin{bmatrix}x_1\cdots x_{i-1}x_{i+1}\cdots x_n & x_1\cdots x_{j-1}x_{j+1}\cdots x_n\\ \\ 1 & 1\end{bmatrix}=x_1\cdots x_{i-1}x_{i+1}\cdots x_{j-1}x_{j+1}\cdots x_n(x_j-x_i).$$

Therefore, $(x_1,\dots,x_n)\in\mathscr{M}$ is critic for $f$ on $\mathscr{M}$ iff

$$x_1\cdots x_{i-1}x_{i+1}\cdots x_{j-1}x_{j+1}\cdots x_n(x_j-x_i)=0,\ \forall i\neq j=1,\dots,n.$$

This produces points where a coordinate is null (hence $f=0$) and, if $x_j>0$ for any $j$, $x_i-x_j=0$ for all $i,j$, and this means that $(x_1,\dots,x_n)=(\alpha,\alpha,\dots,\alpha)$. Imposing that this belongs to $\mathscr{M}$ we find the point $(\frac{L}{n},\dots,\frac{L}{n})$ where $f>0$: therefore this is the maximum! The moral is

$$\max_{x_1+\ldots+x_n=L,\ x_1,\ldots,x_n>0} x_1\cdots x_n=\left(\frac{L}{n}\right)^n.$$

In particular, recalling that $x_1+\ldots+x_n=L$, this can be rewritten as

$$x_1\cdots x_n\leqslant\left(\frac{x_1+\ldots+x_n}{n}\right)^n,\iff\sqrt[n]{x_1\cdots x_n}\leqslant\frac{x_1+\ldots+x_n}{n},$$

that is just the classical inequality between arithmetic and geometric means. ■

Example 3.4.6. *Between all the convex polygons inscribed into a circumference, find those of maximum perimeter.*

Sol. — Let $r>0$ be the radius of the circumference, $\theta_1,\dots,\theta_n$ the subsequent angles formed by the vertexes of the polygon. Then

$$\text{perimeter}\ =\ P(\theta_1,\dots,\theta_n)=\sum_{j=1}^n 2r\sin\frac{\theta_j}{2}.$$

Of course $0<\theta_j<2\pi$ and $\theta_1+\cdots+\theta_n=2\pi$. So, we have to find

$$\max_{\theta_1+\ldots+\theta_n=2\pi,\ 0<\theta_j<2\pi,\ j=1,\ldots,n}\ \sum_{j=1}^n 2r\sin\frac{\theta_j}{2}.$$

Let

$$\mathscr{M}:=\{(\theta_1,\dots,\theta_n)\in]0,2\pi[^n:\ \theta_1+\cdots+\theta_n=2\pi\}.$$

Clearly $\mathscr{M}$ is and $n-1$ dimensional differential manifold in $\mathbb{R}^n$. An argument similar to that one of the previous example, shows that the maximum exists. Let's find stationary points of $P$ on $\mathscr{M}$. These fulfills

$$\text{rank}\begin{bmatrix} r\cos\frac{\theta_1}{2} & \cdots & r\cos\frac{\theta_n}{2}\\ 1 & \cdots & 1\end{bmatrix}=1,\iff r\cos\frac{\theta_i}{2}=r\cos\frac{\theta_j}{2},\ \forall i,j,\iff\theta_i=\theta_j,\ \forall i,j.$$

Therefore, the polygon with maximum perimeter has $\theta_1=\theta_2=\ldots=\theta_n=\frac{2\pi}{n}$, so it is a regular polygon. ■

## 3.5. Exercises

Exercise 3.5.1. Show that the equation

$$y\log x-x\cos y=0,$$

is a graph of a certain function in a neighborhood of $(1,\frac{\pi}{2})$.

Exercise 3.5.2. Let $E:=\left\{(x,y)\in\mathbb{R}^2\ :\ xe^y+y=1\right\}$. Find the set of the points of $E$ where, in a suitable neighborhood, it is possible to explicit $y$ as function of $x$.

Exercise 3.5.3. Let $\alpha \in \mathbb{R}$ and

$$E_\alpha := \left\{(x,y) \in \mathbb{R}^2 \ : \ x^4 + y^4 + 2y^2 - 2x^2 = \alpha\right\}.$$

Find, in function of $\alpha$ the eventual points where, in a neighborhood, it is not possible to express $E_\alpha$ as a graph $y = \varphi(x)$ or $x = \psi(y)$.

Exercise 3.5.4. Consider the system

$$\begin{cases} x + \log y + z = 2, \\ \\ 2x - y^2 + z = 1. \end{cases}$$

Check that $(0,1,2)$ is a solution and show that in a neighborhood of this point the couple $(y,z)$ may be expressed as a local function of $x$.

Exercise 3.5.5. Consider the system in the unknown $(x,y,u,v)$

$$\begin{cases} (x+y)^2 + u^2 y + 2v - 6 = 0, \\ \\ (x-y)^2 + u^2 v - 2xy - 3 = 0. \end{cases}$$

Noticed that $(x,y,u,v) = (0,1,1,2)$ is an its solution, show that in a neighborhood of $(0,1,1,2)$ it is possible to express $(u,v)$ as functions of $(x,y)$, that is $u = u(x,y)$ and $v = v(x,y)$. Use the system to compute $\nabla u$ and $\nabla v$ at $(0,1)$.

Exercise 3.5.6. Determine min/max of $f$ on the set $D$ in the following cases:

i $f = x + y$, $D = \{(x,y) \ : \ x^2 + y^2 = 1\}$;
ii) $f = 2x^2 + y^2 - x$, $D = \{(x,y) \ : \ x^2 + y^2 = 1\}$;
iii) $f = xy$, $D = \{(x,y) \ : \ x^2 + y^2 + xy - 1 = 0\}$;
iv) $f = x^2 + 5y^2 - \frac{1}{2}xy$, $D = \{(x,y) \ : \ x^2 + 4y^2 = 4\}$;
v) $f = x - 2y + 2z$, $D = \{(x,y,z) \ : \ x^2 + y^2 + z^2 = 9\}$;
vi) $f = z^2 e^{xy}$, $D = \{(x,y,z) \ : \ x^2 + y^2 + z^2 = 1\}$.

Exercise 3.5.7. Let $\mathscr{M} := \{(x,y,z) \in \mathbb{R}^3 \ : \ z^2 = x^2 + y^2 + 1,\ z = 2x^2 + y^2\}$. Show that i) $\mathscr{M} \neq \emptyset$ is a differential manifold of dimension . . . ii) $\mathscr{M}$ is compact. iii) $\mathscr{M}$ has points of maximum quote: find them.

Exercise 3.5.8. Let $\mathscr{M} := \{(x,y,z) \in \mathbb{R}^3 \ : \ z^2 = xy + 1\}$. Show that i) $\mathscr{M} \neq \emptyset$ is a differential manifold of dimension . . . ii) $\mathscr{M}$ is not compact. iii) Show that there exists points of $\mathscr{M}$ at minimum distance to the origin and find them.

Exercise 3.5.9. Let $\mathscr{M} := \{(x,y,z) \in \mathbb{R}^3 \ : \ (x^2 + y^2 + z^2)^2 - xyz = 1\}$. i) Show that $\mathscr{M} \neq \emptyset$ is a differential manifold of dimension . . . ii) Say if $\mathscr{M}$ is compact or not. iii) Determine, if they exists, points on $\mathscr{M}$ at maximum distance to the origin.

Exercise 3.5.10. Let $\mathscr{M} := \left\{(x,y,z) \in \mathbb{R}^3 \ : \ x^2 + y^2 - z^2 = 0,\ x^2 - y^2 = 1\right\}$. i) Show that $\mathscr{M} \neq \emptyset$ is a differential manifold of dimension . . . ii) Say if $\mathscr{M}$ is compact or less. iii) Noticed that $0_3$ is not on $\mathscr{M}$, show that exists points of $\mathscr{M}$ at minimum distance from $0_3$ and find them.

Exercise 3.5.11. Let $\mathscr{M} := \left\{(x,y,z) \in \mathbb{R}^3 \ : \ x^2 - xy + y^2 - z^2 = 1,\ x^2 + y^2 = 1\right\}$. i) Show that $\mathscr{M} \neq \emptyset$ is a differential manifold of dimension . . . ii) Show that $\mathscr{M}$ is compact. iii) Find stationary points of $f(x,y,z) = xyz$ on $\mathscr{M}$. What can you say about the problem to find extrema of $f$ on $\mathscr{M}$?

Exercise 3.5.12. Let $\mathscr{M} := \{(x,y,z) \in \mathbb{R}^3 \ : \ x^2 + y^2 - z^2 = 1\}$. i) Show that $\mathscr{M} \neq \emptyset$ is a differential manifold of dimension . . . ii) Is $\mathscr{M}$ compact? iii) Find points of $\mathscr{M}$ at minimum distance from the origin $0_3$.

Exercise 3.5.13. Find the stationary points of $f(x, y, z) := xyz$, $(x, y, z) \in \mathbb{R}^3$ on the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ (here $a, b, c > 0$). Deduce min/max of $f$ on the ellipsoid.

Exercise 3.5.14. Compute the eventual min/max of $f(x, y, z) = xy + yz + zx$ on the plane $x + y + z = 3$.

Exercise 3.5.15. Compute the min/max distance of the point $(0, 1, 0)$ to the following subset of $\mathbb{R}^3$:

$$\begin{cases} x^2 + y^2 + z^2 = 1, \\ \\ x^2 + y^2 = x. \end{cases}$$

Exercise 3.5.16. Consider the set $\mathscr{M} := \left\{(x, y, z) \in \mathbb{R}^3 \ : \ z = x^2 + y^2, \ x + y + z = 0\right\}$. Show that $\mathscr{M}$ is not empty and is a . . . Find points of $\mathscr{M}$ with min/max quote.

Exercise 3.5.17. Let $\mathscr{M} := \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 + z^2 = 1, \ 2z - 3x = 0\}$ and $f(x, y, z) := xz$. i) Show that $\mathscr{M}$ is non empty and say if it is a differential manifold and what is its dimension. ii) Show that $\mathscr{M}$ is compact. iii) Find extrema of $f$ on $\mathscr{M}$.

Exercise 3.5.18. Let $\mathscr{M} := \{(x, y, z) \in \mathbb{R}^3 \ : \ 2x^2 + 2y^2 - z^2 = 1, \ (x - y)^2 + z = 2\}$. i) Show that $\mathscr{M}$ is a differential manifold of dimension. . . ii) Show that $\mathscr{M}$ is not compact. iii) Find stationary points of $f(x, y, z) := z$ on $\mathscr{M}$.

Exercise 3.5.19. Let

$$f(x, y, z) := \frac{\sqrt{x^2 + \frac{y^2}{4}} - 3}{4} + z^2, \ (x, y, z) \in \mathbb{R}^3.$$

i) Compute $\lim_{(x,y,z) \to \infty_3} f(x, y, z)$: what can you deduce by this about min/max $f$? ii) Find and classify all the stationary points of $f$ on $\mathbb{R}^3$. Find, if there exist, min/max of $f$ on $\mathbb{R}^3$. What is $f(\mathbb{R}^3)$? iii) Let $\mathscr{M} := \{(x, y, z) \in \mathbb{R}^3 \ : \ f(x, y, z) = 1\}$. Prove that $\mathscr{M}$ is a non empty differential manifold of dimension. . . Is $\mathscr{M}$ compact? iv) Show that there exists points of $\mathscr{M}$ at min/max distance to the origin. Find them.

Exercise 3.5.20. Among all the parallelepipedes of sides $x, y, z > 0$ with fixed total surface find those with maximum volume.

Exercise 3.5.21. Find

$$\max\{xy^2z^3 \ : \ x, y, z > 0, \ x + y + z = 6\}.$$

Exercise 3.5.22 ($\star$). Let $a_1, \ldots, a_d \in \mathbb{R}$ such that $a_1^2 + \ldots + a_d^2 > 0$. Find

$$\max\{x_1^2 + \ldots + x_d^2 \ : \ a_1 x_1 + \ldots + a_d x_d = 1\}.$$

CHAPTER 4

# Integration

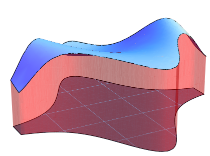
In the first Mathematical Analysis course the concept of *integral* for functions depending on one real variable has been introduced. This is a central notion in Analysis with several applications: from the calculus of areas of plane figures to a fundamental tool in Probability, Physics, Engineering. We recall that if $f = f(x) : [a,b] \longrightarrow [0,+\infty[$, the integral was defined with the following geometrical interpretation:

$$\int_{[a,b]} f(x)\,dx = \text{Area}\,(\text{Trap}(f)), \quad \text{where } \text{Trap}(f) := \{(x,y) \in \mathbb{R}^2 \;:\; x \in [a,b],\ 0 \leqslant y \leqslant f(x)\}.$$

The definition of Area passes through a formally complex *exhaustion method* based on filling the trapezoid Trap(()$f$) with rectangles.

Relevance of integration both in Mathematics and applied disciplines, pushes for the extension of this operation to functions of several variables. In this case, however, we don't have a preferred type of domains as intervals in case of functions of one real variable. For instance, if $f = f(x,y) : E \subset \mathbb{R}^2 \longrightarrow [0,+\infty[$ we might expect

$$\int_E f(x,y)\,dxdy = \text{Volume}\,(\text{Trap}(f)), \text{ where } \text{Trap}(f) := \{(x,y,z) \in \mathbb{R}^3 \;:\; (x,y) \in E,\ 0 \leqslant z \leqslant f(x,y)\}.$$



Extending this idea, if now $f : E \subset \mathbb{R}^3 \longrightarrow [0,+\infty[$, we might expect

$$\int_E f(x,y,z)\,dxdydz = \text{Hyper-Volume}\,(\text{Trap}(f)),$$

where of course

$$\text{Trap}(f) := \{(x,y,z,w) \in \mathbb{R}^4 \;:\; (x,y,z) \in E,\ 0 \leqslant w \leqslant f(x,y,z)\}.$$

Area, Volume and Hyper-Volume are different forms of what we could name *measure* of a set. As it is complicate to construct a solid concept of Area, harder is to define a concept of volume and, more in general, a concept of *$n-$dimensional measure of a subset of* $\mathbb{R}^n$. This construction being totally out of our scope, we will take for granted focussing on the properties of multidimensional integrals. Yet, these are too technical to be proved here, thus we will limit to heuristic arguments for their justification, providing however proper statements.

## 4.1. Integral

In the introduction, we mentioned the concept of *$n-$dimensional measure*. This is in fact a function $\lambda_n$ that assigns to $S \subset \mathbb{R}^n$ a number in $[0, +\infty]$ (we improperly will treat $+\infty$ as a number). In general, it is not possible to assign a measure to every set. However it is possible to do this on a large class of sets of $\mathbb{R}^n$, called *measurable sets*.

Theorem 4.1.1. *There exists a function*

$$\lambda_n \ : \ \mathscr{M}_n \longrightarrow [0, +\infty],$$

*where $\mathscr{M}_n \subset \mathscr{P}(\mathbb{R}^n)$ is called* **class ($n-$dimensional) measurable sets** *fulfilling the following properties:*

i) *Open and closed sets of $\mathbb{R}^n$ are measurable (that is are elements of $\mathscr{M}_n$), in particular $\emptyset$ and $\mathbb{R}^n$ are measurable and*

$$\lambda_n(\emptyset) = 0, \quad \lambda_n(\mathbb{R}^n) = +\infty.$$

*Moreover, if a set S differs by an open (closed) set for a measure 0 set, S is measurable. Precisely,*

$$\text{if } \exists E \subset \mathbb{R}^n \text{ open (or closed) } : \ \lambda_n\left((S \backslash E) \cup (E \backslash S)\right) = 0, \implies E \in \mathscr{M}_n.$$

ii) *$\lambda_n$ factorizes in the sense that*

$$\text{if } S = A \times B \in \mathscr{M}_n, \ A \in \mathscr{M}_k, \ B \in \mathscr{M}_{n-k}, \implies \lambda_n(S) = \lambda_k(A)\lambda_{n-k}(B).$$

*In particular, $\lambda_n$ is coherent with elementary geometry, in the sense that*

$$\lambda_n\left([a_1, b_1] \times \cdots \times [a_n, b_n]\right) = (b_1 - a_1) \cdots (b_n - a_n).$$

iii) *$\lambda_n$ is invariant by translations, rotations, reflections and, in general, the following holds true:*

$$\lambda_n(L(S) + v) = |\det L| \lambda_n(S), \ \forall L \in M_{n \times n} \text{ invertible}, \ \forall v \in \mathbb{R}^n. \tag{4.1.1}$$

*(notice that $L = \mathbb{I}_n$ is translation invariance; $L =$orthogonal matrix $LL^t = \mathbb{I}_n$ and $v = 0$ is rotation invariance).*

iv) *$\lambda_n$ is countably additive, that is if $(S_j)_{j \in \mathbb{N}} \subset \mathscr{M}_n$ are disjoint, that is $S_i \cap S_j = \emptyset$ if $i \neq j$, then*

$$\lambda_n\left(\bigcup_{j=1}^{\infty} S_j\right) = \sum_{j=1}^{\infty} \lambda_n(S_j).$$

Some remarks may be useful to understand previous statement. i) says that large classes of common sets (like open or closed sets) are measurable. This is a good news, because most of sets we consider are open or closed. Just think to the case of sets like

$$O = \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ g_j(x_1, \ldots, x_n) > 0, \ j = 1, \ldots, k\right\} \ (\text{open if } g_j \in \mathscr{C}(\mathbb{R}^n), \ j = 1, \ldots, k),$$

$$C = \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ g_j(x_1, \ldots, x_n) \geqslant 0, \ j = 1, \ldots, k\right\} \ (\text{closed if } g_j \in \mathscr{C}(\mathbb{R}^n), \ j = 1, \ldots, k)$$

Furthermore, if our set $S$ differs by an open/closed set by a measure 0 set, then $S$ is measurable. Measure zero (or null) sets are particularly important. How can we imagine these sets? Here some examples:

- singletons $S = \{x^*\}$ are measure 0 sets: indeed, if $x^* = (x_1^*, \dots, x_n^*)$ we may see
$$\{x^*\} = [x_1^*, x_1^*] \times \cdots \times [x_n^*, x_n^*], \implies \lambda_n(\{x^*\}) = (x_1^* - x_1^*) \cdots (x_n^* - x_n^*) = 0.$$
- countable sets $S = \{x_j^* \ : \ j \in \mathbb{N}\}$ (like naturals or rationals in reals) are null sets: indeed, by countable additivity
$$\lambda_n(S) = \sum_j \lambda_n(\{x_j^*\}) = \sum_j 0 = 0.$$
- "lower dimensional" sets are in general measure zero sets: imagine a segment in the plane, say for simplicity $S = \{(x, c) \ : \ x \in [a, b]\}$. In $\mathbb{R}^2$ this is a null set: indeed
$$\lambda_2(S) = \lambda_2([a, b] \times [c, c]) = (b - a)(c - c) = 0.$$
To give a more precise and general statement we have
$$\text{if } g \in \mathscr{C}(\mathbb{R}^n) \text{ is non constant, } \implies \{(x_1, \dots, x_n) \ : \ g(x_1, \dots, x_n) = 0\} \text{ is a null set.}$$

Let $f : E \subset \mathbb{R}^n \longrightarrow [0, +\infty[$. We call *trapezoid* delimited by $f$ the set
$$\text{Trap}(f) := \left\{(x, y) \in \mathbb{R}^{n+1} \ : \ x \in E, \ 0 \leqslant y \leqslant f(x)\right\}.$$

Definition 4.1.2. *Let $f : E \longrightarrow [0, +\infty[$. If* $\text{Trap}(f) \in \mathscr{M}_{n+1}$ *we pose*
$$\int_E f := \lambda_{n+1}(\text{Trap}(f)). \tag{4.1.2}$$

First question: under which conditions is $\text{Trap}(f)$ is measurable? Here some important cases:

Proposition 4.1.3. *Let $f : E \longrightarrow [0, +\infty[$ be continuous on $E$ closed or open. Then* $\text{Trap}(f)$ *is measurable. Thus, in particular,*
$$\int_E f$$
*is well defined.*

Proof — For $E$ closed it is easy to prove that $\text{Trap}(f)$ is closed too. For $E$ open, $\text{Trap}(f)$ is not open in general. However, one can prove that
$$S := \{(x, y) \in \mathbb{R}^{n+1} \ : \ x \in E, \ 0 < y < f(x)\} \text{ is open,}$$
thus $S \in \mathscr{M}_n$. Clearly $S \subset \text{Trap}(f)$ (thus $S \backslash \text{Trap}(f) = \emptyset$) while
$$\text{Trap}(f) \backslash S = \left\{(x, y) \in \mathbb{R}^{n+1} \ : \ y = 0 \ \vee \ y = f(x)\right\}.$$
Now, because
$$\{(x, y) \in \mathbb{R}^{n+1} \ : \ y = 0\}, \ \{(x, y) \in \mathbb{R}^{n+1} \ : \ y = f(x)\} = \{(x, y) \in \mathbb{R}^{n+1} \ : \ y - f(x) = 0\}$$
are both defined by equations involving continuous functions ($g(x, y) = y$ in the first case, $g(x, y) = y - f(x)$ in the second), they have $\lambda_{n+1} = 0$. We conclude that $\lambda_{n+1}(\text{Trap}(f) \backslash S) = 0$ because $\text{Trap}(f) \backslash S$ is made of two measure zero sets and $\lambda_{n+1}$ is additive. The conclusion is that $\text{Trap}(f)$ differs by $S$ open for a measure zero set, thus it is measurable. ■

Of course, we might need to consider discontinuous functions, but for purposes of this course $f \in \mathscr{C}(E)$ on $E$ open or closed is more than sufficient. Throughout this Chapter we will develop a number of methods to compute integrals. These can be used to compute measures:

Proposition 4.1.4. *If $E$ is open or closed,*

$$\lambda_n(E) = \int_E 1. \tag{4.1.3}$$

Proof — Just notice that if $f \equiv 1$,

$$\mathrm{Trap}(f) = \left\{(x, y) \in \mathbb{R}^{n+1} \ : \ x \in E,\ 0 \leqslant y \leqslant 1\right\} \equiv E \times [0, 1].$$

Thus,

$$\int_E 1\, dx = \lambda_{n+1}\left(E \times [0, 1]\right) \overset{factor.}{=} \lambda_n(E)\lambda_1([0, 1]) = \lambda_n(E). \quad \blacksquare$$

So far, we defined the integral of a positive function. Let's now consider $f : E \longrightarrow \mathbb{R}$ and define

$$f_+(x) := \begin{cases} f(x), & \text{if } f(x) \geqslant 0, \\ 0, & \text{if } f(x) < 0, \end{cases} \qquad f_-(x) := \begin{cases} -f(x), & \text{if } f(x) \leqslant 0, \\ 0, & \text{if } f(x) > 0. \end{cases}$$

Functions $f_\pm$ are called, respectively, **positive part** and **negative part** of $f$. Easily it turns out that

$$f \in \mathscr{C}(E), \implies f_\pm \in \mathscr{C}(E).$$

Moreover, both $f_\pm$ are positive and, finally,

$$f = f_+ - f_-, \ \ |f| = f_+ + f_-.$$

Notice that, if $f \in \mathscr{C}(E)$, $E$ open/closed,

$$\int_E |f| < +\infty, \iff \int_E f_+ < +\infty,\ \int_E f_- < +\infty.$$

This justifies the following

Definition 4.1.5. *Let $f \in \mathscr{C}(E)$, $E$ open or closed in $\mathbb{R}^n$. We say that $f$ is* **integrable** *if*

$$\int_E |f| < +\infty.$$

*We pose*

$$\int_E f := \int_E f_+ - \int_E f_-.$$

*and we denote the set of integrable functions with $L^1(E)$.*

The properties of the integral are very similar to those of one dimensional integral:

Proposition 4.1.6. *The following properties holds:*

i) *(linearity) if $f, g \in L^1(E)$ then $\alpha f + \beta g \in L^1(E)$ for any $\alpha, \beta \in \mathbb{R}$ and*

$$\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g;$$

ii) *(isotonicity) if $f \leqslant g$ on $E$ with $f, g \in L^1(E)$ then $\int_E f \leqslant \int_E g$;*

iii) *(triangular inequality) if $f \in L^1(E)$ then $\left|\int_E f\right| \leqslant \int_E |f|$;*

iv) *(decomposition) if* $f \in L^1(E)$ *and* $E = A \cup B$ *with* $A, B \in \mathscr{M}_n$, $\int_E f = \int_A f + \int_B f$.

It remains now to develop an efficient method of calculus for integrals. This is based on two fundamental tools: *reduction formula* and *change of variables formula*.

## 4.2. Reduction formula

In this Section we introduce the technique based on the *reduction formula* that allows to reduce the calculus of a multiple variables integral to iterated one variable integrals. For pedagogical reasons we present first the case of double integrals, then we will extend to the general case.

**4.2.1. Double Integrals.** To understand the idea, let's consider the problem to compute

$$\int_E f(x,y)\, dxdy.$$

As we know, integrals are continuous versions of discrete sums. Thinking to these we could write

$$\sum_{(x,y)\in E} f(x,y) = \sum_{x\in\mathbb{R}} \left( \sum_{y\in\mathbb{R}\,:\,(x,y)\in E} f(x,y) \right) = \sum_{y\in\mathbb{R}} \left( \sum_{x\in\mathbb{R}\,:\,(x,y)\in E} f(x,y) \right).$$

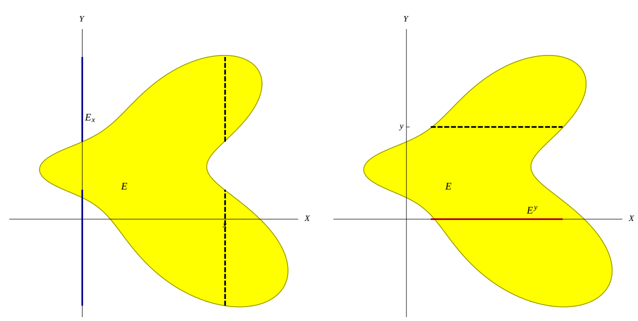
If the sums were finite, there wouldn't be any problem in reordering terms and summing as we prefer, so the previous formula would be a consequence of associativity and commutativity. However, when sum contains infinitely many terms the story is much more complicate. Leaving aside for a moment this problem, we might expect that the analogous for integrals of the previous formula is

$$\int_E f(x,y)\, dxdy = \int_{\mathbb{R}} \left( \int_{E^y} f(x,y)\, dy \right) dx = \int_{\mathbb{R}} \left( \int_{E_x} f(x,y)\, dx \right) dy.$$

The two sets $E_x$ and $E^y$ are defined as

$$E_x := \{ y \in \mathbb{R} \,:\, (x,y) \in E \},\ (x - \textbf{section}), \quad E^y := \{ x \in \mathbb{R} \,:\, (x,y) \in E \},\ (y - \textbf{section}).$$

Notice that, fixed $x \in \mathbb{R}$, $E_x$ is the set of ordinates $y$ of points of $E$ with abscissa $x$, that is $y$ such that $(x,y) \in E$. In other words, $E_x$ is the projection on the $y-$axis of the "slice" of $E$ along the vertical straight line at abscissa $x$. It turns out that if $f \in L^1$ the reduction formula holds true.



Proposition 4.2.1. *Let $f \in L^1(E)$, then*

$$\int_E f(x,y)\,dxdy = \int_{\mathbb{R}}\left(\int_{E_x} f(x,y)\,dy\right)\,dx = \int_{\mathbb{R}}\left(\int_{E^y} f(x,y)\,dx\right)\,dy. \tag{4.2.1}$$

Remark 4.2.2. *Notice that $E_x$ (and $E^y$) may be empty for certain values of $x$ (resp $y$). For such $x$ ($y$), clearly $\int_{E_x} f = 0$ ($\int_{E^y} f = 0$). Therefore*

$$\int_{\mathbb{R}}\left(\int_{E_x} f(x,y)\,dy\right)\,dx = \int_{x\in\mathbb{R}\,:\,E_x\neq\emptyset}\int_{E_x} f(x,y)\,dy\,dx.$$

*However, for future use we prefer to keep a lighter notation as in (4.2.1).* ■

The (4.2.1) requires $f \in L^1(E)$, that is

$$\int_E |f(x,y)|\,dxdy < +\infty.$$

To check this, in principle one should compute a double integral. Notice that if we know $f \in L^1(E)$ then, by the reduction formula,

$$\int_E |f(x,y)|\,dxdy = \int_{\mathbb{R}}\left(\int_{E_x} |f(x,y)|\,dy\right)\,dx = \int_{\mathbb{R}}\left(\int_{E^y} |f(x,y)|\,dx\right)\,dy,$$

so in particular

$$\int_{\mathbb{R}}\left(\int_{E_x} |f(x,y)|\,dy\right)\,dx,\ \int_{\mathbb{R}}\left(\int_{E^y} |f(x,y)|\,dx\right)\,dy \ < +\infty.$$

It turns out that also the vice versa holds true:

Proposition 4.2.3. *Let $f \in \mathscr{C}(E)$, $E \subset \mathbb{R}^2$ open/closed set. If one of the following iterated integrals*

$$\int_{\mathbb{R}}\left(\int_{E_x} |f(x,y)|\,dy\right)\,dx,\ \int_{\mathbb{R}}\left(\int_{E^y} |f(x,y)|\,dx\right)\,dy \tag{4.2.2}$$

*is finite, then $f \in L^1(E)$ and reduction formula (4.2.1) holds.*

Combining the previous Propositions we have an algorithm to check if $f \in L^1(E)$ and to compute its integral by using the reduction formula: *first, one check if one of the (4.2.2) is finite (which one of the two is indifferent and the choice could be done in terms of computational ease); second, one uses (4.2.1) to compute the integral*. Notice that, in particular, if $f \geqslant 0$ the check (4.2.2) leads at same time to the calculation of the integral by (4.2.1).

Example 4.2.4. *Discuss if $f(x,y) := x^3 e^{-yx^2} \in L^1([0,+\infty[\times[1,2])$ and compute its integral.*

Sol. — Clearly $f \in \mathscr{C}(E)$ where $E = [0,+\infty[\times[1,2]$ is closed. Applying (4.2.2), notice that if $E = [0,+\infty[\times[1,2]$, $E_x = \emptyset$ if $x < 0$, $E_x = [1,2]$ if $x \geqslant 0$, therefore

$$\begin{aligned}\int_{\mathbb{R}}\int_{E_x} |f|\,dy\,dx &= \int_0^{+\infty}\left(\int_1^2 x^3 e^{-yx^2}\,dy\right)\,dx = \int_0^{+\infty} x\left[-e^{-yx^2}\right]_{y=1}^{y=2}\,dx \\ &= \int_0^{+\infty} xe^{-x^2} - xe^{-2x^2}\,dx = \left[\frac{-e^{-x^2}}{2}\right]_{x=0}^{x=+\infty} - \left[\frac{-e^{-2x^2}}{4}\right]_{x=0}^{x=+\infty} = \frac{1}{4}.\end{aligned}$$

We deduce $f \in L^1$ and because $f \geqslant 0$, thus $|f| = f$, the same calculation and (4.2.1) gives $\int_{[0,+\infty[\times[1,2]} f = \frac{1}{4}$. ■

Example 4.2.5. *Discuss if $f(x,y) := e^{-x} \in L^1(E)$ where $E = \{(x,y) \in \mathbb{R}^2 \ : \ x \geqslant 0,\ 0 \leqslant y \leqslant x^2\}$. In such case compute the integral of $f$ on $E$.*

Sol. — Clearly $f \in \mathscr{C}(E)$ where $E$ is closed (defined by large inequalities on continuous functions). Applying (4.2.2), notice that $E_x = \emptyset$ if $x < 0$, $E_x = [0, x^2]$ if $x \geqslant 0$, therefore

$$\begin{aligned}\int_{\mathbb{R}} \int_{E_x} |f| \, dy \, dx &= \int_0^{+\infty} \left( \int_0^{x^2} e^{-x} \, dy \right) dx = \int_0^{+\infty} x^2 e^{-x} \, dx = \int_0^{+\infty} x^2 (-e^{-x})' \, dx \\ &= \left[ -x^2 e^{-x} \right]_{x=0}^{x=+\infty} + \int_0^{+\infty} 2x e^{-x} \, dx = 2 \int_0^{+\infty} x(-e^{-x})' \, dx \\ &= 2 \left[ [-xe^{-x}]_{x=0}^{x=+\infty} + \int_0^{+\infty} e^{-x} \, dx \right] = 2 \, [-e^{-x}]_{x=0}^{x=+\infty} = 2.\end{aligned}$$

Therefore $f \in L^1(E)$ and because $f \geqslant 0$ the same calculation and (4.2.1) gives $\int_E f = 2$. ■

A remarkable example of (4.2.1) is obtained by taking $f \equiv 1$. Recalling that $\int_E 1 = \lambda_2(E)$ we obtain

(4.2.3) $$\lambda_2(E) = \int_{\mathbb{R}} \left( \int_{E_x} 1 \, dy \right) dx = \int_{\mathbb{R}} \lambda_1(E_x) \, dx = \int_{\mathbb{R}} \lambda_1(E^y) \, dy.$$

Example 4.2.6. *Compute the area of a disk of radius $r$.*

Sol. — Because the measure is translations–invariant, we can center the disk into the origin. So, let's consider

$$E = \left\{ (x,y) \in \mathbb{R}^2 \ : \ x^2 + y^2 \leqslant r^2 \right\}.$$

This set $E$ is closed, hence measurable. Applying the integration by slices we have

$$\lambda_2(E) = \int_{\mathbb{R}} \lambda_1(E_x) \, dx.$$

Let's determine an $x-$section. We have

$$y \in E_x, \iff (x,y) \in E, \iff x^2 + y^2 \leqslant r^2, \iff y^2 \leqslant r^2 - x^2, \iff y \in \left[ -\sqrt{r^2 - x^2}, \sqrt{r^2 - x^2} \right].$$

Of course we need $r^2 - x^2 \geqslant 0$, that is $x^2 \leqslant r^2$, $|x| \leqslant r$, otherwise $E_x = \emptyset$. Therefore

$$\lambda_2(E) = \int_{\mathbb{R}} \lambda_1(E_x) \, dx = \int_{|x| \leqslant r} \lambda_1 \left( \left[ -\sqrt{r^2 - x^2}, \sqrt{r^2 - x^2} \right] \right) dx = \int_{|x| \leqslant r} 2\sqrt{r^2 - x^2} \, dx.$$

Being $x \longmapsto \sqrt{r^2 - x^2}$ continuous on $[-r, r]$, the last integral is equal to a Riemann one, so

$$\lambda_2(E) = \int_{-r}^{r} 2\sqrt{r^2 - x^2} \, dx = 4 \int_0^r \sqrt{r^2 - x^2} \, dx = 4r \int_0^r \sqrt{1 - \frac{x^2}{r^2}} \, dx.$$

Therefore, setting $\frac{x}{r} = \sin\theta$, $\theta \in [0, \frac{\pi}{2}]$,

$$\lambda_2(E) = 4r \int_0^{\pi/2} \sqrt{1 - (\sin\theta)^2} \, r\cos\theta \, d\theta = 4r^2 \int_0^{\pi/2} (\cos\theta)^2 \, d\theta.$$

Now $\int(\cos\theta)^2 = \int\cos\theta(\sin\theta)' = \cos\theta\sin\theta + \int(\sin\theta)^2 = \frac{1}{2}\sin(2\theta) + \theta - \int(\cos\theta)^2$ hence

$$\lambda_2(E) = 4r^2\left[\frac{1}{4}\sin(2\theta) + \frac{\theta}{2}\right]_{\theta=0}^{\theta=\pi/2} = \pi r^2. \quad \blacksquare$$

**Warning!** If $f \notin L^1$ the reduction formula might be false even if the iterated integral are finite.

Example 4.2.7. *Let*

$$f(x,y) = \frac{x-y}{(x+y)^3},\ (x,y) \in E := [0,1]^2.$$

*Then* $\int_{\mathbb{R}}\left(\int_{E_x} f\ dy\right)\ dx \neq \int_{\mathbb{R}}\left(\int_{E^y} f\ dx\right)\ dy$. *Hence, in particular,* $f \notin L^1([0,1]^2)$.

Sol. — Notice first that

$$E_x = \{y \in \mathbb{R}\ :\ (x,y) \in [0,1]^2\} = \begin{cases} \emptyset, & x \notin [0,1], \\ \\ [0,1] & x \in [0,1] \end{cases}$$

and similarly for $E^y$. Therefore

$$\int_{E^y} f(x,y)\ dx = \begin{cases} 0, & y \notin [0,1], \\ \\ \displaystyle\int_0^1 \frac{x-y}{(x+y)^3}\ dx = \int_0^1 \frac{1}{(x+y)^2}\ dx - 2y\int_0^1 \frac{1}{(x+y)^3}\ dx. & y \in [0,1]. \end{cases}$$

Except for $y = 0$ (therefore for a measure 0 set) both integrals are finite and their value is

$$\left[\frac{(x+y)^{-1}}{-1}\right]_{x=0}^{x=1} - 2y\left[\frac{(x+y)^{-2}}{-2}\right]_{x=0}^{x=1} = \frac{1}{y} - \frac{1}{y+1} + y\left(\frac{1}{(y+1)^2} - \frac{1}{y^2}\right) = -\frac{1}{(y+1)^2}.$$

Hence

$$\int_{\mathbb{R}}\left(\int_{E^y} f(x,y)\ dx\right)\ dy = \int_0^1\left(-\frac{1}{(y+1)^2}\right)\ dy = \left[(y+1)^{-1}\right]_{y=0}^{y=1} = \frac{1}{2} - 1 = -\frac{1}{2}.$$

Exchanging $x$ with $y$ we obtain the same result except for the sign: $\int_{\mathbb{R}}\left(\int_{E_x} f(x,y)dy\right)\ dx = \frac{1}{2}$. $\blacksquare$

**4.2.2. General Multiple Integrals.** The previous mechanism can be extended to functions $f$ of $n$ variables. Let $f = f(z_1,\dots,z_n)$ and imagine we group $(z_1,\dots,z_n)$ into two blocks, one of $k$ variables and the remaining of $n-k$ variables. For simplicity with notations we write

$$f = f(x,y), \text{ where } x = (x_1,\dots,x_k) \in \mathbb{R}^k,\ y = (x_{k+1},\dots,x_n) \in \mathbb{R}^{n-k}.$$

As above, we will denote by $E_x$ (resp. $E^y$) the $x-$section (resp. $y-$section) of $E$ defined as

$$E_x := \{y \in \mathbb{R}^n\ :\ (x,y) \in E\},\quad E^y := \{x \in \mathbb{R}^m\ :\ (x,y) \in E\}.$$

Be careful because now $E_x \subset \mathbb{R}^n$ while $E^y \subset \mathbb{R}^m$. With these notations we have the

THEOREM 4.2.8 (FUBINI–TONELLI). *Let $f \in L^1(E)$, $E \subset \mathbb{R}^{m+n}$. Then the* **reduction formula** *holds*

$$\int_E f = \int_{\mathbb{R}^m} \left( \int_{E_x} f(x,y)\, dy \right) dx = \int_{\mathbb{R}^n} \left( \int_{E^y} f(x,y)\, dx \right) dy. \tag{4.2.4}$$

*Moreover, if $f \in \mathscr{C}(E)$ and one among*

$$\int_{\mathbb{R}^m} \left( \int_{E_x} |f(x,y)|\, dy \right) dx, \quad \int_{\mathbb{R}^n} \left( \int_{E^y} |f(x,y)|\, dx \right) dy,$$

*is finite, then $f \in L^1(E)$ (and the reduction formula (4.2.4) holds). In particular, by taking $f = 1$ we have the* **slicing formula**

$$\lambda_{m+n}(E) = \int_{\mathbb{R}^m} \lambda_n(E_x)\, dx = \int_{\mathbb{R}^n} \lambda_m(E^y)\, dy. \tag{4.2.5}$$

Fubini–Tonelli theorem is a versatile tool to integrate functions of several variables. For instance: consider a function of three variables $f = f(x,y,z) \in \mathscr{C}(E)$, $E \subset \mathbb{R}^3$ open/closed. In this common case, the three variables may be grouped is six different ways, this leading to six different possible applications of reduction formula:

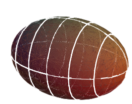
$$x \text{ and } (y,z), \quad \int_E f = \int_{\mathbb{R}} \left( \int_{(y,z)\in E_x} f\, dydz \right) dx = \int_{\mathbb{R}^2} \left( \int_{x \in E_{(y,z)}} f\, dx \right) dydz,$$

$$y \text{ and } (x,z), \quad \int_E f = \int_{\mathbb{R}} \left( \int_{(x,z)\in E_y} f\, dxdz \right) dy = \int_{\mathbb{R}^2} \left( \int_{y \in E_{(x,z)}} f\, dy \right) dxdz,$$

$$z \text{ and } (x,y), \quad \int_E f = \int_{\mathbb{R}} \left( \int_{(y,z)\in E_x} f\, dydz \right) dx = \int_{\mathbb{R}^2} \left( \int_{x \in E_{(y,z)}} f\, dx \right) dydz,$$

Which choice is the best one depends by the complexity of calculus. ■

EXAMPLE 4.2.9. *Compute the volume of a rugby ball $E = \left\{ (x,y,z) \in \mathbb{R}^3 \ : \ \frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} \leqslant 1 \right\}$, $(a,b > 0)$.*



SOL. — Clearly $E$ is closed in $\mathbb{R}^3$, hence measurable. Slicing $E$ along the $z$-axis,

$$m_3(E) = \int_{\mathbb{R}} m_2(E_z)\, dz.$$

Now,

$$(x,y,z) \in E, \iff \frac{x^2+y^2}{a^2} \leqslant 1 - \frac{z^2}{b^2}, \iff (x,y) \in B\left(0_2, \sqrt{1 - \frac{z^2}{b^2}}\right] =: E_z$$

This of course if $1-\frac{z^2}{b^2}\geqslant 0$, that is $z^2\leqslant b^2$, namely $|z|\leqslant b$, otherwise $E_z=\emptyset$. It follows that

$$\begin{aligned} m_3(E) &= \int_{|z|\leqslant b} m_2\left(B\left(0_2, a\sqrt{1-\frac{z^2}{b^2}}\right]\right)\,dz = \int_{|z|\leqslant b}\pi a^2\left(1-\frac{z^2}{b^2}\right)\,dz \overset{R=L}{=} \int_{-b}^{b}\pi a^2\left(1-\frac{z^2}{b^2}\right)\,dz \\ &= \pi a^2\left([z]_{-b}^{b}-\left[\frac{z^3}{3b^2}\right]_{-b}^{b}\right) = \pi a^2\left(2b-\frac{2}{3}b\right) = \pi\frac{4}{3}a^2 b. \end{aligned}$$

Taking $a=b=r$ we obtain the volume of a sphere of radius $r$, the well known $\frac{4}{3}\pi r^3$. ■

## 4.3. Change of variables

Let $T:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ a transformation. If $T$ is a linear bijection we know that

$$\lambda_n(T(E)) = |\det T|\lambda_n(E).$$

What happens if $T$ is a general (non linear) bijection?

Theorem 4.3.1. *Let $T:E\subset\mathbb{R}^n\longrightarrow T(E)$ be a diffeomorphism (that is $T,T^{-1}\in\mathscr{C}^1$) on $E$ open/closed set. Then*

$$\lambda_n(T(E)) = \int_E |\det T'(\xi)|\,d\xi. \tag{4.3.1}$$

Proof — (sketch): We decompose $E$ as disjoint union of neighborhoods of some of its points, let's say

$$E=\bigcup_j U_{x_j}.$$

Then, by countable additivity,

$$\lambda_n(T(E)) = \sum_j \lambda_n(T(U_{x_j})).$$

If $T$ is regular (differentiable) then $T(x)=T(x_0)+T'(x_0)(x-x_0)+o(x-x_0)\sim_{x_0} T(x_0)+T'(x_0)(x-x_0)$. The sense of $\sim_{x_0}$ is that $T(x)$ can be replaced by $T(x_0)+T'(x_0)(x-x_0)$ in a neighborhood of $x_0$ and the approximation is more precise smaller is this neighborhood. Therefore, we may expect that

$$\begin{aligned} \lambda_n(T(U_{x_j})) &\approx \lambda_n(T(x_j)+T'(x_j)(U_{x_j}-x_j)) = \lambda_n(T'(x_j)(U_{x_j}-x_j)) = |\det T'(x_j)|\lambda_n(U_{x_j}-x_j) \\ &= |\det T'(x_j)|\lambda_n(U_{x_j}). \end{aligned}$$

Hence

$$\lambda_n(T(E)) \approx \sum_j |\det T'(x_j)|\lambda_n(U_{x_j}) \approx \int_E |\det T'(\xi)|\,d\xi.$$

Of course this is not yet a proof and it needs a lot of technical work to make it a rigorous argument, but this is the main idea on which is based. ■

We may review (4.3.1) as

$$\int_{T(E)} dx = \int_E |\det T'(\xi)|\, d\xi, \quad \overset{\Phi:=T^{-1}}{\Longleftrightarrow} \quad \int_F dx = \int_{\Phi(F)} |\det(\Phi^{-1})'(\xi)|\, d\xi.$$

This last is a special case of change of variable formula:

THEOREM 4.3.2 (CHANGE OF VARIABLES FORMULA). *Let $f : F \longrightarrow \mathbb{R}$, $f \in \mathscr{C}(F)$, $F$ open/closed, and $\Phi : F \longrightarrow \Phi(F)$ be a diffeomorphism, that is $\Phi, \Phi^{-1} \in \mathscr{C}^1$. Then*

$$\boxed{\int_F f(x)\, dx \overset{y=\Phi(x),\ x=\Phi^{-1}(y)}{=} \int_{\Phi(F)} f(\Phi^{-1}(\xi))|\det(\Phi^{-1})'(\xi)|\, d\xi.} \tag{4.3.2}$$

PROOF — (sketch) As in the previous proof, divide $F$ in a disjoint union of small neighbourhoods,

$$F = \bigcup_j U_{x_j},$$

in such a way that

$$\int_F f = \sum_j \int_{U_{x_j}} f.$$

We may choose $U_{x_j}$ in such a way that $f(x) \approx f(x_j)$ for all $x \in U_{x_j}$ (by continuity). Thus

$$\int_{U_{x_j}} f \approx \int_{U_{x_j}} f(x_j) = f(x_j) \int_{U_{x_j}} = f(x_j) \int_{\Phi(U_{x_j})} |\det(\Phi^{-1})'|.$$

Now, because $\Phi$ is a diffeomorphism, we may imagine that $\Phi(U_{x_j}) = V_{\Phi(x_j)}$ is a neighbourhood of $\Phi(x_j)$. Thus, if $\xi \in V_{\Phi(x_j)}$, by continuity $f(\Phi^{-1}(\xi)) \approx f(\Phi^{-1}(\Phi(x_j))) = f(x_j)$ for all $\xi \in V_{\Phi(x_j)}$. Furthermore,

$$\bigcup_j V_{\Phi(x_j)} = \bigcup_j \Phi(U_{x_j}) = \Phi\left(\bigcup_j U_{x_j}\right) = \Phi(F).$$

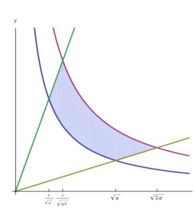
Summing up all previous remarks we would have

$$\int_F f \approx \sum_j f(x_j) \int_{\Phi(U_{x_j})} |\det(\Phi^{-1})'| = \sum_j \int_{V_{\Phi(x_j)}} f(\Phi^{-1}(\xi))|\det(\Phi^{-1})'(\xi)| = \int_{\Phi(F)} f(\Phi^{-1}(\xi))|\det(\Phi^{-1})'(\xi)|\, d\xi. \quad \blacksquare$$

EXAMPLE 4.3.3. *Compute*

$$\int_{1\leqslant xy\leqslant 2,\ 0<ax\leqslant y\leqslant \frac{x}{a}} \frac{y^4 \arctan(xy)}{(x^2+y^2)^2}\, dxdy, \quad 0 < a < 1.$$

SOL. — The domain is closed in $\mathbb{R}^2$, hence measurable and $f \in \mathscr{C}$.



Notice that

$$\frac{y^4 \arctan(xy)}{(x^2+y^2)^2} = \left(\frac{y}{x}\right)^4 \frac{\arctan(xy)}{\left(1+\left(\frac{y}{x}\right)^2\right)^2}.$$

It seems therefore natural to introduce the new variables

$$\xi = xy, \quad \eta = \frac{y}{x}, \quad (\xi,\eta) := \Psi(x,y),$$

where $\Psi : ]0,+\infty[^2 \longrightarrow ]0,+\infty[^2$, $\Psi(x,y) = (xy, \frac{y}{x})$ is clearly $\mathscr{C}^1$. We need $\Psi^{-1}$. If $(\xi,\eta) \in ]0,+\infty[^2$ then

$$\begin{cases} \xi = xy, \\ \eta = \frac{y}{x}, \end{cases} \iff \begin{cases} \xi = \eta x^2, \\ y = \eta x, \end{cases} \iff \begin{cases} x = \sqrt{\frac{\xi}{\eta}}, \\ y = \sqrt{\xi\eta}, \end{cases} \iff \Psi^{-1}(\xi,\eta) = \left(\sqrt{\frac{\xi}{\eta}}, \sqrt{\xi\eta}\right).$$

Therefore

$$I(a) := \int_{1\leqslant xy\leqslant 2,\ 0<ax\leqslant y\leqslant \frac{x}{a}} \frac{y^4 \arctan(xy)}{(x^2+y^2)^2}\, dxdy = \int_{1\leqslant \xi\leqslant 2,\ a\leqslant\eta\leqslant\frac{1}{a}} \frac{\eta^4}{(1+\eta^2)^2} \arctan\xi \; |\det(\Psi^{-1})'(\xi,\eta)|\; d\xi d\eta.$$

and because

$$|\det(\Psi^{-1})'(\xi,\eta)| = \frac{1}{|\det \Psi'(\Psi^{-1}(\xi,\eta))|},$$

with

$$\Psi'(x,y) = \begin{bmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{bmatrix}, \implies \det \Psi'(x,y) = \frac{y}{x} + x\frac{y}{x^2} = 2\frac{y}{x} = 2\eta.$$

we have

$$I(a) = \int_{1\leqslant\xi\leqslant 2,\ a\leqslant\eta\leqslant\frac{1}{a}} \frac{\eta^4}{(1+\eta^2)^2} \arctan\xi \frac{1}{2\eta}\, d\xi d\eta = \frac{1}{2}\left(\int_1^2 \arctan\xi\, d\xi\right)\left(\int_a^{\frac{1}{a}} \frac{\eta^3}{(1+\eta^2)^2}\, d\eta\right).$$

Now

$$\int_1^2 \arctan\xi\, d\xi = [\xi\arctan\xi]_1^2 - \int_1^2 \frac{\xi}{1+\xi^2}\, d\xi = 2\arctan 2 - \frac{\pi}{4} - \frac{1}{2}\log\frac{5}{2},$$

while

$$\int_a^{\frac{1}{a}} \frac{\eta^3}{(1+\eta^2)^2}\, d\eta = \int_a^{\frac{1}{a}} \frac{\eta}{1+\eta^2}\, d\eta - \int_a^{\frac{1}{a}} \frac{\eta}{(1+\eta^2)^2}\, d\eta = -\log a + \frac{1}{2}\frac{1-a^2}{1+a^2}. \quad \blacksquare$$

**4.3.1. Polar coordinates in $\mathbb{R}^2$.** A very important change of variable in plane integration is

$$\begin{cases} x = \rho\cos\theta, \\ y = \rho\sin\theta, \end{cases} \iff (x,y) = \Psi(\rho,\theta) = (\rho\cos\theta, \rho\sin\theta).$$

Here we may notice that change of variable is defined in the form $(x,y) = \Psi(\rho,\theta)$. This means that, referring to notations of (4.3.2), present $\Psi$ is just $\Phi^{-1}$. Thus

$$\det(\Phi^{-1})' = \det\Psi' = \det\begin{bmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{bmatrix} = \rho(\cos^2\theta + \sin^2\theta) = \rho,$$

and (4.3.2) becomes

$$\int_E f(x,y)\,dxdy = \int_{E_{pol}} f(\rho\cos\theta, \rho\sin\theta)\rho\,d\rho d\theta, \tag{4.3.3}$$

where $E_{pol}$ is $E$ in polar coordinates.

EXAMPLE 4.3.4. *Compute*

$$\int_{\mathbb{R}^2} e^{-\sqrt{x^2+y^2}}\,dxdy.$$

SOL. — We have

$$\begin{aligned}\int_{\mathbb{R}^2} e^{-\sqrt{x^2+y^2}}\,dxdy &= \int_{\rho\geqslant 0,\theta\in[0,2\pi]} e^{-\rho}\rho\,d\rho d\theta = \int_0^{+\infty}\left(\int_0^{2\pi} e^{-\rho}\rho\,d\theta\right)\,d\rho = 2\pi\int_0^{+\infty}\rho e^{-\rho}\,d\rho \\ &= 2\pi\left([-\rho e^{-\rho}]_{\rho=0}^{\rho=+\infty} + \int_0^{+\infty} e^{-\rho}\,d\rho\right) = 2\pi.\ \blacksquare\end{aligned}$$

EXAMPLE 4.3.5 (GAUSSIAN INTEGRAL). *A very beautiful (and relevant) application of the (4.3.3) is the formula*

$$\int_{\mathbb{R}} e^{-\frac{x^2}{2}}\,dx = \sqrt{2\pi}.$$

*More in general: if $C$ is a $d\times d$ positive symmetric matrix,*

$$\int_{\mathbb{R}^d} e^{-\frac{1}{2}C^{-1}x\cdot x}\,dx = \sqrt{(2\pi)^d \det C}. \tag{4.3.4}$$

SOL. — Let's start by the integral

$$\int_{\mathbb{R}^2} e^{-\frac{x^2+y^2}{2}}\,dxdy = \int_{\mathbb{R}}\left(\int_{\mathbb{R}} e^{-\frac{x^2+y^2}{2}}\,dx\right)\,dy = \int_{\mathbb{R}} e^{-\frac{x^2}{2}}\left(\int_{\mathbb{R}} e^{-\frac{y^2}{2}}\,dx\right)\,dy = \left(\int_{\mathbb{R}} e^{-\frac{x^2}{2}}\,dx\right)^2.$$

On the other hand, by (4.3.3)

$$\int_{\mathbb{R}^2} e^{-\frac{x^2+y^2}{2}}\,dxdy = \int_0^{+\infty}\left(\int_0^{2\pi} e^{-\frac{\rho^2}{2}}\rho\,d\theta\right)\,d\rho = 2\pi\int_0^{+\infty} e^{-\frac{\rho^2}{2}}\rho\,d\rho = 2\pi\left[e^{-\frac{\rho^2}{2}}\right]_{\rho=0}^{\rho=+\infty} = 2\pi,$$

and by this the conclusion follows.

To compute (4.3.4) notice first that, being $C$ symmetric, it is diagonalizable: this means that there exists $T$ invertible such that $T^{-1}CT = \operatorname{diag}(\sigma_1,\dots,\sigma_d)$. Furthermore, because $C$ is symmetric, $T$ is also orthogonal, that is $T^{-1} = T^t$ (transposed matrix). Therefore $C = TDT^{-1}$, hence

$$\int_{\mathbb{R}^d} e^{-\frac{1}{2}C^{-1}x\cdot x}\,dx = \int_{\mathbb{R}^d} e^{-\frac{1}{2}(TDT^{1-})^{-1}x\cdot x}\,dx = \int_{\mathbb{R}^d} e^{-\frac{1}{2}(TD^{-1}T^{1-})x\cdot x}\,dx = \int_{\mathbb{R}^d} e^{-\frac{1}{2}D^{-1}T^{-1}x\cdot T^{-1}x}\,dx.$$

Now, set $y = T^{-1}x$, in such a way that $x = Ty$ and

$$\int_{\mathbb{R}^d} e^{-\frac{1}{2}D^{-1}T^t x\cdot T^t x}\,dx = \int_{\mathbb{R}^d} e^{-\frac{1}{2}D^{-1}y\cdot y}\,|\det T|\,dy = \int_{\mathbb{R}^d} e^{-\frac{1}{2}D^{-1}y\cdot y}\,dy.$$

Last = is justified because, being $T$ orthogonal, $TT^t = \mathbb{I}$, hence $1 = \det(TT^t) = \det T \det T^t = (\det T)^2$ by which $|\det T| = 1$. Moreover,

$$D^{1-} y \cdot y = \sum_j \frac{1}{\sigma_j} y_j^2,$$

therefore

$$\int_{\mathbb{R}^d} e^{-\frac{1}{2}D^{-1}y\cdot y}\, dy = \int_{\mathbb{R}^d} \prod_{j=1}^d e^{-\frac{y_j^2}{2\sigma_j}}\, dy_j = \prod_{j=1}^d \int_{\mathbb{R}} e^{-\frac{y_j^2}{2\sigma_j}}\, dy_j \overset{x_j = \frac{y_j}{\sqrt{\sigma_j}}}{=} \prod_{j=1}^d \sqrt{\sigma_j} \int_{\mathbb{R}} e^{-\frac{x^2}{2}}\, dx = \sqrt{(2\pi)^d \sigma_1 \cdots \sigma_d}.$$

To conclude just notice that

$$\sigma_1 \cdots \sigma_d = \det D = \det(T^{-1}CT) = \det T^{-1} \det C \det T = \det C. \quad \blacksquare$$

**4.3.2. Spherical and cylindrical coordinates.** The analogous of polar coordinates for functions of three variables are *spherical coordinates*:

$$\begin{cases} x = \rho\cos\theta\sin\varphi, \\ y = \rho\sin\theta\sin\varphi, \quad (\rho,\theta,\varphi) \in [0,+\infty[\times[0,2\pi]\times[0,\pi]. \\ z = \rho\cos\varphi. \end{cases}$$

Also in this case the change of variable is defined in the form

$$(x, y, z) = \Psi(\rho, \theta, \varphi),$$

thus, referring to (4.3.2), $\Psi = \Phi^{-1}$. Hence,

$$\det(\Phi^{-1})' = \det \begin{bmatrix} \cos\theta\sin\varphi & -\rho\sin\theta\sin\varphi & \rho\cos\theta\cos\varphi \\ \sin\theta\sin\varphi & \rho\cos\theta\sin\varphi & \rho\sin\theta\cos\varphi \\ \cos\varphi & 0 & -\rho\sin\varphi \end{bmatrix} = \rho^2\sin\varphi.$$

Therefore, (4.3.2) reads as

$$\int_E f(x, y, z)\, dxdydz = \int_{E_{spher}} f(\rho\cos\theta\sin\varphi, \rho\sin\theta\sin\varphi, \rho\cos\varphi)\rho^2\sin\varphi\, d\varphi\, d\theta\, d\rho.$$

Here $E_{spher}$ is $E$ in spherical coordinates. This type of change of variable is often useful when $f$ has some spherical symmetry, that is it depends on $x^2 + y^2 + z^2$.

Example 4.3.6. *Using spherical coordinates, compute the volume of a sphere of radius $r$.*

Sol. — We have

$$\begin{aligned} \lambda_3\left(\{x^2+y^2+z^2 \leqslant r^2\}\right) &= \int_{x^2+y^2+z^2\leqslant r^2} dxdydz = \int_{0\leqslant\rho\leqslant r,\ 0\leqslant\theta\leqslant 2\pi,\ 0\leqslant\varphi\leqslant\pi} \rho^2\sin\varphi\, d\rho d\theta d\varphi \\ &= 2\pi\left(\int_0^\pi \sin\varphi\, d\varphi\right)\left(\int_0^r \rho^2\, d\rho\right) = \frac{4}{3}\pi r^3. \quad \blacksquare \end{aligned}$$

When $f$ has not a central symmetry but it is symmetric respect to some of the axes, a further variant of polar coordinates may be useful. Let first introduce this system of coordinates defined as

$$\begin{cases} x = \rho\cos\theta, \\ y = \rho\sin\theta, \qquad (\rho,\theta,z) \in [0,+\infty[\times[0,2\pi]\times\mathbb{R}. \\ z = z. \end{cases}$$

Also in this case the change of variables is defined in the form

$$(x,y,z) = \Psi(\rho,\theta,z), \text{ where } \Psi = \Phi^{-1}.$$

Being,

$$\det\Psi' = \det\begin{bmatrix} \cos\theta & -\rho\sin\theta & 0 \\ \sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \rho,$$

according to (4.3.2) we have

$$\int_E f(x,y,z)\, dxdydz = \int_{E_{cil}} f(\rho\cos\theta,\rho\sin\theta,z)\rho\, d\rho d\theta dz.$$

This change of variables is particularly useful in the case of functions symmetric respect to the $z$ axis (that is depending on $x^2+y^2$ that becomes $\rho^2$ in new coords).

EXAMPLE 4.3.7. *Compute the volume of the rugby ball* $E = \{(x,y,z)\in\mathbb{R}^3 \,:\, \frac{x^2+y^2}{a^2}+\frac{z^2}{b^2}\leqslant 1\}$ *by adapting cylindrical coordinates.*

PROOF — Adapting the cylindrical coords $(x,y,z) = \Psi^{-1}(\rho,\theta,z) := (a\rho\cos\theta, a\rho\sin\theta, bz)$ we have

$$\det(\Psi^{-1})' = \det\begin{bmatrix} a\cos\theta & -a\rho\sin\theta & 0 \\ a\sin\theta & a\rho\cos\theta & 0 \\ 0 & 0 & b \end{bmatrix} = ba^2\rho,$$

therefore

$$m_3(E) = \int_{\rho^2+\tilde{z}^2\leqslant 1,\ \rho\geqslant 0,\ \theta\in[0,2\pi],\ \tilde{z}\in\mathbb{R}} ba^2\rho\, d\rho d\theta dz = 2\pi a^2 b\int_{\rho^2+z^2\leqslant 1,\ \rho\geqslant 0}\rho\, d\rho dz.$$

To compute the last integral we may use polar coords for $(\rho,z) = (r\cos\alpha, r\sin\alpha)$. Then

$$\int_{\rho^2+z^2\leqslant 1,\ \rho\geqslant 0}\rho\, d\rho dz = \int_{-\frac{\pi}{2}\leqslant\alpha\leqslant\frac{\pi}{2},\ 0\leqslant r\leqslant 1}(r\cos\alpha)r\, drd\alpha = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\alpha\, d\alpha\int_0^1 r^2\, dr = \frac{2}{3}.$$

Moral: $m_3(E) = \frac{4\pi}{3}a^2 b$. ■

## 4.4. Barycenter, center of mass, inertia moments

Through multiple integrals we can define several quantities relevant in Geometry and Physics. To fix ideas consider a set $E\subset\mathbb{R}^3$. We call **barycenter** of $E$ the point $(\overline{x},\overline{y},\overline{z})$ defined as

$$\overline{x} = \frac{1}{\lambda_3(E)}\int_E x\, dx\, dy\, dz, \quad \overline{y} = \frac{1}{\lambda_3(E)}\int_E y\, dx\, dy\, dz, \quad \overline{z} = \frac{1}{\lambda_3(E)}\int_E z\, dx\, dy\, dz.$$

In other words, the barycenter is the point whose coords are the mean values of the coords of $E$. With special symmetries some of the coords of the barycenter may vanish. For instance, if $E$ is symmetric with respect to the plane $yz$, that is $(x, y, z) \in E$ iff $(-x, y, z) \in E$), then $\overline{x} = 0$. Indeed, if $\Phi(x, y, z) = (-x, y, z)$ we have $\Phi(E) = E$ therefore, by change of variables,

$$\int_E x\, dxdydz = \int_{\Phi(E)} x\, dxdydz = \int_E (-x)|\det \Phi'(x, y, z)|\, dxdydz = -\int_E x\, dxdydz$$

by which $\int_E x\, dxdydz = 0$.

If $E$ represents a solid body with density of mass $\varrho = \varrho(x, y, z)$, the total mass is, by definition,

$$\mu(E) := \int_E \varrho(x, y, z)\, dxdydz.$$

In Physics it is then important the **center of mass**: it is the point where the sum of all the forces acting on $E$ could be applied to get the same effect. This point has coords $(x_G, y_G, z_G)$

$$x_G = \frac{1}{\mu(E)} \int_E x\varrho(x, y, z)\, dxdydz, \quad y_G = \frac{1}{\mu(E)} \int_E y\varrho(x, y, z)\, dxdydz, \quad z_G = \frac{1}{\mu(E)} \int_E z\varrho(x, y, z)\, dxdydz.$$

If the body is homogeneous (that is $\varrho \equiv \varrho_0 \in \mathbb{R}$) the center of mass coincide with the barycenter as it is easy to see.

Another important quantity for Physics is the **inertia moment with respect to some axis**. For instance, if the axis is the $z$ one, this is defined by

$$I_z := \int_E (x^2 + y^2)\varrho(x, y, z)\, dxdydz.$$

Example 4.4.1. *Determine the barycenter of a spherical cap* $E := \{(x, y, z) \; : \; x^2 + y^2 + z^2 \leqslant r^2,\ z \geqslant h\}$ *con* $0 \leqslant h < r$.

Sol. — By symmetries, it is evident that $\overline{x} = \overline{y} = 0$. Let's compute

$$\overline{z} = \frac{1}{\lambda_3(E)} \int_E z\, dxdydz.$$

It seems convenient to slice $E$ perpendicularly to the $z-$axis:

$$\begin{aligned}
\lambda_3(E) &= \int_h^r \left( \iint_{x^2+y^2 \leqslant r^2 - z^2} dxdy \right) dh = \int_h^r \pi(r^2 - z^2)\, dz = \pi r^2 (r - h) - \pi \left[ \frac{z^3}{3} \right]_{z=h}^{z=r} \\
&= \pi(r - h)\left( r^2 - \frac{1}{3}(r^2 + rh + h^2) \right).
\end{aligned}$$

Similarly

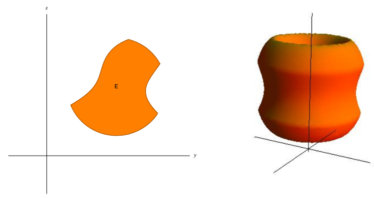
$$\begin{aligned}
\int_E z\, dxdydz &= \int_h^r \left( \iint_{x^2+y^2 \leqslant r^2 - z^2} z\, dxdy \right) dz = \int_h^r z \left( \iint_{x^2+y^2 \leqslant r^2 - z^2} dxdy \right) dz = \int_h^r z\pi(r^2 - z^2)\, dz \\
&= \pi r^2 \left[ \frac{z^2}{2} \right]_{z=h}^{z=r} - \pi \left[ \frac{z^4}{4} \right]_{z=h}^{z=r} = \pi r^2 \frac{r^2 - h^2}{2} - \pi \frac{r^4 - h^4}{4} = \pi \frac{r^2 - h^2}{2} \left( r^2 - \frac{r^2 + h^2}{2} \right) \\
&= \pi \frac{(r^2 - h^2)^2}{4}.
\end{aligned}$$

By this we get $\overline{z}$. In the case $h = 0$ (that is when $E$ is the half-sphere) we have $\overline{z} = \frac{3}{8} r$. ■

Let $D \subset \mathbb{R}^3$ by a domain obtained by a rotation around one of the axes of a plane set $E$. To fix ideas, let's assume that the rotation be around the $z-$axis of a domain $E$ in the plane $yz$. This domain can be identified by $\{(0, y, z) \; : \; (y, z) \in E\} \subset \mathbb{R}^3$. Therefore, $D$ can be represented as

$$D = \{(y\cos\theta, y\sin\theta, z) \; : \; (y, z) \in E, \; \theta \in [0, 2\pi]\} = \Phi(E \times [0, 2\pi]),$$

where $\Phi$ is nothing but the cylindrical coords map.



By the formula of change of variables

$$\lambda_3(D) = \int_{E\times[0,2\pi]} J_\Phi(y, \theta, z)\, dy\, d\theta\, dz = \int_{E\times[0,2\pi]} y\, dy\, d\theta\, dz = 2\pi \int_E y\, dydz$$

that gives the **Pappo's Theorem**:

(4.4.1) $$\lambda_3(D) = 2\pi\lambda_2(E)\overline{y}.$$

EXAMPLE 4.4.2. *Let's compute the volume of a thorus* $\mathbb{T}_{r,R} := \left\{(x, y, z) \in \mathbb{R}^3 \; : \; (\sqrt{x^2 + y^2} - R)^2 + z^2 \leqslant r^2\right\}$ *($0 < r < R$).*

SOL. — By (4.4.1)

$$\lambda_3(\mathbb{T}_{r,R}) = 2\pi m_2 \left(\{(y - R)^2 + z^2 \leqslant r^2\right) \overline{y} = 2\pi 1\pi r^2 \overline{y} = 4\pi^2 r^2 \overline{y}.$$

Here $\overline{y}$ it's the ordinate of the barycenter of the disk $E := \{(y - R)^2 + z^2 \leqslant r^2\}$, so

$$\overline{y} = \frac{1}{\lambda_2(E)} \int_E y\, dydz = \frac{1}{\pi r^2} \int_{(y-R)^2+z^2\leqslant r^2} y\, dydz.$$

Changing to polar coord $y - R = \rho\cos\theta$, $z = \rho\sin\theta$, we have easily

$$\overline{y} = \frac{1}{\pi r^2} \int_0^{2\pi} \left( \int_0^r \rho(R + \rho\cos\theta)\, d\rho \right) d\theta = \frac{1}{\pi r^2} 2\pi \frac{r^2}{2} R = R,$$

(as it is natural!). Hence $\lambda_3(\mathbb{T}_{r,R}) = 4\pi^2 r^2 R$. ■

## 4.5. Exercises

EXERCISE 4.5.1. Compute

1. $\int_{0\leqslant y\leqslant 1,\ 0\leqslant x\leqslant 1-y^2} xe^y\, dxdy$.
2. $\int_{0\leqslant y\leqslant 1-x^2} \frac{x}{2+y}\, dxdy$.
3. $\int_{|y|\leqslant 1-x^2} \frac{1}{1+y}\, dxdy$
4. $\int_{0\leqslant x,y\leqslant 1,\ 0\leqslant z\leqslant 3-x+y} x\sin(\pi y)\, dxdydz4$.
5. $\int_{x\geqslant 0,\ y\geqslant 0,\ x+y+z\leqslant 1} xyz\, dxdydz$.
6. $\int_{0\leqslant x\leqslant 1,\ 0\leqslant y\leqslant 2,\ 0\leqslant z\leqslant 6-x^2-y^2} x\log(1 + y)\, dxdydz$.

EXERCISE 4.5.2. Compute:

1. $\int_{[0,1]^2} e^{\max\{x^2,y^2\}}\,dxdy$. 2. $\int_{[0,1]\times[2,4]} \frac{1}{(x-y)^2}\,dxdy$. 3. $\int_{[0,+\infty[\times[1,+\infty[} e^{-xy^4}\,dxdy$.

4. $\int_{1\leqslant x\leqslant 2,\ \frac{1}{x}\leqslant y\leqslant x} \frac{x}{y}\,dxdy$. 5. $\int_{[0,1]^3} e^{\max\{x,y,z\}}\,dxdy$. 6. $\int_{[1,+\infty[^3} y^3z^8e^{-xy^2z^3}\,dxdydz$.

EXERCISE 4.5.3. Compute

1. $\int_D x\sqrt{y^2-x^2}\,dxdy,\ \ D=\{(x,y)\in\mathbb{R}^2\ :\ 0\leqslant y\leqslant 1,\ 0\leqslant x\leqslant y\}$.

2. $\int_D \frac{x^2e^{-x^2}}{1+(xy)^2}\,dxdy,\ \ D=\{(x,y)\in\mathbb{R}^2\ :\ |xy|\leqslant 1\}$.

3. $\int_D zy^2\sqrt{x^2+zy}\,dxdydz,\ \ D=\{(x,y,z)\in\mathbb{R}^3\ :\ 0\leqslant z\leqslant x^2,\ 0\leqslant x\leqslant 1,\ 0\leqslant y\leqslant 1\}$.

EXERCISE 4.5.4. For which values $\alpha\in\mathbb{R}$ the function $f_\alpha(x,y):=\frac{1}{(x-y)^\alpha}$ belongs to $L^1([1,+\infty[\times[0,1])$? In this case compute the integral $\int_{[1,+\infty[\times[0,1]} f_\alpha$.

EXERCISE 4.5.5. Let $D:=\{(x,y)\in\mathbb{R}^2\ :\ x\geqslant 0,\ y\geqslant 0,\ x^2+y^2\leqslant r^2\}$. Draw $D$ and describe it in polar coords. Determine its barycenter and compute the integral

$$\int_D \frac{x+y}{x^2+y^2}\,dxdy.$$

EXERCISE 4.5.6 (POLAR, SPHERICAL, CYLINDRICAL COORDS). Draw (if possible) and compute the volume of

1. $\left\{(x,y,z) : 9(1-\sqrt{x^2+y^2})^2+4z^2\leqslant 1\right\}$.
2. $\left\{(x,y,z)\ :\ x^2+y^2+z^2\leqslant r^2,\ \left(x-\frac{r}{2}\right)^2+y^2\leqslant\frac{r^2}{4}\right\}$.
3. $\left\{(x,y,z)\in\mathbb{R}^3\ :\ \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\leqslant 1\right\}$, $(a,b,c>0)$.
4. $\left\{(x,y,z)\ :\ x^2+y^2\leqslant 1,\ x^2+z^2\leqslant 1,\ y^2+z^2\leqslant 1\right\}$.
5. $\left\{(x,y,z)\in\mathbb{R}^3\ :\ x^2+y^2\leqslant 4,\ 4x^2+4y^2+z^2\leqslant 64\right\}$.
6. $\left\{(x,y,z)\in\mathbb{R}^3\ :\ x^2+y^2+z^2\leqslant 16,\ x^2+y^2\geqslant 4\right\}$
7. $\left\{(x,y,z)\in\mathbb{R}^3\ :\ z\geqslant\sqrt{x^2+y^2},\ x^2+y^2+z^2\leqslant 1\right\}$.
8. $\left\{(x,y,z)\in\mathbb{R}^3\ :\ z\geqslant x^2+y^2,\ z\leqslant 18-x^2-y^2\right\}$.

Exercise 4.5.7. Discuss existence and (if possible) the value of the following integrals:

1. $\displaystyle\int_{\mathbb{R}^2} \frac{1}{\cosh(x^2+y^2)}\,dxdy.$ 2. $\displaystyle\int_{x^2+y^2\leqslant 4} \sqrt{4-x^2-y^2}\,dxdy$

3. $\displaystyle\int_{x^2+2y^2\leqslant 1} \frac{1}{1+x^2+2y^2}\,dxdy.$ 4. $\displaystyle\int_{\mathbb{R}^2} \frac{1}{1+(x^2+2y^2)^2}\,dxdy.$

5. $\displaystyle\int_{x^2+y^2\leqslant 16,\ -5\leqslant z\leqslant 4} \sqrt{x^2+y^2}\,dxdydz.$ 6. $\displaystyle\int_{\mathbb{R}^3} \sqrt{x^2+y^2+z^2}e^{-(x^2+y^2+z^2)}\,dxdydz.$

7. $\displaystyle\int_{\mathbb{R}^3} \frac{1}{1+(x^2+2y^2+3z^2)^2}\,dxdydz.$ 8. $\displaystyle\int_{\mathbb{R}^2} \frac{1}{1+(x^2+xy+y^2)^2}\,dxdy.$

9. $\displaystyle\int_{\mathbb{R}^3} e^{-(x^2+y^2+z^2-xy+yz-xz)}\,dxdydz.$ 10. $\displaystyle\int_{\mathbb{R}^3} \frac{1}{1+x^4+y^4+z^4}\,dxdydz.$

Exercise 4.5.8. By using carefully the suggested change of variables, compute

1. $\displaystyle\int_D xy\,dxdy,\ D=\{(x,y)\in\mathbb{R}^2\ :\ 1\leqslant xy\leqslant 3,\ x\leqslant y\leqslant 3x\},\ (u=xy,\ v=\frac{y}{x}).$

2. $\displaystyle\int_D y^2\,dxdy,\ D=\{(x,y)\in\mathbb{R}^2\ :\ 1\leqslant xy\leqslant 2,\ 1\leqslant xy^2\leqslant 2\}.\ (u=xy,\ v=xy^2).$

3. $\displaystyle\int_D \sqrt{x^2-y^2}\,dxdy,\ D=\{(x,y)\in\mathbb{R}^2\ :\ 1\leqslant x^2-y^2\leqslant 2,\ px\leqslant y\leqslant qx\}\ (-1<p<q<1).\ (u=x^2-y^2,\ v=\frac{y}{x}).$

Exercise 4.5.9. Let $a>1$ and

$$E_a := \left\{(x,y)\in\mathbb{R}^2\ :\ \frac{1}{ax}\leqslant y\leqslant \frac{1}{x},\ x^2\leqslant y\leqslant ax^2\right\}.$$

Draw $E_a$. Show that $\Phi(x,y) := \left(xy, \frac{y}{x^2}\right)$ is a diffeomorphism modulo null sets on $E_a$. Use this to compute

$$I(a) := \int_{E_a} \frac{x^2}{y}e^{xy}\,dxdy.$$

Compute $\lim_{a\to+\infty} I(a)$ and check if $I = \int_{x^2\leqslant y\leqslant \frac{1}{x}} \frac{x^2}{y}e^{xy}\,dxdy$.

Exercise 4.5.10 (★). Let $D := \{(x,y)\in\mathbb{R}^2\ :\ 0\leqslant x\leqslant y\}$ and

$$f(x,y) := \frac{x^{3/2}}{\sqrt{y-x}}e^{-(xy)^{3/2}},\ \ (x,y)\in D.$$

Show that $\Phi(x,y) := (xy, x/y)$ is a diffeomorphism modulo null sets on $D$. Use this to say if $f\in L^1(D)$. In such case compute $\int_D f$.

Exercise 4.5.11 (★). Let $D := \{(x,y)\in[0,+\infty[^2\ :\ xy\geqslant 1\}$, and $f(x,y) := \frac{\log(xy)}{y(x+y^2)^7}$, $(x,y)\in D$. Show that $\Phi(x,y) := (xy, \frac{y^2}{x})$ is a diffeomorphism modulo null sets on $E$. Use this to say if $f\in L^1(D)$. In such case compute $\int_D f$.

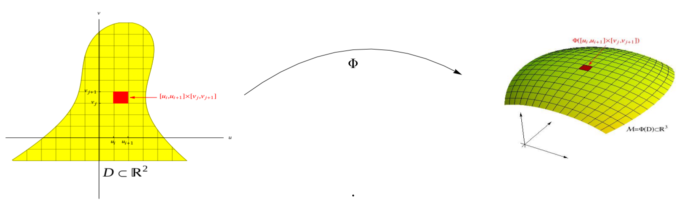
CHAPTER 5

# Surface integrals

In this Chapter we introduce the important concept of integral on a surface in $\mathbb{R}^3$. This concept is justified initially to give a measure to the area of a surface in a similar way as for the length of a curve. Furthermore, this tool is fundamental to give a precise definition of *flux of a vector field through a surface*, a very important concept for Physics and Engineering.

## 5.1. Definition

Let's start by the problem of computing the area of a surface $\mathscr{M} \subset \mathbb{R}^3$. Do describe a surface we will use here a *parametrization*, that is a function $\Phi = \Phi(u,v) : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ such that $\mathscr{M} = \Phi(D)$.




A natural idea consists in dividing $D$ in small rectangles $[u_i, u_{i+1}] \times [v_j, v_{j+1}]$. It seems then reasonable that

$$\text{Area}(\mathscr{M}) = \sum_{i,j} \text{Area}\left(\Phi([u_i, u_{i+1}] \times [v_j, v_{j+1}])\right).$$

Now, if $u_i \sim u_{i+1}$ and $v_j \sim v_{j+1}$ we could say that $\Phi([u_i, u_{i+1}] \times [v_j, v_{j+1}])$ is almost a parallelogram of sides $\Phi(u_i, v_{j+1}) - \Phi(u_i, v_j)$ and $\Phi(u_{i+1}, v_j) - \Phi(u_i, v_j)$ (see next picture). By Lagrange approximation

$$\Phi(u_{i+1}, v_j) - \Phi(u_i, v_j) \approx \partial_u \Phi(u_i, v_j)(u_{i+1} - u_i), \quad \Phi(u_i, v_{j+1}) - \Phi(u_i, v_j) \approx \partial_v \Phi(u_i, v_j)(v_{j+1} - v_j),$$

so

$$\text{Area}\left(\Phi([u_i, u_{i+1}] \times [v_j, v_{j+1}])\right) \approx \text{Area}\left([\partial_u \Phi(u_i, v_j)\ \partial_v \Phi(u_i, v_j)]\right)(u_{i+1} - u_i)(v_{j+1} - v_j),$$

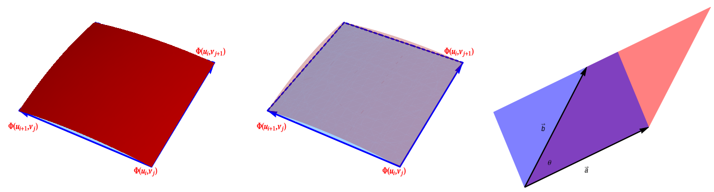
where we settled $[\vec{a}\ \vec{b}]$ the parallelogram of sides $\vec{a}$ and $\vec{b}$. Notice incidentally that $\text{Area}[\vec{a}, \vec{b}] \neq 0$ iff $\vec{a}$ and $\vec{b}$ are linearly independent. It is therefore natural to require on $\Phi$ that

$$\partial_u \Phi(u,v),\ \partial_v \Phi(u,v) \text{ be linearly independent, } \forall (u,v) \in D.$$

We say that $\Phi$ is an **immersion**.

Let's come now to the problem to compute the area of a parallelogram into the space. Elementary geometry gives the formula

$$\text{Area}[\vec{a}, \vec{b}] = \|\vec{a}\| \|\vec{b}\| \sin \theta,$$



where $\theta$ is the angle formed by $\vec{a}$ and $\vec{b}$. By recalling the vector product,

$$\text{Area}[\vec{a},\vec{b}] = \|\vec{a}\|\|\vec{b}\|\sin\theta = \|\vec{a}\times\vec{b}\|, \quad \text{where } \vec{a}\times\vec{b} = \det\begin{bmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}.$$

Returning to the initial problem

$$\text{Area}\left(\Phi([u_i,u_{i+1}]\times[v_j,v_{j+1}])\right) \approx \|\partial_u\Phi(u_i,v_i)\times\partial_v\Phi(u_i,v_i)\|(u_{i+1}-u_i)(v_{j+i}-v_j)$$

hence

$$\text{Area}(\mathscr{M}) \approx \sum_{i,j}\|\partial_u\Phi(u_i,v_i)\times\partial_v\Phi(u_i,v_i)\|(u_{i+1}-u_i)(v_{j+i}-v_j) \approx \int_D \|\partial_u\Phi(u,v)\times\partial_v\Phi(u,v)\|\,dudv.$$

This informal argument justifies the

Definition 5.1.1. *Let $\mathscr{M} = \Phi(D)$ be a parametric surface with $\Phi\in\mathscr{C}^1$ immersive on $D$. We set*

$$\sigma_2(\mathscr{M}) := \int_D \|\partial_u\Phi(u,v)\times\partial_v\Phi(u,v)\|\,dudv \equiv \int_D \|\partial_u\Phi\times\partial_v\Phi\|.$$

*More generally: if $f:\mathscr{M}\longrightarrow\mathbb{R}$ we set*

$$\int_{\mathscr{M}} f\,d\sigma_2 := \int_D f(\Phi(u,v))\|\partial_u\Phi(u,v)\times\partial_v\Phi(u,v)\|\,dudv \equiv \int_D f(\Phi)\|\partial_u\Phi\times\partial_v\Phi\|. \tag{5.1.1}$$

*The symbol $d\sigma_2 = \|\partial_u\Phi(u,v)\times\partial_v\Phi(u,v)\|\,dudv$ is called* **area element**.

Example 5.1.2. *Let's compute the area of a sphere of radius $r$, $\mathbb{S}^2_r := \{(x,y,z)\in\mathbb{R}^3 \;:\; x^2+y^2+z^2 = r^2\}$.*

Sol. — The natural parametrization of the sphere is given by the map

$$\Phi(\theta,\varphi) := r\left(\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi\right),\ (\theta,\varphi)\in[0,2\pi]\times[0,\pi].$$

We have

$$\partial_\theta\Phi = r(-\sin\theta\sin\varphi, \cos\theta\sin\varphi, 0), \quad \partial_\varphi\Phi = r(\cos\theta\cos\varphi, \sin\theta\cos\varphi, -\sin\varphi),$$

hence (by shortening $S=\sin\theta$, $C=\cos\theta$, $s=\sin\varphi$, $c=\cos\varphi$)

$$\partial_\theta\Phi\times\partial_\varphi\Phi = \det\begin{bmatrix} i & j & k \\ -rSs & rCs & 0 \\ rCc & rSc & -rs \end{bmatrix} = r^2(-Cs^2, -Ss^2, -S^2sc - C^2sc) = -r^2(Cs^2, Ss^2, sc),$$

so finally

$$\|\partial_\theta\Phi\times\partial_\varphi\Phi\| = r^2\sqrt{C^2s^4+S^2s^4+s^2c^2} = r^2\sqrt{s^2(s^2+c^2)} = r^2\sqrt{s^2} = r^2|s| = r^2|\sin\varphi|.$$

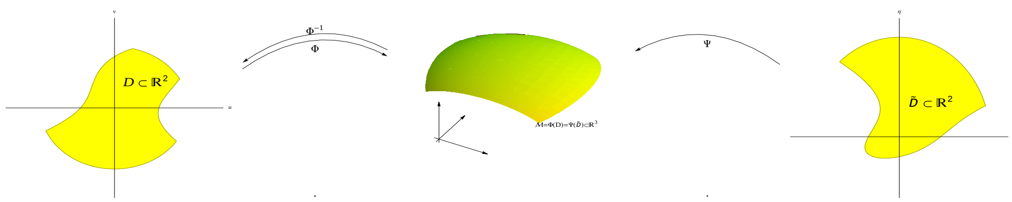
Therefore

$$\sigma_2(\mathbb{S}^2(r)) = \int_{0\leqslant\theta\leqslant 2\pi,\ 0\leqslant\varphi\leqslant\pi} r^2|\sin\varphi|\, d\theta d\varphi = 2\pi r^2 \int_0^\pi \sin\varphi\, d\varphi = 4\pi r^2. \quad \blacksquare$$

Apparently the surface integral $\int_{\mathscr{M}} f\, d\sigma$ depends by the parametrization $\Phi$. Actually the value doesn't change if $\mathscr{M} = \Psi(\widetilde{D})$ and the change of parametrization $\Phi^{-1}\circ\Psi$ is a diffeomorphism.

Proposition 5.1.3. *Let $\mathscr{M} = \Phi(D) = \Psi(\widetilde{D})$, with $\Phi, \Psi \in \mathscr{C}^1$ immersive and such that $\Phi^{-1}\circ\Psi$ be a diffeomorphism. Then*

$$\int_D f(\Phi(u,v))\|\partial_u\Phi(u,v)\times\partial_v\Phi(u,v)\|\, dudv = \int_{\widetilde{D}} f(\Psi(\xi,\eta))\|\partial_\xi\Psi(\xi,\eta)\times\partial_\eta\Psi(\xi,\eta)\|\, d\xi d\eta \tag{5.1.2}$$




Proof — Call $x = (u,v)$ and $y = (\xi,\eta)$ and set $\Gamma := \Phi^{-1}\circ\Psi$ be the change of parametrization. Then, by chain rule,

$$\Psi(y) = \Phi(\Phi^{-1}(\Psi(y)) = \Phi(\Gamma(y)), \implies \Psi'(y) = \Phi'(\Gamma(y))\Gamma'(y),$$

that is

$$[\partial_\xi\Psi\ \partial_\eta\Psi] = [\partial_u\Phi(\Gamma)\ \partial_v\Phi(\Gamma)]\Gamma'.$$

Therefore

$$\begin{aligned}\|\partial_\xi\Psi\times\partial_\eta\Psi\| &= \text{Area}[\partial_\xi\Psi\ \partial_\eta\Psi] = \text{Area}\left([\partial_u\Phi(\Gamma)\ \partial_v\Phi(\Gamma)]\Gamma'\right) = \text{Area}\left([\partial_u\Phi(\Gamma)\ \partial_v\Phi(\Gamma)]\right)|\det\Gamma'(y)| \\ &= \|\partial_u\Phi(\Gamma)\times\partial_v\Phi(\Gamma)\||\det\Gamma'|.\end{aligned}$$

To finish, by the change of variables

$$\begin{aligned}\int_{\widetilde{D}} f(\Psi(y))\|\partial_\xi(y)\Psi\times\partial_\eta\Psi(y)\|\, dy &= \int_{\widetilde{D}} f(\Phi(\Gamma(y)))\|\partial_u\Phi(\Gamma(y))\times\partial_v\Phi(\Gamma(y))\||\det\Gamma'(y)|\, dy \\ &\overset{x=\Gamma(y)}{=} \int_D f(\Phi(x))\|\partial_u\Phi(x)\times\partial_v\Phi(x)\|\, dx. \quad \blacksquare\end{aligned}$$

Similarly to the case of multidimensional integral, the **barycentre** of a surface $\mathscr{M}$ is the point

$$(\overline{x},\overline{y},\overline{z}) = \frac{1}{\sigma_2(\mathscr{M})}\left(\int_{\mathscr{M}} x\, d\sigma_2, \int_{\mathscr{M}} y\, d\sigma_2, \int_{\mathscr{M}} z\, d\sigma_2\right).$$

If $\varrho(x,y,z)$ is a mass density on $\mathscr{M}$, the **total mass** of $\mathscr{M}$ is

$$\mu(\mathscr{M}) := \int_{\mathscr{M}} \varrho\, d\sigma_2.$$

If the distribution is homogeneous, that is if $\varrho \equiv \varrho_0$, then $\mu(\mathscr{M}) = \varrho_0 \sigma_2(\mathscr{M})$. The **center of mass** of $\mathscr{M}$ is the point

$$\frac{1}{\mu(\mathscr{M})}\left(\int_{\mathscr{M}} x\varrho \, d\sigma_2, \int_{\mathscr{M}} y\varrho \, d\sigma_2, \int_{\mathscr{M}} z\varrho \, d\sigma_2\right).$$

For homogeneous bodies the center of mass is the barycenter.

**5.1.1. Surface integral over the graph of a regular function.** A very important particular case is the following:

Proposition 5.1.4. *Let $\beta : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function of class $\mathscr{C}^1$ and let*

$$\mathscr{M} := G(\beta) \equiv \{(u, v, \beta(u, v)) \ : \ (u, v) \in D\}.$$

*Then*

$$\int_{\mathscr{M}} f \, d\sigma_2 = \int_D f(u, v, \beta(u, v))\sqrt{1 + \|\nabla\beta(u, v)\|^2} \, dudv. \tag{5.1.3}$$

Proof — Just compute the area element: here $\Phi : D \longrightarrow \mathbb{R}^3$ is given by $\Phi(u, v) = (u, v, \beta(u, v))$. Then

$$\partial_u \Phi = (1, 0, \partial_u\beta), \ \partial_v \Phi = (0, 1, \partial_v\beta), \ \Longrightarrow \ \partial_u\Phi \times \partial_v\Phi = \det\begin{bmatrix} i & j & k \\ 1 & 0 & \partial_u\beta \\ 0 & 1 & \partial_v\beta \end{bmatrix} = (-\partial_u\beta, -\partial_v\beta, 1),$$

so

$$\|\partial_u\Phi \times \partial_v\Phi\| = \sqrt{(\partial_u\beta)^2 + (\partial_v\beta)^2 + 1} = \sqrt{1 + \|\nabla\beta\|^2}. \quad \blacksquare$$

Example 5.1.5. *Let's compute again the area of $\mathbb{S}^2_r$.*

Sol. — We can see the are of $\mathbb{S}^2_r$ as twice the area of an half sphere. This is a graph: for instance in the half plane $z \geqslant 0$, $z = \sqrt{r^2 - (x^2 + y^2)} =: \beta(x, y)$. Then

$$\sigma_2(\mathbb{S}^2_r) = 2\int_{x^2+y^2 \leqslant r^2} \sqrt{1 + |\nabla\beta(x, y)|^2} \, dxdy.$$

Being $\nabla\beta = -\frac{(x,y)}{\sqrt{r^2-(x^2+y^2)}}$ we have

$$\begin{aligned}
\sigma_2(\mathbb{S}^2(r)) &= 2\int_{x^2+y^2\leqslant r^2} \sqrt{1 + \frac{x^2 + y^2}{r^2 - (x^2 + y^2)}} \, dxdy = 2\int_{0\leqslant\rho\leqslant r, \ 0\leqslant\theta\leqslant 2\pi} \sqrt{1 + \frac{\rho^2}{r^2 - \rho^2}}\rho \, d\rho d\theta \\
&= 4\pi r \int_0^r \frac{\rho}{\sqrt{r^2 - \rho^2}} \, d\rho = 4\pi r\left[-\sqrt{r^2 - \rho^2}\right]_{\rho=0}^{\rho=r} = 4\pi r^2. \quad \blacksquare
\end{aligned}$$

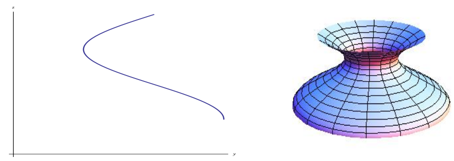
Example 5.1.6. *Compute the barycenter of an half sphere.*

Sol. — The half sphere is the graph of $\beta : D = \{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 \leqslant r^2\} \longrightarrow \mathbb{R}$, $\beta(x, y) = \sqrt{r^2 - (x^2 + y^2)}$. By simmetry it is evident that $\int_{\mathscr{M}} x \, d\sigma_2 = \int_{\mathscr{M}} y \, d\sigma_2 = 0$. It remains

$$\int_{\mathscr{M}} z \, d\sigma_2 = \int_{x^2+y^2\leqslant r^2} z\frac{r}{\sqrt{r^2 - (x^2 + y^2)}} \, dxdy = r\int_{x^2+y^2\leqslant r^2} dxdy = \pi r^3.$$

Being $\sigma_2(\mathscr{M}) = 2\pi r^2$ we obtain the barycenter as the point $\frac{1}{2\pi r^2}\left(0,0,\pi r^3\right) = \left(0,0,\frac{r}{2}\right)$. ■

**5.1.2. Guldino's formula.** This is an analogous for surfaces of the thm of Pappo. Let's consider a rotation surface $\mathscr{M}$ obtained by a rotation of a curve respect to one of the axes. To fix ideas let $\gamma = (y(t), z(t)) \in \mathscr{C}^1([a,b])$ be in the plan $yz$ with $y(t) > 0$.



We may describe the rotation around the $z$ axis as

$$\mathscr{M} = \{(y(t)\cos\theta, y(t)\sin\theta, z(t)) \;:\; t \in [a,b],\; \theta \in [0,2\pi]\} = \Phi([a,b]\times[0,2\pi]),$$

where of course $\Phi(t,\theta) = (y(t)\cos\theta, y(t)\sin\theta, z(t))$. Let's prove the

Theorem 5.1.7 (Guldino). *Let $\gamma(t) = (y(t), z(t)) \in \mathscr{C}^1([a,b])$ with $\|\gamma'\| \neq 0$ and let*

$$\mathscr{M} := \{(y(t)\cos\theta, y(t)\sin\theta, z(t)) \;:\; t \in [a,b],\; \theta \in [0,2\pi]\},$$

*be the rotation surface of $\gamma$ around the $z$ axis. Then*

$$\sigma_2(\mathscr{M}) = 2\pi \int_a^b y(t)\|\gamma'(t)\|\, dt. \tag{5.1.4}$$

Proof — Notice that $\mathscr{M} = \Phi([a,b]\times[0,2\pi])$ where $\Phi(t,\theta) = (y(t)\cos\theta, y(t)\sin\theta, z(t))$, hence

$$\partial_t\Phi = (y'\cos\theta, y'\sin\theta, z'),\; \partial_\theta\Phi = (-y\sin\theta, y\cos\theta, 0),$$

hence the area element is

$$\|\partial_t\Phi\times\partial_\theta\Phi\| = \left\|\det\begin{bmatrix} i & j & k \\ y'\cos\theta & y'\sin\theta & z' \\ -y\sin\theta & y\cos\theta & 0\end{bmatrix}\right\| = \|(yz'\cos\theta, yz'\sin\theta, yy'(\cos^2\theta+\sin^2\theta))\|$$

$$= \sqrt{y^2z'^2(\cos^2\theta+\sin^2\theta)+y^2y'^2} = \sqrt{y^2(y'^2+z'^2)} \overset{y\geqslant 0}{=} y\|\gamma'\|.$$

Finally

$$\sigma_2(\mathscr{M}) = \int_{t\in[a,b],\;\theta\in[0,2\pi]} y\|\gamma'\|\, dt d\theta = 2\pi\int_a^b y\|\gamma'\|\, dt. \;■$$

Example 5.1.8. *Compute the area of a torus $\mathbb{T}_{R,r}$ of radiuses $0 < r < R$.*

Sol. — The torus may be seen as the rotation of a circumference of radius $r$ at distance $R$ by the $z$ axis. Taking the section in the plane $yz$ with $y \geqslant 0$,

$$\gamma \;:\; (y(t), z(t)) = (R + r\cos t, r\sin t),\; t \in [0,2\pi].$$

By Guldino

$$\sigma_2(\mathbb{T}_{R,r}) = 2\pi\int_0^{2\pi}(R + r\cos t)\|\gamma'\|\, dt.$$

Now, $\gamma' = (-r\sin t, r\cos t)$ hence clearly $\|\gamma'\| = r$. Therefore

$$\sigma_2(\mathbb{T}_{R,r}) = 2\pi \int_0^{2\pi} (R + r\cos t) r \, dt = 4\pi^2 r R. \quad \blacksquare$$

## 5.2. Flux of a vector field

As anticipated in the introduction, one of main application of surface integrals is to give a correct definition of *flux of a vector field through a surface*. Let $\mathscr{M} = \Phi(D)$ be a parametric surface with $\Phi \in \mathscr{C}^1$ immersive and let $\vec{F} : \Omega \subset \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a *vector field* defined on $\Omega \supset \mathscr{M}$. Our aim is to define

$$\int_{\mathscr{M}} \vec{F} \cdot \vec{n} \, d\sigma_2, \quad \text{where } \vec{n}(x) = \text{unit normal to } \mathscr{M} \text{ at point } x. \tag{5.2.1}$$

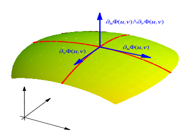
Let's see how to define this normal. Take $x = \Phi(u,v)$ and consider the curve

$$\gamma(t) := \Phi(u+t, v), \quad \Longrightarrow \quad \gamma(0) = \Phi(u,v) = x, \ \gamma \in \mathscr{C}^1, \ \gamma(t) \in \mathscr{M}.$$

In other words $\gamma$ is a regular curve on $\mathscr{M}$. Its velocity when the curve passes at point $x$ is the vector

$$\gamma'(0) = \partial_u \Phi(u,v).$$

Clearly, because the velocity (the derivative) is tangent to the curve, we may expect that $\partial_u \Phi(u,v)$ will be tangent to $\mathscr{M}$. Similarly $\partial_v \Phi(u,v)$ will be tangent to $\mathscr{M}$. Being $\Phi$ immersive, the vectors $\partial_u \Phi$ and $\partial_v \Phi$ are linearly independent.




It is therefore natural to pose

$$\vec{n}_\Phi(u,v) := \frac{\partial_u \Phi(u,v) \times \partial_v \Phi(u,v)}{\|\partial_u \Phi(u,v) \times \partial_v \Phi(u,v)\|}. \tag{5.2.2}$$

Being normalized this is a unit vector and because of the properties of the vector product, this vector will be perpendicular to $\partial_u \Phi$ and $\partial_v \Phi$, hence to the plane generate by them that, reasonably, should be the tangent plane to $\mathscr{M}$ whatever this could be. Notice that also $-\vec{n}$ is unitary and orthogonal to $\mathscr{M}$. To obtain this as normal it is sufficient to change the order of the two parameters. Indeed, by setting $\Psi(u,v) := \Phi(v,u)$, then

$$\partial_u \Psi = \partial_v \Phi, \ \ \partial_v \Psi = \partial_u \Phi, \ \Longrightarrow \ \partial_u \Psi \times \partial_v \Psi = \partial_v \Phi \times \partial_u \Phi = -\partial_u \Phi \times \partial_v \Phi,$$

so

$$\vec{n}_\Psi = -\vec{n}_\Phi.$$

This remark shows that there could be an ambiguity in the notation (5.2.1): indeed the surface integral doesn't depend by the specific parametrization, but the vector $\vec{n}$ depends clearly by this in such a way that we could have different parametrization of the same $\mathscr{M}$ and fluxes with opposite signs. This doesn't seems a dramatic problem but some care is needed when we define the flux:

Definition 5.2.1. *Let $\vec{F} : \Omega \subset \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a continuous vector field on $\Omega \supset \mathscr{M} = \Phi(D)$ parametric surface with $\Phi \in \mathscr{C}^1$ immersive. We call* **flux of $\vec{F}$ through** $\mathscr{M}$

$$\langle \vec{F} \rangle_{\mathscr{M}} := \int_{\mathscr{M}} \vec{F} \cdot \vec{n}_\Phi \, d\sigma_2. \tag{5.2.3}$$

Notice that

$$\langle \vec{F} \rangle_{\mathscr{M}} = \int_D \vec{F}(\Phi) \cdot \frac{\partial_u \Phi \times \partial_v \Phi}{\|\partial_u \Phi \times \partial_v \Phi\|} \|\partial_u \Phi \times \partial_v \Phi\| \, dudv = \int_D \vec{F}(\Phi) \cdot (\partial_u \Phi \times \partial_v \Phi) \, dudv.$$

It is easy to check that

$$\langle \vec{F} \rangle_{\mathscr{M}} = \int_D \det \left[ \vec{F}(\Phi(u,v)) \; \partial_u \Phi(u,v) \; \partial_v \Phi(u,v) \right] \, dudv. \tag{5.2.4}$$

Example 5.2.2. *Compute the flux of $\vec{F} = (x,y,z)$ through the paraboloid $z = x^2 + y^2$ included between $z = 1$ and $z = 2$.*

Sol. — On a $\mathscr{M} = \Phi(D)$ où $\Phi(u,v) = (u,v,u^2+v^2)$ et $(u,v) \in D = \{1 \leqslant u^2+v^2 \leqslant 2\}$. Par la (5.2.4)

$$\begin{aligned} \langle \vec{F} \rangle_{\mathscr{M}} &= \int_{1 \leqslant u^2+v^2 \leqslant 2} \det \begin{bmatrix} u & 1 & 0 \\ v & 0 & 1 \\ u^2+v^2 & 2u & 2v \end{bmatrix} dudv = \int_{1 \leqslant u^2+v^2 \leqslant 2} \left( -2u^2 - 2v^2 - (u^2+v^2) \right) dudv \\ &= -3 \int_{1 \leqslant u^2+v^2 \leqslant 2} (u^2+v^2) \, dudv = -3 \int_0^{2\pi} \left( \int_1^{\sqrt{2}} \rho^2 \cdot \rho \, d\rho \right) d\theta = -6\pi \left[ \frac{\rho^4}{4} \right]_{\rho=1}^{\rho=\sqrt{2}} = -\frac{9}{2}\pi. \quad \blacksquare \end{aligned}$$

A special but important case is when $\mathscr{M}$ is described as graph of a regular function, that is

$$\mathscr{M} := \{(x,y,\varphi(x,y)) \; : \; (x,y) \in D \subset \mathbb{R}^2\}.$$

In this case a natural parametrization is $\Phi(x,y) := (x,y,\varphi(x,y))$ and

$$\vec{n} = \frac{(1,0,\partial_x \varphi) \times (0,1,\partial_y \varphi)}{\|(1,0,\partial_x \varphi) \times (0,1,\partial_y \varphi)\|},$$

and being

$$(1,0,\partial_x \varphi) \times (0,1,\partial_y \varphi) = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \partial_x \varphi \\ 0 & 1 & \partial_y \varphi \end{bmatrix} = (-\partial_x \varphi, -\partial_y \varphi, 1)$$

we obtain

$$\vec{n} = \frac{(-\partial_x \varphi, -\partial_y \varphi, 1)}{\sqrt{1 + \|\nabla \varphi\|^2}}.$$

The (5.2.4) take the form

$$\langle \vec{F} \rangle_{\mathscr{M}} = \int_D \vec{F} \cdot (-\partial_x \varphi, -\partial_y \varphi, 1) \, dxdy, \tag{5.2.5}$$

where, of course, $\vec{F} = \vec{F}(x,y,\varphi(x,y))$.

## 5.3. Outward flux

Of great importance is the concept of *outward flux of a vector field* $\vec{F}$ *by a domain* $\Omega \subset \mathbb{R}^3$. Let $\Omega \subset \mathbb{R}^3$ be an *open and bounded domain* such that $\partial\Omega$ be a parametric manifold, that is $\partial\Omega = \Phi(D)$ with $\Phi : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be $\mathscr{C}^1$ and immersive. We would like to define

$$\int_{\partial D} \vec{F} \cdot \vec{n}_e \, d\sigma_2.$$

where $\vec{n}_e$ stands for the outward normal unit vector to $\partial\Omega$. Intuitively, it is clear what it should be an outward direction, but we need a formal definition and to see under which condition this indeed exists.

To simplify our discussion we will consider the quite common case of an open domain of type

$$\Omega = \left\{(x, y, z) \in \mathbb{R}^3 \ : \ g(x, y, z) < 0\right\},$$

with $g$ regular enough. It is not difficult to check that

$$\partial\Omega = \{(x, y, z) \ : g(x, y, z) = 0\}.$$

For instance, the open ball centered in the origin with radius $r$ is $\Omega = \left\{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 + z^2 - r^2 < 0\right\}$. Here $g(x, y, z) := x^2 + y^2 + z^2 - r^2$ and $\partial\Omega$ is the sphere of radius $r$.

In this setting $\partial\Omega = \{g = 0\}$ is the level set of a numerical function $g$. This type of set sound familiar: going back to the Dini thm, under suitable conditions this set locally looks like the graph of some function. We could have $x = X(y, z)$ or $y = Y(x, z)$ or again $z = Z(x, y)$. The first is possible, according to Dini thm, if $\partial_x g \neq 0$; the second if $\partial_y g \neq 0$ while the third demands $\partial_z g \neq 0$. Therefore: *if one among*

$$\partial_x g, \ \partial_y g, \ \partial_z g$$

*is different from* 0*, the set* $\{g = 0\}$ *appears locally as a graph*. Now, recalling that the three quantities are nothing but the three components of the vector $\nabla g$ the previous condition just reads as

$$\nabla g \neq 0.$$

We recall that in this case we say that $g$ is a *submersion* on $\partial\Omega$ and this is a *differential manifold of dimension* $3 - 1 = 2$*, that is a surface*. Why this is helpful? Because if we know that $\{g = 0\}$ is a graph we can see what is a normal direction to it. Indeed, suppose for instance that

$$\{g = 0\} = \{z = Z(x, y)\} = \Phi(D), \ \ \text{where } \Phi(x, y) = (x, y, Z(x, y)).$$

Therefore a normal direction is given by

$$\partial_x\Phi \times \partial_y\Phi = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \partial_x Z \\ 0 & 1 & \partial_y Z \end{bmatrix} = (-\partial_x Z, -\partial_y Z, 1).$$

Recalling again the Dini thm

$$\partial_x Z = \frac{\partial_x g}{\partial_z g}, \ \ \partial_y Z = \frac{\partial_y g}{\partial_z g},$$

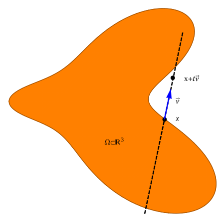
hence

$$\partial_x\Phi \times \partial_y\Phi = \left(\frac{\partial_x g}{\partial_z g}, \frac{\partial_y g}{\partial_z g}, -1\right) = \frac{1}{\partial_z g}\left(\partial_x g, \partial_y g, \partial_z g\right) = c\nabla g.$$

Therefore
$$\vec{n} = \frac{c\nabla g}{\|c\nabla g\|} = \frac{\nabla g}{\|\nabla g\|}$$
is a unit normal to $\partial\Omega$. The computation lead to the same result (of course) if $\{g = 0\} = \{x = X(y,z)\}$ or $\{g = 0\} = \{y = Y(x,z)\}$ (exercise).

Now, the question is: is $\vec{n} = \frac{\nabla g}{\|\nabla g\|}$ outward or inward? Of course we have first to define what is an outward direction:

Definition 5.3.1. *Let $\Omega \subset \mathbb{R}^3$ open and bounded, $P \in \partial\Omega$. We say that $\vec{v} \in \mathbb{R}^3$ is* **outward direction by** *$\Omega$ if*
$$P + t\vec{v} \notin \Omega,\ \forall t \in ]0,T],\ \textit{for some } T > 0.$$




We can now give the answer to the above question.

Proposition 5.3.2. *Let $g \in \mathscr{C}^1(\mathbb{R}^3)$, $\Omega := \{g < 0\}$ and $g$ be such that $\nabla g \neq 0$ on $\partial\Omega = \{g = 0\}$. Then*
$$\vec{n}_e = \frac{\nabla g}{\|\nabla g\|}$$
*is outward normal unit vector on $\partial\Omega$.*

Proof — Let $P \in \partial\Omega$, that is $g(P) = 0$ and consider the straight line passing through $P$ with direction $\vec{n}_e := \frac{\nabla g}{\|\nabla g\|}$, that is
$$\gamma(t) = P + t\vec{n}_e(P) = P + t\frac{\nabla g(P)}{\|\nabla g(P)\|},\ t \in \mathbb{R}.$$
Let's check that $\gamma(t) \notin \Omega$ if $t \in [0,T]$ for some $T$. Notice that $\gamma(t) \notin \Omega$ iff $g(\gamma(t)) \geqslant 0$. First of all $g(\gamma(0)) = g(P) = 0$ and because
$$\frac{d}{dt}g(\gamma(t)) = \nabla g(\gamma(t)) \cdot \gamma'(t) = \nabla g(\gamma(t)) \cdot \vec{n}_e(P),$$
we have
$$\frac{d}{dt}g(\gamma(t))\Big|_{t=0} = \nabla g(P) \cdot \vec{n}_e(P) = \nabla g(P) \cdot \frac{\nabla g(P)}{\|\nabla g(P)\|} = \|\nabla g(P)\| > 0.$$
Therefore $g(\gamma(t)) \nearrow$ on some $[0,T,]$, that is $g(\gamma(t)) > g(\gamma(0)) = g(P) = 0$ if $t \in ]0,T]$. ■

Example 5.3.3. In the case of the ball $\Omega = \{(x,y,z) \in \mathbb{R}^3\ :\ x^2 + y^2 + z^2 < r^2\}$, $g(x,y,z) = x^2 + y^2 + z^2 - r^2$, hence $\nabla g = (2x, 2y, 2z) = 0$ iff $(x,y,z) = 0_3 \notin \partial\Omega$. Therefore
$$\vec{n}_e(x,y,z) = \frac{\nabla g}{\|\nabla g\|} = \frac{2(x,y,z)}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{(x,y,z)}{\sqrt{x^2 + y^2 + z^2}}.$$

It is easy to check directly that $\vec{n}_e$ is outward. ■

What can be said on a general open and bounded set $\Omega \subset \mathbb{R}^3$. Looking carefully to the previous argument we can see that the conclusion ($\vec{n}_e = \frac{\nabla g}{\|\nabla g\|}$) still holds if $\Omega = \{g < 0\}$ *just in a neighborhood of* $P \in \partial\Omega$. Let's give a name to this property:

Definition 5.3.4. *An open and bounded set $\Omega \subset \mathbb{R}^3$ is called* **stokian** *if*

$$\forall P \in \partial\Omega,\ \exists U_P,\ \exists g : \mathbb{R}^3 \longrightarrow \mathbb{R}\ :\ \Omega \cap U_P = \{g < 0\},$$

*with $\nabla g \neq 0$ on the set $\{g = 0\}$.*

## 5.4. Divergence theorem

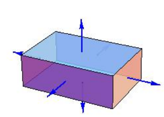
The following thm is a central tool to compute outward fluxes: it allows to transform the computation of an outward flux (which is in general not easy to perform) into a volume integral (in general much easier). In practice, the divergence thm is a multidimensional version of the fundamental formula of integral calculus.

Theorem 5.4.1 (Gauss). *Let $\Omega \subset \mathbb{R}^3$ be a stokian open and bounded set and let $\vec{F} = (f, g, h) : \Omega \cup \partial\Omega \longrightarrow \mathbb{R}^3$ be a $\mathscr{C}^1$ vector field on $\Omega$ continuous on $\partial\Omega$. Then*

$$\int_{\partial\Omega} \vec{F} \cdot \vec{n}_e \, d\sigma_2 = \int_{\Omega} \mathrm{div}\vec{F}. \tag{5.4.1}$$

*where* $\mathrm{div}\vec{F} = \partial_x f + \partial_y g + \partial_z h$ *is called* **divergence of** $\vec{F}$.

Proof — The proof in the general case of $\Omega$ is hard and complex. We will do on a very simple case that however will show the basic idea. Let's consider $\Omega$ be a parallelepiped $\Omega = ]a_1, b_1[\times]a_2, b_2[\times]a_3, b_3[$ and let's call $\mathscr{M}_1^{\pm}$, $\mathscr{M}_2^{\pm}$ and $\mathscr{M}_3^{\pm}$ the opposite sides.



Then

$$\langle \vec{F} \rangle_{\partial\Omega} = \int_{\mathscr{M}_1^+} \vec{F}\cdot\vec{n}_e \, d\sigma_2 + \int_{\mathscr{M}_1^-} \vec{F}\cdot\vec{n}_e \, d\sigma_2 + \int_{\mathscr{M}_2^+} \vec{F}\cdot\vec{n}_e \, d\sigma_2 + \int_{\mathscr{M}_2^-} \vec{F}\cdot\vec{n}_e \, d\sigma_2 + \int_{\mathscr{M}_3^+} \vec{F}\cdot\vec{n}_e \, d\sigma_2 + \int_{\mathscr{M}_3^-} \vec{F}\cdot\vec{n}_e \, d\sigma_2.$$

Let's take the outward flux by $\mathscr{M}_1^{\pm}$ that can be described as

$$\mathscr{M}_1^+ = \{(b_1, y, z)\ :\ y \in [a_2, b_2],\ z \in [a_3, b_3]\}, \quad \mathscr{M}_1^- = \{(a_1, y, z)\ :\ y \in [a_2, b_2],\ z \in [a_3, b_3]\}.$$

It is clear that $\vec{n}_e = (1,0,0)$ on $\mathscr{M}_1^+$ while $\vec{n}_e = (-1,0,0)$ on $\mathscr{M}_1^-$. Therefore

$$\int_{\mathscr{M}_1^+} \vec{F}\cdot\vec{n}_e\, d\sigma_2 + \int_{\mathscr{M}_1^-} \vec{F}\cdot\vec{n}_e\, d\sigma_2$$

$$= \int_{(y,z)\in[a_2,b_2]\times[a_3,b_3]} (f,g,h)\cdot(1,0,0)\, dydz + \int_{(y,z)\in[a_2,b_2]\times[a_3,b_3]} (f,g,h)\cdot(-1,0,0)\, dydz$$

$$= \int_{(y,z)\in[a_2,b_2]\times[a_3,b_3]} f(b_1,y,z)\, dydz - \int_{(y,z)\in[a_2,b_2]\times[a_3,b_3]} f(a_1,y,z)\, dydz$$

$$= \int_{(y,z)\in[a_2,b_2]\times[a_3,b_3]} (f(b_1,y,z) - f(a_1,y,z))\, dydz.$$

Now, by the fundamental formula of integral calculus

$$f(b_1,y,z) - f(a_1,y,z) = \int_{a_1}^{b_1} \partial_x f(x,y,z)\, dx,$$

hence

$$\int_{\mathscr{M}_1^+} \vec{F}\cdot\vec{n}_e\, d\sigma_2 + \int_{\mathscr{M}_1^-} \vec{F}\cdot\vec{n}_e\, d\sigma_2 = \int_{(y,z)\in[a_2,b_2]\times[a_3,b_3]} \left(\int_{a_1}^{b_1} \partial_x f(x,y,z)\, dx\right) dydz = \int_\Omega \partial_x f.$$

Similarly

$$\int_{\mathscr{M}_2^+} \vec{F}\cdot\vec{n}_e\, d\sigma_2 + \int_{\mathscr{M}_2^-} \vec{F}\cdot\vec{n}_e\, d\sigma_2 = \int_\Omega \partial_y g, \quad \int_{\mathscr{M}_3^+} \vec{F}\cdot\vec{n}_e\, d\sigma_2 + \int_{\mathscr{M}_3^-} \vec{F}\cdot\vec{n}_e\, d\sigma_2 = \int_\Omega \partial_z h.$$

Summing up these formulas we obtain finally

$$\int_{\partial\Omega} \vec{F}\cdot\vec{n}_e\, d\sigma = \int_\Omega \left(\partial_x f + \partial_y g + \partial_z h\right) = \int_\Omega \mathrm{div}\vec{F}. \quad \blacksquare$$

Example 5.4.2. *Compute the outward flux by* $D = \left\{(x,y,z)\in\mathbb{R}^3 \;:\; \frac{\sqrt{x^2+(y/2)^2}-3}{4} + z < 1\right\}$ *of the vector field* $\vec{F}(x,y,z) = (x,y,z^2)$.

Sol. — By divergence thm

$$\int_{\partial\Omega} \vec{F}\cdot\vec{n}_e\, d\sigma_2 = \int_\Omega \mathrm{div}\vec{F}\, dxdydz = \int_\Omega (1+1+2z)\, dxdydz = 2\int_\Omega dxdydz + 2\int_D z\, dxdydz.$$

The second integral vanishes being $\Omega$ invariant by symmetry $(x,y,z) \longmapsto (x,y,-z)$ while the integrand is odd. The first integral is instead the volume of $\Omega$: by using adapted cylindrical coordinates,

$$\begin{cases} x = \rho\cos\theta, \\ \frac{y}{2} = \rho\sin\theta, \\ z = z, \end{cases} \iff \begin{cases} x = \rho\cos\theta, \\ y = 2\rho\sin\theta, \\ z = z. \end{cases}$$

we have

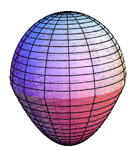
$$\Omega_{(\rho,\theta,z)} = \left\{(\rho,\theta,z)\in\mathbb{R}_+\times[0,2\pi]\times\mathbb{R} \;:\; \frac{\rho-3}{4} + z^2 \leqslant 1, \iff \rho \leqslant 4(1-z^2)\right\}.$$

Notice that this last condition forces $z \in [-1,1]$ whence

$$\begin{aligned}\int_D dxdydz &= \int_{D_{(\rho,\theta,z)}} 2\rho\, d\rho d\theta dz = 2\int_{-1}^{1} dz \left(\int_0^{2\pi} d\theta \left(\int_0^{4(1-z^2)} \rho\, d\rho\right)\right) = 2\pi \int_{-1}^{1} 16(1-z^2)^2\, dz \\ &= 2\pi \int_{-1}^{1} 16(1-2z^2+z^4)\, dz = 2\pi\left(32 - \frac{64}{3} + \frac{32}{5}\right). \quad \blacksquare\end{aligned}$$

Exercise 5.4.3. *Let $\Omega := \{(x,y,z) \in \mathbb{R}^3 \ :\ x^2+y^2 < z < 1 + \sqrt{1-(x^2+y^2)}\}$. Compute the outward flux by $\Omega$ of the vector field $\vec{F}(x,y,z) := (x,y,x^2+y^2)$, computing also all the part of the flux relative to the several portions of $\partial\Omega$.*

Sol. — $\Omega$ is the region included between the paraboloid $z = x^2+y^2$ and the ball centered at $(0,0,1)$ with radius 2.



By divergence thm

$$\langle \vec{F} \rangle_{\partial\Omega} = \int_{\partial\Omega} \vec{F}\cdot\vec{n}_e\, d\sigma = \int_\Omega \mathrm{div}\vec{F}\, dxdydz = \int_\Omega (1+1+0)\, dxdydz = 2m_3(\Omega).$$

Let's compute $m_3(\Omega)$. In cylindrical coordinates

$$m_3(\Omega) = \int_\Omega dxdydz = \int_{\Omega_{cil}} \rho\, d\rho d\theta dz,$$

where

$$\Omega_{cil} = \left\{(\rho,\theta,z) \in [0,+\infty[\times[0,2\pi]\times\mathbb{R} \ :\ \rho^2 \leqslant z \leqslant 1 + \sqrt{1-\rho^2}\right\}.$$

Of course one need that $1-\rho^2 \geqslant 0$, that is $0 \leqslant \rho \leqslant 1$ and also $\rho^2 \leqslant 1+\sqrt{1-\rho^2}$, always true being $\rho \leqslant 1$. Therefore

$$\begin{aligned} m_3(\Omega) &= \int_{0\leqslant\rho\leqslant1,\ 0\leqslant\theta\leqslant2\pi,\ \rho^2\leqslant z\leqslant 1+\sqrt{1-\rho^2}} \rho\, d\rho d\theta dz \overset{FT}{=} \int_0^{2\pi}\left(\int_0^1\left(\int_{\rho^2}^{1+\sqrt{1-\rho^2}} \rho\, dz\right) d\rho\right) d\theta \\ &= 2\pi\int_0^1 \rho\left(1+\sqrt{1-\rho^2}-\rho^2\right) d\rho = 2\pi\left(\left[\frac{\rho^2}{2}\right]_0^1 + \left[-\frac{1}{3}(1-\rho^2)^{3/2}\right]_0^1 - \left[\frac{\rho^4}{4}\right]_0^1\right) \\ &= 2\pi\left(\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\right) = \frac{7}{6}\pi. \end{aligned}$$

Let's compute now the component of the flux outgoing by the paraboloid. This is described by the equation $z = x^2 + y^2$ and the part interior to $\Omega$ is $z > x^2 + y^2$, that is $g(x, y, z) := x^2 + y^2 - z < 0$. Therefore,

$$\vec{n}_e = \frac{\nabla g}{\|\nabla g\|} = \frac{(2x, 2y, -1)}{\sqrt{1 + 4(x^2 + y^2)}}.$$

Hence, calling $\mathscr{P}$ the paraboloid, we have

$$\int_{\mathscr{P}} \vec{F} \cdot \vec{n}_e \, d\sigma_2 = \int_{\mathscr{P}} (x, y, x^2 + y^2) \cdot \frac{(2x, 2y, -1)}{\sqrt{1 + 4(x^2 + y^2)}} \, d\sigma_2 = \int_{\mathscr{P}} \frac{x^2 + y^2}{\sqrt{1 + 4(x^2 + y^2)}} \, d\sigma_2.$$

Now, $\mathscr{P}$ is the graph of $(x, y) \longmapsto x^2 + y^2$ on the domain $D = x^2 + y^2 \leqslant 1$: therefore

$$\begin{aligned} \int_{\mathscr{P}} \frac{x^2 + y^2}{\sqrt{1 + 4(x^2 + y^2)}} \, d\sigma_2 &= \int_D \frac{x^2 + y^2}{\sqrt{1 + 4(x^2 + y^2)}} \sqrt{1 + 4(x^2 + y^2)} \, dxdy = \int_D (x^2 + y^2) \, dxdy \\ &= \int_0^1 \left( \int_0^{2\pi} \rho^3 \, d\theta \right) d\rho = \frac{\pi}{2}. \end{aligned}$$

The remaining component of the flux (that one relative to the half sphere) can be now deduce by difference by the total outward flux. ■

**5.4.1. Gauss theorem on central fields.** A beautiful application of the divergence thm is the Gauss thm concerning the outward flux of a central field. This result is very important for gravitation and electric forces.

THEOREM 5.4.4 (GAUSS). *Let*

$$\vec{F}(x, y, z) = F_0 \frac{(x, y, z)}{\|(x, y, z)\|^3}, \quad (x, y, z) \in \mathbb{R}^3 \backslash \{0_3\}.$$

*Then, for any stokian open set* $\Omega \subset \mathbb{R}^3$,

$$\langle \vec{F} \rangle_{\partial\Omega} = \begin{cases} 0, & \textit{if } 0_3 \in \Omega^c \backslash \partial\Omega, \\ \\ 4\pi F_0, & \textit{if } 0_3 \in \Omega. \end{cases}$$

PROOF — Assume first that $0_3 \in \Omega^c \backslash \partial\Omega$. Then $\vec{F} \in \mathscr{C}^1(\Omega) \cap \mathscr{C}(\partial\Omega)$. By divergence thm

$$\langle \vec{F} \rangle_{\partial\Omega} = \int_{\partial\Omega} \vec{F} \cdot \vec{n}_e \, d\sigma_2 = \int_{\Omega} \text{div}\vec{F} \, dxdydz.$$

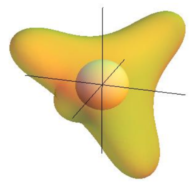
Now: the key point is that $\text{div}\vec{F} = 0$. Indeed,

$$\begin{aligned} \text{div}\vec{F} &= F_0 \left( \partial_x \frac{x}{(x^2 + y^2 + z^2)^{3/2}} + \partial_y \frac{y}{(x^2 + y^2 + z^2)^{3/2}} + \partial_z \frac{z}{(x^2 + y^2 + z^2)^{3/2}} \right) \\ &= F_0 \left( \frac{(x^2 + y^2 + z^2) - 3x^2}{(x^2 + y^2 + z^2)^{5/2}} + \frac{(x^2 + y^2 + z^2) - 3y^2}{(x^2 + y^2 + z^2)^{5/2}} + \frac{(x^2 + y^2 + z^2) - 3z^2}{(x^2 + y^2 + z^2)^{5/2}} \right) = 0. \end{aligned}$$

Therefore we deduce $\langle \vec{F} \rangle_{\partial\Omega} = 0$.

Let's pass to the other case: $0_3 \in \Omega$. In this case we cannot say that $\vec{F} \in \mathscr{C}^1(\Omega)$ because $\Omega$ contains $0_3$ which is the singularity of $\vec{F}$, so we cannot apply the divergence thm on $\Omega$. However we can take out a suitably small ball $B(0_3, \rho] \subset \Omega$ (it exists because $\Omega$ is open and $0_3 \in \Omega$) and define

$$\widetilde{\Omega} := \Omega \backslash B(0, \rho].$$



It is clear that $\widetilde{\Omega}$ is still a stokian open set and $\partial\widetilde{\Omega} = \partial\Omega \cup \partial B(0,\rho]$ and now $\vec{F} \in \mathscr{C}^1(\widetilde{\Omega}) \cap \mathscr{C}(\partial\widetilde{\Omega})$. By the divergence thm then

$$0 = \int_{\widetilde{\Omega}} \text{div}\vec{F} = \int_{\partial\widetilde{\Omega}} \vec{F}\cdot\vec{n}_e \, d\sigma_2 = \int_{\partial\Omega} \vec{F}\cdot\vec{n}_e \, d\sigma_2 - \int_{\partial B(0,\rho]} \vec{F}\cdot\vec{n}_e \, d\sigma_2.$$

The agreement here is that in the second integral $\vec{n}_e$ stand for the outward normal unit vector by $B(0,\rho]$ (therefore the outward normal by the ball is inward for $\widetilde{\Omega}$. Now, because

$$\int_{\partial B(0,\rho]} \vec{F}\cdot\vec{n}_e \, d\sigma_2 = \int_{\partial B(0,\rho]} F_0 \frac{(x,y,z)}{\|(x,y,z)\|^3} \cdot \frac{(x,y,z)}{\|(x,y,z)\|} \, d\sigma_2 = F_0 \int_{\partial B(0,\rho]} \frac{1}{\|(x,y,z)\|^2} \, d\sigma_2 = F_0 \frac{1}{\rho^2} 4\pi\rho^2 = 4\pi F_0,$$

we conclude that $\int_{\partial\Omega} \vec{F}\cdot\vec{n}_e \, d\sigma_2 = 4\pi F_0$. ■

Let's consider for instance the case of the electric field generated by a point charge $q_0$ at point $(x_0, y_0, z_0)$, that is

$$\vec{E}(x,y,z) = \frac{q_0}{\varepsilon} \frac{(x-x_0, y-y_0, z-z_0)}{\|(x-x_0, y-y_0, z-z_0)\|}$$

where $\varepsilon$ is the *permittivity*. By the Gauss thm

$$\langle \vec{E} \rangle_{\partial\Omega} = \begin{cases} 4\pi \frac{q_0}{\varepsilon}, & \text{if } (x_0,y_0,z_0) \in \Omega, \\ \\ 0, & \text{if } (x_0,y_0,z_0) \in \Omega^c \backslash \partial\Omega. \end{cases}$$

Let $\varrho = \varrho(x,y,z)$ be now a charge density. We'll have

$$\langle \vec{E} \rangle_{\partial\Omega} = \frac{4\pi}{\varepsilon} \int_\Omega \varrho.$$

On the other hand, by divergence thm,

$$\langle \vec{E} \rangle_{\partial\Omega} = \int_{\partial\Omega} \vec{E}\cdot\vec{n}_e \, d\sigma = \int_\Omega \text{div}\vec{E},$$

whence

$$\int_\Omega \text{div}\vec{E} = \int_\Omega \frac{4\pi}{\varepsilon}\varrho, \ \forall\Omega, \iff \text{div}\vec{E} = \frac{4\pi}{\varepsilon} q.$$

This is the first of Maxwell equations, the fundamental equations of the electro-magnetic field.

**5.4.2. Gradient theorem.** By the divergence thm we have the

Corollary 5.4.5. *Let $\Omega \subset \mathbb{R}^3$ be a stokian open and bounded set and let $f : \Omega \cup \partial\Omega \longrightarrow \mathbb{R}$ be a function $\mathscr{C}^1$ on $\Omega$ and continuous on $\partial\Omega$. Then*

$$\int_\Omega \nabla f = \int_{\partial\Omega} f\vec{n}_e \, d\sigma_2.$$

Proof — Let's check the first component of the stated identity, that is

$$\int_\Omega \partial_x f = \int_{\partial\Omega} f(\vec{n}_e)_x$$

where $(\vec{n}_e)_x$ is the first component of $\vec{n}_e$. To this aim introduce the vector field $\vec{F} := (f, 0, 0)$. Then,

$$\mathrm{div}\vec{F} = \partial_x f + \partial_y 0 + \partial_z 0 = \partial_x f,$$

whence

$$\int_\Omega \partial_x f = \int_\Omega \mathrm{div}\vec{F} = \int_{\partial\Omega} \vec{F}\cdot\vec{n}_e = \int_{\partial\Omega} (f, 0, 0)\cdot\vec{n}_e = \int_{\partial\Omega} f(\vec{n}_e)_x. \quad \blacksquare$$

A nice application of this Corollary is the well known

Corollary 5.4.6 (Archimedean principle). *A body immersed in an incompressible homogeneous fluid experiences a buoyant force equal to the weight of the displaced fluid.*

Proof — Assume that the fluid has a constant density per unit of volume $\varrho$ and let $p = p(x, y, z)$ the pressure (weight force per unit of surface) exercised by the fluid at point $(x, y, z)$. For simplicity we may assume that the $z$ axis corresponds to the vertical axis and $z = 0$ correspond to the surface of the fluid (assumed to be flat). Therefore, if $p_0$ (constant) is the atmospheric pressure on the surface of the fluid,

$$p(x, y, z) = -\varrho g z + p_0.$$

Now, assume that the body is represented by an open and bounded stokian set $\Omega \subset \{z \leqslant 0\}$. The resultant of the pressure forces on the body is therefore

$$-\int_{\partial\Omega} p\vec{n}_e \, d\sigma_2.$$

The $-$ gives account of the fact that are counted positively pressures inward directed to $\Omega$. Now, by the gradient thm

$$-\int_{\partial\Omega} p\vec{n}_e \, d\sigma_2 = -\int_\Omega \nabla p \, dxdydz = -\int_\Omega (0, 0, -\varrho g) \, dxdydz = \varrho g m_3(\Omega)\vec{k},$$

that is the resultant of the pressure forces is upward oriented with intensity $\varrho g m_3(\Omega)$ which is nothing but the weight of the body. $\blacksquare$

### 5.5. Green formula

The divergence theorem hods also in the case of a plane domain $\Omega \subset \mathbb{R}^2$. In this case $\partial\Omega$ is a $1-$dim. surface, that is a curve that can be parametrized by $\gamma = \gamma(t) : [a, b] \longrightarrow \mathbb{R}^2$. Proceeding similarly to the surface integrals, we define

$$\int_{\partial\Omega} f \, d\sigma_1 := \int_a^b f(\gamma(t))\|\gamma'(t)\| \, dt.$$

With this agreement,

Theorem 5.5.1. *Let $\Omega \subset \mathbb{R}^2$ be a stokian open and bounded set and let $\vec{F} = (f, g) : \Omega \cup \partial\Omega \longrightarrow \mathbb{R}^2$ be a $\mathscr{C}^1$ vector field on $\Omega$ continuous on $\partial\Omega$. Then*

$$\int_{\partial\Omega} \vec{F} \cdot \vec{n}_e \, d\sigma_1 = \int_\Omega \mathrm{div}\vec{F}. \tag{5.5.1}$$

*where* $\mathrm{div}\vec{F} = \partial_x f + \partial_y g$ *is called* **divergence of** $\vec{F}$.

This theorem has an important application in relation with another integral operation on vector fields:

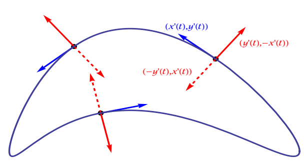
Definition 5.5.2. *Let $\vec{F} \in \mathscr{C}(D)$ be a vector field on $D \subset \mathbb{R}^d$ open set. If $\gamma \in \mathscr{C}^1([a, b])$ is a regular curve in $D$ (that is $\gamma(t) \in D$ for any $t \in [a, b]$), we define the* **line integral of $\vec{F}$ along $\gamma$**

$$\int_\gamma \vec{F} := \int_a^b \vec{F}(\gamma(t)) \cdot \gamma'(t) \, dt \equiv \sum_{k=1}^d \int_a^b f_k(\gamma(t))\gamma_k'(t) \, dt.$$

*We say that $\gamma$ is a* **circuit** *if $\gamma(b) = \gamma(a)$. In this case we call* **circulation** *of $\vec{F}$ along $\gamma$ the integral*

$$\oint_\gamma \vec{F} := \int_\gamma \vec{F}.$$

Now, assume that $\partial\Omega \equiv \gamma([a, b])$ where $\gamma : [a, b] \subset \mathbb{R} \longrightarrow \mathbb{R}^2$ be a regular circuit (that is $\partial\Omega$ be the trace of the circuit $\gamma$).




We notice that if $\gamma(t) = (x(t), y(t))$, then $\gamma'(t) = (x'(t), y'(t))$ is tangent to $\gamma$ hence the vectors

$$(-y'(t), x'(t)), \quad (y'(t), -x'(t))$$

are perpendicular to $\gamma$. Of course, being opposite, one will be outward and the other inward to $\Omega$. The question is, of course: which one is the outward? The intuition suggests that *if $\gamma$ is counter-clock wise oriented, then $(y'(t), -x'(t))$ will be outward by $\Omega$, otherwise it will be $(-y'(t), x'(t))$ the outward vector*. Assuming the counter-clock orientation and that $\gamma' \neq 0$ always,

$$\vec{n}_e(x(t), y(t)) = \frac{(y'(t), -x'(t))}{\|(y'(t), -x'(t))\|} = \frac{(y'(t), -x'(t))}{\sqrt{x'(t)^2 + y'(t)^2}} = \frac{(y'(t), -x'(t))}{\|\gamma'(t)\|}.$$

In this way

$$\int_{\partial\Omega}(f,g)\cdot\vec{n}_e\,d\sigma_1 = \int_a^b (f,g)\cdot\frac{(y'(t),-x'(t))}{\|\gamma'(t)\|}\,\|\gamma'(t)\|\,dt = \int_a^b (f,g)\cdot(y',-x') =$$

$$= \int_a^b (-g,f)\cdot(x',y') = \oint_\gamma (-g,f).$$

By this remark we derive the

Theorem 5.5.3 (Green). *Let $\vec{F}=(f,g)$ be a vector field $\mathscr{C}^1(\Omega)$ with $\Omega\subset\mathbb{R}^2$ open and bounded and such that $\partial\Omega\equiv\gamma$ be a circuit. Then*

$$\oint_\gamma \vec{F} = \int_\Omega \left(\partial_x g - \partial_y f\right) \tag{5.5.2}$$

Proof — By the preliminaries we have that

$$\oint_\gamma \vec{F} = \int_a^b (f,g)\cdot(x',y') = \int_{\partial\Omega}(g,-f)\cdot\vec{n}_e\,d\sigma_1 = \int_\Omega \mathrm{div}(g,-f) = \int_\Omega(\partial_x g-\partial_y f). \quad \blacksquare$$

Example 5.5.4. *Compute the counter-clock circulation of $\vec{F}=(y^3,-x^3)$ on $\gamma=\partial B(0,2]$.*

Sol. — By Green formula

$$\oint_\gamma \vec{F} = \int_{x^2+y^2\leqslant 4}\left(\partial_x(-x^3)-\partial_y(y^3)\right)\,dxdy = -3\int_{x^2+y^2\leqslant 4}(x^2+y^2)\,dxdy = -3\int_0^{2\pi}\left(\int_0^2 \rho^2\,d\rho\right)\,d\theta$$

$$= -6\pi\left[\frac{\rho^3}{3}\right]_{\rho=0}^{\rho=2} = -16. \quad \blacksquare$$

As particular case of the Green formula we obtain the

Corollary 5.5.5 (Area formula). *Let $\Omega\subset\mathbb{R}^2$ be open and bounded such that $\partial\Omega\equiv\gamma$, with $\gamma\in\mathscr{C}^1$. Then*

$$\lambda_2(\Omega) = \oint_\gamma (0,x) \equiv \oint_\gamma x\,dy = -\oint_\gamma y\,dx. \tag{5.5.3}$$

Proof — By the Green formula

$$\oint_\gamma (0,x) = \int_\Omega \mathrm{div}(x,-0) = \int_\Omega 1 = \lambda_2(\Omega). \quad \blacksquare$$

Example 5.5.6. *Compute the area $\Omega$ delimitated by the cycloid $(a(t-\sin t), a(1-\cos t))$, $t\in[0,2\pi]$ and the $x-$axis.*

Sol. — Let's apply the area formula $\lambda_2(\Omega) = -\oint_{\partial\Omega} y\, dx$. The part of this integral relative to the part of $\partial\Omega$ on the axis vanishes because $y \equiv 0$. Considering also the orientation of the cycloid,

$$\lambda_2(\Omega) = \int_0^{2\pi} a(1-\cos t)\cdot a(1-\cos t)\, dt = -a^2 \int_0^{2\pi} (1-\cos t)^2\, dt.$$

Now, $\int_0^{2\pi}(1-\cos t)^2\, dt = 2\pi - 2\int_0^{2\pi}\cos t\, dt + \int_0^{2\pi}(\cos t)^2\, dt = 2\pi + \int_0^{2\pi}(\cos t)^2\, dt$ hence,

$$\int_0^{2\pi}(\cos t)^2\, dt = \int_0^{2\pi}\cos t(\sin t)'\, dt = [\cos t \sin t]_{t=0}^{t=2\pi} - \int_0^{2\pi}(-\sin t)\sin t\, dt = \int_0^{2\pi}(1-(\cos t)^2)\, dt$$

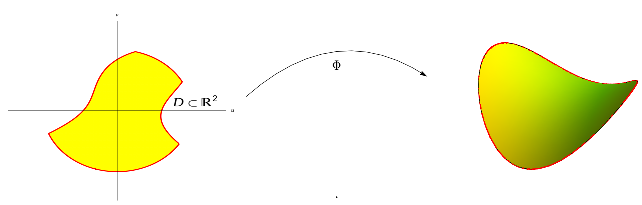
$$= 2\pi - \int_0^{2\pi}(\cos t)^2\, dt,$$

so, finally, $\int_0^{2\pi}(\cos t)^2\, dt = \pi$. We obtain $\lambda_2(\Omega) = 3a^2\pi$. ■

## 5.6. Stokes Formula

The Green formula transform a plane circuitation along $\gamma \subset \mathbb{R}^2$ into a plane integral on a domain $D$ such that its boundary coincides with the circuit $\gamma$. The *Stokes formula* is the extension of Green formula to the case of a circuitation along $\gamma \subset \mathbb{R}^3$. Let's start with the

Definition 5.6.1. *We say that a parametric surface $\mathscr{M} := \Phi(D) \subset \mathbb{R}^3$ has an* **edge** *if $\partial D \equiv \gamma$ circuit. The image of $\gamma$ through $\Phi$ is called edge of $\mathscr{M}$ and it is denoted by $\partial\mathscr{M}$ [1]. We say that $\partial\mathscr{M}$ is counter clock wise oriented iff $\gamma$ is counter-clock wise oriented w.r.t. $D$.*




Theorem 5.6.2 (Stokes Formula). *Let $\mathscr{M}$ be a parametric surface with edge counter-clock wise oriented and let $\vec{F} : D \subset \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a $\mathscr{C}^1$ vector field on $D \supset \mathscr{M}$. Then*

$$\oint_{\partial\mathscr{M}} \vec{F} = \int_{\mathscr{M}} (\nabla\times\vec{F})\cdot\vec{n}\, d\sigma_2, \tag{5.6.1}$$

*where*

$$\nabla\times\vec{F} := \det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ f & g & h \end{bmatrix} = \left(\partial_y h - \partial_z g, \partial_z f - \partial_x h, \partial_x g - \partial_y f\right).$$

*In other words: the circulation of $\vec{F}$ is the flux of $\nabla\times\vec{F}$ (called* **curl** *of $\vec{F}$) through $\mathscr{M}$.*

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[1]Warning: here $\partial\mathscr{M}$ is not the boundary of $\mathscr{M}$.

PROOF — Let's limit to a special (but important) case: $\mathscr{M}$ be the graph of a regular function,

$$\mathscr{M} = \{(x, y, \varphi(x, y)) \ : \ (x, y) \in D \subset \mathbb{R}^2\}, \text{ with } \partial D \equiv \gamma.$$

Assume that $\gamma = \gamma(t) = (x(t), y(t))$ be counter-clock wise oriented. Then

$$\oint_{\partial\mathscr{M}} \vec{F} = \int_a^b (f, g, h) \cdot (x, y, \varphi(x, y))'.$$

Noticed that

$$(x, y, \varphi(x, y))' = (x', y', \partial_x\varphi x' + \partial_y\varphi y'),$$

we have

$$\oint_{\partial\mathscr{M}} \vec{F} = \int_a^b \left(f x' + g y' + h(\partial_x\varphi x' + \partial_y\varphi y')\right) = \int_a^b (f + h\partial_x\varphi, g + h\partial_y\varphi) \cdot (x', y') = \oint_{\partial D} (f + h\partial_x\varphi, g + h\partial_y\varphi).$$

By Green formula then

$$\oint_{\partial\mathscr{M}} \vec{F} = \int_D \left(\partial_x \left(g + h\partial_y\varphi\right) - \partial_y \left(f + h\partial_x\varphi\right)\right).$$

Now, recall that every of $f, g, h$ is evaluated at $(x, y, \varphi(x, y))$. Therefore

$$\partial_x \left(g + h\partial_y\varphi\right) = \partial_x g + \partial_z g\partial_x\varphi + (\partial_x h + \partial_z h\partial_x\varphi)\partial_y\varphi + h\partial_{xy}\varphi,$$

$$\partial_y \left(f + h\partial_x\varphi\right) = \partial_y f + \partial_z f\partial_y\varphi + (\partial_y h + \partial_z h\partial_y\varphi)\partial_x\varphi + h\partial_{xy}\varphi,$$

and by taking their difference we obtain the quantity

$$(\partial_x g - \partial_y f) + \partial_x\varphi(\partial_z g - \partial_y h) + \partial_y\varphi(\partial_x h - \partial_z f) = \nabla \times (f, g, h) \cdot (-\partial_x\varphi, -\partial_y\varphi, 1),$$

hence

$$\oint_{\partial\mathscr{M}} \vec{F} = \int_D (\nabla \times (f, g, h)) \cdot (-\partial_x\varphi, -\partial_y\varphi, 1)\, dxdy \overset{(5.2.5)}{=} \int_{\mathscr{M}} (\nabla \times (f, g, h)) \cdot \vec{n}\, d\sigma_2. \quad \blacksquare$$

EXAMPLE 5.6.3. *Compute the circuitation of* $\vec{F}(x, y, z) := (x^2 z, xy^2, z^2)$ *along* $\gamma = \{x + y + z = 1,\ x^2 + y^2 = 9\}$.

SOL. — Clearly $\gamma$ is an ellipse that we may think as edge of

$$\mathscr{M} = \Phi(D),\ \ \Phi : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3,\ \ \Phi(x, y) = (x, y, 1 - (x + y)),\ \ D := \{(x, y) : x^2 + y^2 \leqslant 9\}.$$

It is easy to take the normal unit vector to $\mathscr{M}$ in such a way that $\gamma = \partial\mathscr{M}$ be counter-clock wise oriented: $\vec{n} = \frac{(1,1,1)}{\sqrt{3}}$. Moreover, being $\mathscr{M}$ graph of $f(x, y) := 1 - (x + y)$ $(x, y) \in D$, we have

$$d\sigma_2 = \sqrt{1 + \|\nabla f(x, y)\|^2}\, dxdy = \sqrt{1 + \|(-1, -1)\|^2}\, dxdy = \sqrt{3}\, dxdy.$$

Therefore being $\nabla \times \vec{F} = (0, x^2, y^2)$, by the curl theorem

$$\oint_\gamma \vec{F} = \int_{\mathscr{M}} \nabla \times \vec{F} \cdot \vec{n}\, d\sigma = \int_{x^2+y^2\leqslant 9} (0, x^2, y^2) \cdot \frac{(1, 1, 1)}{\|(1, 1, 1)\|} \sqrt{3}\, dxdy = \int_{x^2+y^2\leqslant 9} (x^2 + y^2)\, dxdy$$

$$= \int_0^{2\pi} d\theta \int_0^3 \rho^3\, d\rho = 2\pi \left[\frac{\rho^4}{4}\right]_{\rho=0}^{\rho=3} = \pi\frac{3^4}{2}. \quad \blacksquare$$

## 5.7. Exercices

EXERCISE 5.7.1. Compute the area of each of the following surfaces:

- $\mathscr{M} := \left\{(x; y; z) \in \mathbb{R}^3 \ : \ x \geqslant 0,\ z \geqslant 0,\ x + z = 2,\ x^2 + y^2 - 2x \leqslant 0\right\}$.
- $\mathscr{M} := \left\{(x, y, z) \ : z \geqslant 0,\ x^2 + y^2 + z^2 = r^2, \left(x - \frac{r}{2}\right)^2 + y^2 \leqslant \frac{r^2}{4}\right\}$ (Viviani's vault).
- $\mathscr{M}$ rotation surface around the $x-$axis of the graph of the function $f(x) = \alpha x^2$, $x \in [0, a]$, $a, \alpha > 0$.
- $\mathscr{M}$ rotation surface around the $z-$axis of the graph of the function $f(y) = 2 - \cosh y$, $y \in [0, a]$, $a$ such that $f \geqslant 0$.

EXERCISE 5.7.2. Let $\mathscr{M}$ be the rotation surface around the $z-$axis of the curve $y = ze^{-z}$, $z \in [0, h]$ $(h > 0)$. Compute the outward flux of $\vec{F}(x, y, z) := (x + y^2, y + x^2, z)$ through $\mathscr{M}$.

EXERCISE 5.7.3. Let $\Omega := \{(x, y, z) \in \mathbb{R}^3 \ : \ 1 \leqslant z \leqslant 2,\ \sqrt{x^2 + y^2} \leqslant z\}$. Draw $\Omega$ and compute the outward flux of $\vec{F}(x, y, z) := (x^3, 0, z^2)$ by $\Omega$.

EXERCISE 5.7.4. Let $\Omega := \left\{(x, y, z) \in \mathbb{R}^3 \ : \ (x - z)^2 + (y + z)^2 \leqslant z^2,\ 0 \leqslant z \leqslant b\right\}$ $(b > 0)$. Compute the outward flux of $\vec{F}(x, y, z) = (xy, x - y, yz)$ by $\Omega$.

EXERCISE 5.7.5. Let $\Omega := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leqslant z \leqslant 1\}$. Draw $\Omega$, compute its volume and the area of $\partial\Omega$. Hence, compute the outward flux of $\vec{F}(x, y, z) := (x, y, z)$ by $\Omega$, computing the components of the flux by each part of $\partial\Omega$.

EXERCISE 5.7.6. Let $\Omega := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leqslant 1,\ a \leqslant z \leqslant 1\}$ with $0 < a < 1$. Draw $\Omega$, compute its volume and its barycenter. Compute the outward flux of $\vec{F}(x, y, z) := (x^2, -y^2, z)$ by $\Omega$ and each of its components on the several parts of $\partial\Omega$.

EXERCISE 5.7.7. Let $\Omega := \{(x, y, z) \in \mathbb{R}^3 : x^2 + z^2 \leqslant e^y, 0 \leqslant y \leqslant a\}$ with $a > 0$. Draw $\Omega$ and compute outward flux and relative components of $\vec{F}(x, y, z) := (-z, (x^2 + z^2)y, x)$ by $\Omega$.

EXERCISE 5.7.8. Let $\mathscr{M}$ the rotation surface around the $z-$axis of the curve $z = 4 - \cosh(y - 1)$, $y \in [1, 2]$. Calculate its area. Let now

$$\vec{F}(x, y, z) := \frac{(x, y, z)}{\|(x, y, z)\|^3} = \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}},\ \ (x, y, z) \in \mathbb{R}^3 \backslash \{0_3\}.$$

Compute the flux of $\vec{F}$ through $\mathscr{M}$ (hint: introduce an auxiliary disk $\left\{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 = r^2,\ z = a\right\}$ and apply suitably the divergence theorem. . . ).

EXERCISE 5.7.9. Let $\mathscr{M}$ the rotation surface around the $z-$axis of the curve $y = z^2$ $z \in [0, h]$ $(h > 0)$. Compute the area of $\mathscr{M}$ and the flux of $\vec{F}(x, y, z) := (xz, yz, -z)$ through $\mathscr{M}$.

EXERCISE 5.7.10. Let $\Omega := \left\{(x, y, z) \in \mathbb{R}^3 \ : \ z^2 + 6 \leqslant x^2 + y^2 \leqslant 5z\right\}$. Is $\Omega$ is a rotation solid? Draw $\Omega$, compute its volume and the outward flux of $\vec{F}(x, y, z) := (x^2, y, z^2)$ by $\Omega$.

EXERCISE 5.7.11. Let $D := \{(x, y) \in \mathbb{R}^2 : x \geqslant 0,\ \sqrt{1 + x^2} \leqslant y \leqslant 2\}$ and let $\Omega$ the solid obtained by rotating $D$ around the $y-$axis. Compute the area of $\partial\Omega$ and the outward flux by $\Omega$ of $\vec{F}(x, y, z) := (x^2, y, z^2)$.

EXERCISE 5.7.12. Let $\Omega := \{(x, y, z) \in [0, +\infty[^3 : \ 1 \leqslant x^2 + y^2 \leqslant 4 - z^2\}$. Compute the outward flux by $\Omega$ of $\vec{F}(x, y, z) := (x, y, z^2)$. Determine in particular the component of this flux on $\partial\Omega \cap \{x^2 + y^2 + z^2 = 4\}$.

EXERCISE 5.7.13. Compute the outward flux by $\Omega = \{(x, y, z) \in \mathbb{R}^3 \ : \ \max\{|x|, |y|, |z|\} \leqslant 1\}$ of the vector field $\vec{F}(x, y, z) = \frac{(x,y,z)}{|(x,y,z)|^3}$.

EXERCISE 5.7.14. Let $\vec{F}(x,y,z) := (y,x,z)$. Show that $g$ is conservative and find all its potentials on $\mathbb{R}^3$. Now, let

$$\vec{H}(x,y,z) = u\left(x^2+y^2\right)\vec{F}(x,y,z),\ (x,y,z) \in \mathbb{R}^3 \backslash \{(0,0,z)\ :\ z \in \mathbb{R}\},$$

where $u \in \mathscr{C}^1(]0,+\infty[)$.

i) Compute $\oint_\gamma \vec{H}$ where $\gamma(t) := (r\cos t, r\sin t, k)$, $t \in [0,2\pi]$, $r > 0$ and $k \in \mathbb{R}$ fixed parameters.
ii) Find all the possible $u$ such that $\vec{H}$ be irrotational.
iii) Find all the possible $u$ such that $\vec{H}$ be conservative.

EXERCISE 5.7.15. By applying the Green formula compute

1) $\displaystyle\oint_\gamma (xy, 3x+2y),\ \ \gamma \equiv \partial[-1,1]^2.$

2) $\displaystyle\oint_\gamma (\cos x + 6y^2, 3x - e^{-y^2}),\ \ \gamma \equiv \partial B(0,1].$

3) $\displaystyle\oint_\gamma \left(x^2+y, xy\right),\ \ \gamma(t) := (1+\cos t, \sin t),\ t \in [0,2\pi].$

4) $\displaystyle\oint_\gamma (x^3 - y^3, x^3 + y^3),\ \ \gamma = \partial\left(B(0,r] \cap [0,+\infty[^2\right).$

EXERCISE 5.7.16. Compute the area delimitated by the following curves:

1) $t(\cos t, \sin t),\ t \in [0,2\pi].$ 2) $(\sin t + \sin^2 t, -\cos t - \sin t\cos t),\ t \in [0,2\pi].$ 3) $(\cos^2 t, \sin^2 t),\ t \in [0,2\pi].$

4) $(\cos^3 t, \sin^3 t),\ t \in [0,2\pi].$ 5) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ (a,b>0).$ 6)

EXERCISE 5.7.17. By applying the curl theorem compute the $\oint_\gamma \vec{F}$ in the following cases:

1) $\gamma \equiv \begin{cases} x^2+y^2 = 1, \\ x+z = 1, \end{cases}$ $\vec{F} = (y-z, z-x, x-y).$

2) $\gamma \equiv \begin{cases} x+y+z = 1, \\ x^2+y^2 = 9, \end{cases}$ $\vec{F} = (x^2 z, xy^2, z^2)$

3) $\gamma \equiv \begin{cases} x^2+z^2 = 1, \\ y = x^2, \end{cases}$ $\vec{F} = (x^3+z^3, y^2, x^3+z^3).$

4) $\gamma \equiv \begin{cases} x^2+y^2 = 1, \\ z = xy, \end{cases}$ $\vec{F} = (y,z,x)$

5) $\gamma \equiv \begin{cases} x^2+y^2+z^2 = 1, \\ z = \sqrt{x^2+y^2}. \end{cases}$ $\vec{F} = (y^2+z^2, z^2+x^2, x^2+y^2).$

CHAPTER 6

# Vector fields

Consider a function $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$ differentiable on $D$ open set. We know that

$$\nabla f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^d.$$

In this Chapter we will study the following

**Problem:** *given a function $\vec{F} : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^d$ find a function $f : D \longrightarrow \mathbb{R}^d \longrightarrow \mathbb{R}$ differentiable on $D$ such that*

$$\nabla f = \vec{F}, \text{ on } D.$$

Notice that, in dimension $d = 1$, this problem is the analogous of finding a *primitive* for function of one real variable: given $F$, determine $f$ such that $f' = F$. By the Fundamental Theorem of Integral Calculus, if $f \in \mathscr{C}([a, b])$ then the solution always exists and it is given by

$$f(x) = \int_c^x F(y)\, dy.$$

As we will see, the multidimensional version we're going to study in this Chapter, is more involved.

In Physics, a function like $\vec{F}$ is called **vector field** while a function $f$ such that $\nabla f = \vec{F}$ is called *potential* of $\vec{F}$. For instance, the *gravitational field* induced by a point mass $m$ positioned at point $x_0 \in \mathbb{R}^3$ is

$$\vec{F}(x) = -Gm \frac{x - x_0}{\|x - x_0\|^3}, \quad x \in \mathbb{R}^3 \backslash \{x_0\} =: D,$$

where $G$ is the *universal gravitational constant*. A potential is the function

$$f(x) = Gm \frac{1}{\|x - x_0\|}, \ x \in D,$$

as it can be easily checked.

## 6.1. Preliminaries

We start with the

Definition 6.1.1. *A continuous function $\vec{F} : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^d$ on $D$ open set is called* **vector field on** $D$. *We say that $\vec{F}$ is* **conservative on** $D$ *if there exists $f \in \mathscr{C}^1(D)$ such that $\vec{F} = \nabla f$ on $D$. The function $f$ is called* **potential** *of $\vec{F}$.*

If $\vec{F} = (f_1, \dots, f_d)$ then

$$\nabla f = \vec{F}, \iff \begin{cases} \partial_1 f(x) = f_1(x), \\ \vdots \\ \partial_d f(x) = f_d(x), \end{cases} \quad \forall x \in D.$$

Clearly, if $f$ is a potential for $\vec{F}$, also $f + c$, where $c$ is a real constant, is a potential for $\vec{F}$ because $\nabla(f + c) = \nabla f + \nabla c = \vec{F} + \vec{0} = \vec{F}$. In dimension 1, because potentials are the primitives, if the domain is an interval all the potentials differ by an additive constant. In higher dimension this remains true if the domain $D$ is made of one piece, that is if it is connected.

Theorem 6.1.2. *Let $D$ be a connected set and $f, g$ potentials of the vector field $\vec{F} \in \mathscr{C}(D)$. Then $f - g \equiv$constant.*

Let's see how these concepts works on some easy examples.

Example 6.1.3. *Find the potentials for the field*

$$\vec{F}(x, y) = (y, x), \ \ (x, y) \in \mathbb{R}^2 =: D.$$

Sol. — Of course $g \in \mathscr{C}(\mathbb{R}^2)$. We have to find $f \in \mathscr{C}^1(\mathbb{R}^2)$ such that

$$\begin{cases} \partial_x f(x, y) = y, \\ \\ \partial_y f(x, y) = x, \end{cases} \quad (x, y) \in \mathbb{R}^2.$$

Consider the first equation, $\partial_x f(x, y) = y$. We can see this as a problem of one variable primitive and say that

$$f(x, y) = \int y \, dx = yx + c,$$

where $c$ is a free constant: of course constant w.r.t. $x$. Then we may imagine $c = c(y)$, that is

$$f(x, y) = xy + c(y).$$

To find $c$ we use the second equation, $\partial_y f(x, y) = x$. Indeed

$$\partial_y f(x, y) = x, \iff x + c'(y) = x, \iff c'(y) = 0, \iff c(y) \equiv c,$$

and we deduce $f(x, y) = xy + c, \ c \in \mathbb{R}$. ■

## 6.2. Irrotational fields

As reminded, in dimension 1 the problem to find a primitive of a function $F$ has always an answer if $F$ is continuous. Moving to dimensions $\geqslant 2$, things are different: even if $\vec{F}$ is continuous, the problem $\nabla f = \vec{F}$ might not have any solution!

Example 6.2.1. *Show that the field*

$$\vec{F}(x, y) = (y, -x), \ \ (x, y) \in \mathbb{R}^2 =: D,$$

*has not potentials.*

Sol. — Of course $\vec{F} \in \mathscr{C}(\mathbb{R}^2)$. We have to find $f \in \mathscr{C}^1(\mathbb{R}^2)$ such that

$$\begin{cases} \partial_x f(x,y) = y, \\ \\ \partial_y f(x,y) = -x, \end{cases} \quad (x,y) \in \mathbb{R}^2.$$

Consider the first equation, $\partial_x f(x,y) = y$. We can see this as a problem of one variable primitive and say that

$$f(x,y) = \int y\, dx = xy + c(y).$$

Now, by the second equation

$$\partial_y f(x,y) = -x, \iff x + c'(y) = -x, \iff c'(y) = -2x.$$

Now this is impossible because $c$ must be constant in $x$! We deduce that it is impossible that $f$ exists. ■

In the previous Example actually $\vec{F} \in \mathscr{C}^\infty$ (its components are polynomials). The reason why this is not sufficient to ensure the existence of a potential has to be found in a consequence of Schwarz Thm 2.3.9:

Theorem 6.2.2. *Let $\vec{F} \in \mathscr{C}^1(D)$, $\vec{F} = (f_1, \dots, f_d)$ be a conservative vector field. Then*

$$\partial_i f_j \equiv \partial_j f_i, \ \text{ on } D,\ \forall i,j = 1,\dots,n. \tag{6.2.1}$$

Proof — It's just a simple check. Because $\vec{F} = \nabla f$ for some $f \in \mathscr{C}^1(D)$, that is $f_i = \partial_i f$, being $\vec{F} \in \mathscr{C}^1$ it follows that $\partial_i f \in \mathscr{C}^1$, that is $f \in \mathscr{C}^2$. Therefore, by Schwarz Thm,

$$\partial_j f_i = \partial_j \partial_i f \overset{Schwarz}{=} \partial_i \partial_j f = \partial_i f_j. \ ■$$

Definition 6.2.3. *A $\mathscr{C}^1(D)$ vector field $\vec{F}$ fulfilling (6.2.1) is said* **irrotational**.

Remark 6.2.4. Of course, (6.2.1) is interesting for $i \neq j$, because for $i = j$ it is a tautology. For instance, a planar vector field $\vec{F} = (u(x,y), v(x,y))$ is irrotational iff $\partial_y u \equiv \partial_x v$. ■

Example 6.2.5. It is easy to check that in the Examples 6.2.1 the proposed fields are not irrotational. For instance,

$$(y,-x) \text{ is irrotational } \iff \partial_y(y) = \partial_x(-x), \iff 1 = -1,$$

which is false. ■

Therefore, to be conservative the vector field $\vec{F} \in \mathscr{C}^1$ must be first of all irrotational. The natural question is if to be irrotational is a sufficient condition to be conservative? The answer is no!

Example 6.2.6 (Important!). *The field*

$$\vec{F}(x,y) := \left( -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right), \ (x,y) \in \mathbb{R}^2 \backslash \{0_2\}$$

*is irrotational but not conservative on* $\mathbb{R}^2\backslash\{0_2\}$.

Sol. — Let's check first that $\vec{F}$ is irrotational. It is evident that $\vec{F} \in \mathscr{C}^1$ and $\vec{F}$ is irrotational iff

$$\partial_y \left( -\frac{y}{x^2+y^2} \right) \equiv \partial_x \left( \frac{x}{x^2+y^2} \right)$$

We have

$$\partial_y\left(-\frac{y}{x^2+y^2}\right) = -\frac{x^2+y^2-y2y}{(x^2+y^2)^2} = -\frac{x^2-y^2}{x^2+y^2},\ \ \partial_x\left(\frac{x}{x^2+y^2}\right) = -\frac{x^2-y^2}{x^2+y^2}.$$

Therefore $\vec{F}$ is irrotational. Let's assume that a potential $f$ exists. Then

$$\partial_x f(x,y) = -\frac{y}{x^2+y^2},\ \iff\ f(x,y) = \int -\frac{y}{x^2+y^2}\,dx + c(y) = -\frac{1}{y}\int \frac{1}{1+\left(\frac{x}{y}\right)^2}\,dx + c(y) = -\arctan\frac{x}{y} + c(y).$$

This if $y \neq 0$. If $y = 0$,

$$\partial_x f(x,0) = 0,\ \iff\ f(x,0) = c(0),$$

Therefore the candidate is

$$f(x,y) = \begin{cases} -\arctan\frac{x}{y} + c(y), & y \neq 0, \\ \\ c(0), & y = 0. \end{cases}$$

On the other hand, if $y \neq 0$,

$$\partial_y f(x,y) = \frac{x}{x^2+y^2},\ \iff\ \partial_y\left(-\arctan\frac{x}{y} + c(y)\right) = \frac{x}{x^2+y^2},\ \iff\ c'(y) \equiv 0,$$

so $c \equiv C$. We derive then that

$$f(x,y) = \begin{cases} -\arctan\frac{x}{y} + C, & y \neq 0, \\ \\ C, & y = 0. \end{cases}$$

We are done apparently. But... looking carefully to $f$ we see that the $f$ we found is not even continuous! To see this consider a point $(x,0)$ with $x > 0$. If $(x,y) \longrightarrow (x,0)$ with $y \longrightarrow 0+$ then

$$f(x,y) = -\arctan\frac{x}{y} + C \longrightarrow -\arctan(+\infty) + C = -\frac{\pi}{2} + C.$$

But if $(x,y) \longrightarrow (x,0)$ with $y \longrightarrow 0-$ we have

$$f(x,y) = -\arctan\frac{x}{y} + C \longrightarrow -\arctan(-\infty) + C = +\frac{\pi}{2} + C.$$

The conclusion is that $\lim_{(x,y)\longrightarrow(x,0)} f(x,y)$ doesn't exists, for any $x > 0$. ■

## 6.3. Line Integral

According to the Fundamental Theorem of Integral Calculus, in dimension 1 a solution of the problem $\nabla f = \vec{F}$, that is $f' = F$, is

$$f(x) = f(x_0) + \int_{x_0}^{x} F(y)\,dy.$$

Through a suitable definition of integral for a vector field $\vec{F}$ over a curve, this formula has a natural extension to multiple dimensions. Let's start by the

Proposition 6.3.1. *Let $\vec{F} \in \mathscr{C}^1(D)$ be a conservative vector field with potential $f$. Then*

$$\int_\gamma \vec{F} = f(\gamma(b)) - f(\gamma(a)), \;\; \forall \gamma \in \mathscr{C}^1([a,b]),\; \gamma \subset D. \tag{6.3.1}$$

*In particular, if $\gamma$ is a* **circuit** *in D,*

$$\oint_\gamma \vec{F} = 0. \tag{6.3.2}$$

Proof — If $\vec{F} = \nabla f$ then

$$\int_\gamma \vec{F} = \int_a^b \vec{F}(\gamma(t)) \cdot \gamma'(t)\, dt = \int_a^b \nabla f(\gamma(t)) \cdot \gamma'(t)\, dt = \int_a^b \frac{d}{dt} f(\gamma(t))\, dt = f(\gamma(b)) - f(\gamma(a)),$$

because of the fundamental thm of integral calculus. The (6.3.3) follows immediately because for a circuit $\gamma$ we have $\gamma(b) = \gamma(a)$. ■

Example 6.3.2. Considering again the field presented in the example 6.2.6, that is $\vec{F} = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$ we can check that not all the circuitations vanish. For instance, if $\gamma$ is the unit circle centered into the origin, $\gamma(t) = (\cos t, \sin t)$, then applying the definition we get

$$\oint_\gamma \vec{F} = \int_0^{2\pi} \frac{-\sin t}{1}(-\sin t) + \frac{\cos t}{1} \cos t\; dt = \int_0^{2\pi} dt = 2\pi. \;\; ■$$

The condition (6.3.3) of the null circulations turns out to be also sufficient because a field $\vec{F}$ would be conservative. This is the announced extension of the Fundamental Thm of Integral Calculus:

Theorem 6.3.3 (fundamental thm of calculus for fields). *Let $\vec{F} \in \mathscr{C}(D)$ be a vector field on $D \subset \mathbb{R}^d$ open and connected and such that*

$$\oint_\gamma \vec{F} = 0, \;\; \forall \gamma \in \mathscr{C}^1,\; \gamma \subset D. \tag{6.3.3}$$

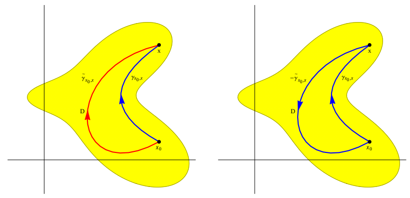
*Then $\vec{F}$ is conservative and all the possible potentials are*

$$f(x) = \int_{\gamma_{x_0,x}} \vec{F} + c,\; x \in D,\; c \in \mathbb{R}, \tag{6.3.4}$$

*where $\gamma_{x_0,x}$ is a regular path contained in D joining $x_0$ to $x$.*

Proof — First of all we have to prove that the (6.3.4) is well defined, that is: the definition doesn't depend on the particular path connecting $x_0$ to $x$. So, let $\widetilde{\gamma}_{x_0,x}$ be a second path connecting $x_0$ to $x$. We want to prove that

$$\int_{\gamma_{x_0,x}} \vec{F} = \int_{\widetilde{\gamma}_{x_0,x}} \vec{F}.$$



Let's use the notation $-\widetilde{\gamma}_{x_0,x}$ for the path with same "trace" of $\widetilde{\gamma}_{x_0,x}$ but opposite orientation (see the picture). Then if we consider the path formed by $\gamma_{x_0,x}$ first and then by $-\widetilde{\gamma}_{x_0,x}$ we get a circuit that we will denote by the symbol $\gamma_{x_0,x} - \widetilde{\gamma}_{x_0,x}$. Then, by our assumption

$$0 = \oint_{\gamma_{x_0,x}-\widetilde{\gamma}_{x_0,x}} \vec{F} = \int_{\gamma_{x_0,x}} \vec{F} + \int_{-\widetilde{\gamma}_{x_0,x}} \vec{F} = \int_{\gamma_{x_0,x}} \vec{F} - \int_{\widetilde{\gamma}_{x_0,x}} \vec{F}, \implies \int_{\gamma_{x_0,x}} \vec{F} = \int_{\widetilde{\gamma}_{x_0,x}} \vec{F}.$$

(we accept that $\int_{\gamma_1+\gamma_2} \vec{F} = \int_{\gamma_1} \vec{F} + \int_{\gamma_2} \vec{F}$ and that $\int_{-\gamma} \vec{F} = -\int_{\gamma} \vec{F}$).

Let's prove now that $f$ defined by (6.3.4) is a potential for $\vec{F}$, that is $\nabla f = \vec{F}$. We have to prove that

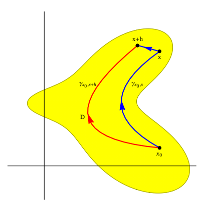
$$f(x+h) - f(x) = \vec{F}(x) \cdot h + o(h).$$

First,

$$f(x+h) - f(x) = \int_{\gamma_{x_0,x+h}} \vec{F} - \int_{\gamma_{x_0,x}} \vec{F}.$$

Because the integral doesn't depend on the particular path connecting $x_0$ to $x+h$ we have that the path integral along $\gamma_{x_0,x+h}$ must be identical to the path integral along $\gamma_{x_0,x}$ followed by the segment $[x, x+h]$. In symbols

$$f(x+h) = \int_{\gamma_{x_0,x_0+h}} \vec{F} = \int_{\gamma_{x_0,x}+[x,x+h]} \vec{F} = \int_{\gamma_{x_0,x}} \vec{F} + \int_{[x,x+h]} \vec{F} = f(x) + \int_{[x,x+h]} \vec{F}.$$



Therefore

$$f(x+h) - f(x) = \int_{[x,x+h]} \vec{F}.$$

Now, the natural parametrization of the segment $[x, x+h]$ is $\gamma(t) = x + th$, $t \in [0,1]$ so $\gamma'(t) = h$. Hence

$$\int_{[x,x+h]} \vec{F} = \int_0^1 \vec{F}(x+th) \cdot h \, dt = \int_0^1 \left(\vec{F}(x+th) - \vec{F}(x) + \vec{F}(x)\right) \cdot h \, dt$$

$$= \vec{F}(x) \cdot h + \int_0^1 \left(\vec{F}(x+th) - \vec{F}(x)\right) \cdot h \, dt.$$

To conclude, we need to prove that

$$\int_0^1 \left(\vec{F}(x+th) - \vec{F}(x)\right) \cdot h \, dt = o(h). \tag{6.3.5}$$

Notice that

$$\left|\int_0^1 \left(\vec{F}(x+th)-\vec{F}(x)\right)\cdot h\,dt\right| \overset{\triangle}{\leqslant} \int_0^1 \left\|\left(\vec{F}(x+th)-\vec{F}(x)\right)\cdot h\right\|\,dt \overset{C-S}{\leqslant} \int_0^1 \left\|\vec{F}(x+th)-\vec{F}(x)\right\|\|h\|\,dt.$$

Because $\vec{F}$ is continuous on $D$, it is continuous at $x$. Hence, taking $\varepsilon > 0$ there exists $\delta(\varepsilon)$ such that $\|\vec{F}(y)-\vec{F}(x)\| \leqslant \varepsilon$ for any $y \in D$ such that $\|y-x\| \leqslant \delta(\varepsilon)$. Let now $y = x+th$, $\|y-x\| = \|th\| = |t|\|h\| \leqslant \|h\|$ with $t \in [0,1]$. If $\|h\| \leqslant \delta(\varepsilon)$ we have $\|\vec{F}(x+th)-\vec{F}(x)\| \leqslant \varepsilon$, so

$$\left|\int_0^1 \left(\vec{F}(x+th)-\vec{F}(x)\right)\cdot h\,dt\right| \leqslant \varepsilon\|h\|\int_0^1 dt = \varepsilon\|h\|,\ \forall\|h\| \leqslant \delta(\varepsilon),$$

by which the (6.3.5) easily follows. ■

This Theorem gives a necessary and sufficient condition in order to have $\vec{F}$ conservative and a formula for a potential. However, to check the hypothesis is practically impossible: we cannot check that *every* circuitation vanishes. Because a conservative field $\vec{F}$ is necessarily irrotational, the check can be considerably simplified in this case. We will limit here to state some important result in the case of vector fields in $\mathbb{R}^2$. Let first introduce a definition:

Definition 6.3.4. *Let $D \subset \mathbb{R}^2$ connected. We say that $D$ is* **simply connected** *if the support of every regular circuit $\gamma \subset D$ is the boundary of a subset $\Omega \subset D$, that is* $\text{Supp}(\gamma) = \partial\Omega$.

Theorem 6.3.5. *Every $\vec{F}$ irrotational on $D \subset \mathbb{R}^2$ simply connected is conservative.*

Proof — We prove that (6.3.3) holds. Let $\vec{F} = (f_1, f_2)$ and let $\gamma$ be a circuit in $D$. Because $\text{Supp}(\gamma) = \partial\Omega$, by the Green formula (5.5.2),

$$\oint_\gamma \vec{F} = \int_\Omega (\partial_x f_2 - \partial_y f_1)\,dxdy = 0,$$

being $\vec{F}$ irrotational. ■

In practice, in the previous statement $D$ must be a connected set without "holes". Imagine $D = \mathbb{R}^2\backslash\{0_2\}$. If $\gamma \subset D$ is such that its "interior" does not contain $0_2$, then $\text{Supp}(\gamma) = \partial\Omega$ with $\Omega \subset D$. This is false if $\gamma$ contains $0_2$. For instance: let $\gamma(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$, the unit circle. Then $\text{Supp}(\gamma) = \partial\Omega$ iff $\Omega = B(0,1]$, but $B(0,1] \ni 0_2 \notin D$. In cases like this we have

Theorem 6.3.6. *Let $D \subset \mathbb{R}^2$ simply connected and $\vec{F} \in C^1(D\backslash\{x_1,\dots,x_\ell\})$ irrotational and such that*

$$\oint_{\partial B(x_k,r_k]} \vec{F} = 0,\ \ k = 1,\dots,\ell,$$

*where $\partial B(x_k, r_k]$ are circumferences centered at $x_k$ and radius $r_k$ such that $B(x_k, r_k]$ doesn't contain other points $x_j$ (this is clearly possible if $r_k$ is small enoug). Then $\vec{F}$ is conservative.*

Example 6.3.7. *Show that the field*

$$\vec{F}(x,y) = \left(\frac{y^2+2xy-x^2}{(x^2+y^2)^2}, -\frac{x^2+2xy-y^2}{(x^2+y^2)^2}\right),\ \ (x,y) \in \mathbb{R}^2\backslash\{0_2\} =: D,$$

*is conservative and compute the potentials.*

Sol. — The check that $\vec{F} \in \mathscr{C}^1$ is irrotational, that is

$$\partial_y \frac{y^2+2xy-x^2}{(x^2+y^2)^2} = \partial_x \left(-\frac{x^2+2xy-y^2}{(x^2+y^2)^2}\right),$$

is left as exercise. Let's check that thm 6.3.6 can be applied. We have to control if $\oint_{\partial B(0,1]} \vec{F} = 0$. Parametrizing $\partial B(0,1]$ in the standard way $\gamma(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$ we have

$$\oint_{\partial B(0,1]} \vec{F} = \int_0^{2\pi} \vec{F}(\gamma(t)) \cdot \gamma'(t)\, dt = \int_0^{2\pi} f_1(\cos t, \sin t)(\cos t)' + f_2(\cos t, \sin t)(\sin t)'\, dt,$$

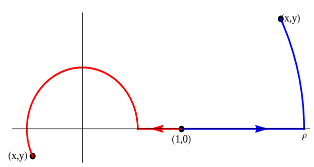
where $\vec{F} = (f_1, f_2)$. We have

$$f_1(\cos t, \sin t) = (\sin t)^2 + 2\sin t \cos t - (\cos t)^2, \quad f_2(\cos t, \sin t) = -\left((\cos t)^2 + 2\cos t \sin t - (\sin t)^2\right).$$

Let's write $C = \cos t$, $S = \sin t$. Then

$$\oint_\gamma \vec{F} = \int_0^{2\pi} -(S^2+2SC-C^2)S - (C^2+2SC-S^2)C\, dt = 0.$$

Therefore, by thm 6.3.6, it follows that $\vec{F}$ is conservative on $D$. Let's compute the potentials by using the (6.3.4). Let's fix $(1,0)$ as initial point. If $(x,y) \in \mathbb{R}^2$ we connect $(1,0)$ to $(x,y)$ with a path done first by a segment $[(1,0),(\rho,0)]$ where $\rho := \sqrt{x^2+y^2}$ followed by an arch of circumference of radius $\rho$ up to $(x,y)$.




Suppose, to fix ideas, that $\rho = \sqrt{x^2+y^2} \geqslant 1$ and let $\arg(x,y)$ the angle corresponding to the point $(x,y)$. The path is then

$$\gamma_{(1,0),(x,y)} = \gamma_1 + \gamma_2, \;\text{ where }\; \gamma_1(t) := (t,0),\; t \in [1,\rho], \quad \gamma_2(t) = (\rho\cos\theta, \rho\sin\theta),\; \theta \in [0, \arg(x,y)].$$

We have $\gamma_1'(t) = (1,0)$ so

$$\int_{\gamma_1} \vec{F} = \int_1^\rho f_1(t,0)\, dt = \int_1^\rho -\frac{t^2}{t^4}\, dt = -\int_1^\rho \frac{1}{t^2}\, dt = \frac{1}{t}\Big|_{t=1}^{t=\rho} = \frac{1}{\rho} - 1.$$

Similarly, $\gamma_2'(\theta) = (-\rho\sin\theta, \rho\cos\theta) = (-\rho S, \rho C)$, so

$$\begin{aligned}
\int_{\gamma_2} \vec{F} &= \int_0^{\arg(x,y)} \frac{\rho^2S^2+2\rho^2CS-\rho^2C^2}{(\rho^2C^2+\rho^2S^2)^2}(-\rho S) - \frac{\rho^2C^2+2\rho^2CS-\rho^2S^2}{(\rho^2C^2+\rho^2S^2)^2}(\rho C)\, d\theta \\
&= \frac{1}{\rho}\int_0^{\arg(x,y)} \left(-S^3-2CS^2+C^2S-C^3-2C^2S+S^2C\right)\, d\theta \\
&= \frac{1}{\rho}\int_0^{\arg(x,y)} \left(-S^3-C^3-S^2C-C^2S\right)\, d\theta.
\end{aligned}$$

Let's compute the integrals:

$$\int_0^{\arg(x,y)} S^2C\,d\theta = \int_0^{\arg(x,y)} (\sin\theta)^2\cos\theta\,d\theta = \frac{1}{3}(\sin\theta)^3\Big|_{\theta=0}^{\theta=\arg(x,y)} = \frac{1}{3}(\sin\arg(x,y))^3.$$

Notice that $\sin\arg(x,y) = \frac{y}{\sqrt{x^2+y^2}}$ et, similarly, $\cos\arg(x,y) = \frac{x}{\sqrt{x^2+y^2}}$, hence

$$\int_0^{\arg(x,y)} S^2C\,d\theta = \frac{1}{3}\frac{y^3}{(x^2+y^2)^{3/2}}.$$

Similarly

$$\int_0^{\arg(x,y)} C^2S\,d\theta = -\frac{1}{3}(\cos\theta)^3\Big|_{\theta=0}^{\theta=\arg(x,y)} = -\frac{1}{3}\frac{x^3}{(x^2+y^2)^{3/2}} + \frac{1}{3}.$$

Integrating by parts,

$$\begin{aligned}\int_0^{\arg(x,y)} S^3\,d\theta &= \int_0^{\arg(x,y)} (-C)'S^2\,d\theta = -CS^2\Big|_{\theta=0}^{\theta=\arg(x,y)} + 2\int_0^{\arg(x,y)} C^2S\,d\theta\\ &= -\frac{xy^2}{(x^2+y^2)^{3/2}} - \frac{2}{3}\frac{x^3}{(x^2+y^2)^{3/2}} + \frac{2}{3}.\end{aligned}$$

Similarly

$$\begin{aligned}\int_0^{\arg(x,y)} C^3\,d\theta &= \int_0^{\arg(x,y)} S'C^2\,d\theta = SC^2\Big|_{\theta=0}^{\theta=\arg(x,y)} + 2\int_0^{\arg(x,y)} S^2C\,d\theta\\ &= \frac{x^2y}{(x^2+y^2)^{3/2}} + \frac{2}{3}\frac{y^3}{(x^2+y^2)^{3/2}}.\end{aligned}$$

Therefore,

$$\int_{\gamma_2}\vec{F} = \frac{xy^2}{(x^2+y^2)^2} + \frac{x^3}{(x^2+y^2)^2} - \frac{x^2y}{(x^2+y^2)^2} - \frac{y^3}{(x^2+y^2)^2} + \frac{1}{(x^2+y^2)^{1/2}},$$

and finally

$$f(x,y) = \frac{x^3-y^3+xy^2-yx^2}{(x^2+y^2)^2} - 1. \quad \blacksquare$$

## 6.4. Exercises

Exercise 6.4.1. Compute $\int_\gamma \vec{F}$ in the following cases:

(1) $\vec{F}(x,y) := (y^3+x, -\sqrt{x})$ on $D = [0,+\infty[\times\mathbb{R}$, $\gamma$ of equation $x = y^2$ connecting $(0,0)$ to $(1,1)$.
(2) $\vec{F}(x,y) := (y^2, 2xy+1)$ on $D = \mathbb{R}^2$, $\gamma$ of equation $y = \sqrt{|x-1|}$, $x\in[0,2]$.
(3) $\vec{F}(x,y) := (\sqrt{y}, x^3+y)$ on $D = \mathbb{R}\times[0,+\infty[$ along $y = x^2$ connecting $(1,1)$ to $(2,4)$.
(4) $\vec{F}(x,y) := \left(\frac{x+1}{y-1}, \frac{y+1}{x-1}\right)$ on $D = \{(x,y)\in\mathbb{R}^2 \;:\; y\neq 1,\ x\neq 1\}$, along the segment connecting $(0,0)$ to $(1/2,1/2)$.
(5) $\vec{F}(x,y) := (\log(1+y), \log(1+x))$, on $D = ]-1,+\infty[^2$, along the segment connecting $(1,0)$ to $(0,1)$.
(6) $\vec{F}(x,y,z) := (y+z, x+z, x+y)$ on $D = \mathbb{R}^3$ along the helix $\gamma(t) = (r\cos t, r\sin t, kt)$, $t\in[0,2\pi]$.

Exercise 6.4.2. For each of the following vector fields on the given domains, check if they are irrotational, conservative and, in this case, find a potential:

(1) $\vec{F}(x,y) := (x, y-1)$, $(x,y) \in \mathbb{R}^2$;
(2) $\vec{F}(x,y) := (y,x)$, $(x,y) \in \mathbb{R}^2$;
(3) $\vec{F}(x,y) := (x,-y)$, $(x,y) \in \mathbb{R}^2$;
(4) $\vec{F}(x,y,z) := (y+z, x+z, x+y)$, $(x,y,z) \in \mathbb{R}^3$;

EXERCISE 6.4.3. Find al possible values for $a,b,c \in \mathbb{R}$ such that the field

$$\vec{F}(x,y) := \left(ax^3 + by + 3x^2y^2, cx^4 + 2x^3y + 1\right),$$

be irrotational on $\mathbb{R}^2$. In the case say if it is also conservative and find all the potentials.

EXERCISE 6.4.4. Consider the vector field

$$\vec{F}(x,y) := \left(\sin\frac{x}{y} + \frac{x}{y}\cos\frac{x}{y}, -\frac{x^2}{y^2}\cos\frac{x}{y} + 3\right), \text{ on } D = \mathbb{R}\times]0,+\infty.$$

Check that $\vec{F}$ is irrotational and say if it is also conservative. In this case compute a potential $f$ such that $f(\pi,1) = 3$.

EXERCISE 6.4.5. Consider the vector field

$$\vec{F}(x,y) := \left(\frac{y}{x^2}\cos\frac{y}{x}, -\frac{a}{x}\cos\frac{y}{x}\right), \text{ on } D = ]0,+\infty[\times\mathbb{R}.$$

Find values of $a \in \mathbb{R}$ such that $\vec{F}$ is irrotational. For such value say if $\vec{F}$ is also conservative and, in the case, find the potentials.

EXERCISE 6.4.6. Consider the vector field

$$\vec{F}(x,y) := \left(\frac{1}{1+y^2}, -\frac{2xy}{(1+y^2)^2}\right), \ (x,y) \in \mathbb{R}^2.$$

Is $\vec{F}$ irrotational? Conservative? Compute, the path integral $\int_\gamma \vec{F}$ where $\gamma(t) = \left(e^{\sin t}, \frac{2\cos t}{1+(\cos t)^2}\right)$, $t \in [0,\pi]$.

EXERCISE 6.4.7. Consider the vector field

$$\vec{F}(x,y) := \left(-\frac{axy}{(x^2+y^2)^2}, \frac{bx^2 - y^2}{(x^2+y^2)^2}\right), \text{ on } D = \mathbb{R}^2\backslash\{0_2\}.$$

Find values of $a,b \in \mathbb{R}$ such that $\vec{F}$ be irrotational. For such value say if $\vec{F}$ is also conservative and, in the case, find the potentials.

EXERCISE 6.4.8. Consider the vector field

$$\vec{F}(x,y,z) := \left(a(x,y,z), x^2 + 2yz, y^2 - z^2\right), \ (x,y,z) \in \mathbb{R}^3,$$

where $a$ is a $\mathscr{C}^1$ function. Find all the possible $a$ in such a way that $\vec{F}$ be irrotational. Show that there is a unique $a$ null as $y = z = 0$. In that case find all the potentials of $\vec{F}$.

EXERCISE 6.4.9. Find $a,b,c,d \in \mathbb{R}$ in such a way that the vector field

$$\vec{F}(x,y) := \left(\frac{ax+by}{x^2+y^2}, \frac{cx+dy}{x^2+y^2}\right), \ (x,y) \in \mathbb{R}^2\backslash\{0_2\}$$

be irrotational. For such values, find those such that $\vec{F}$ is conservative and find also its potentials.

Exercise 6.4.10. Let $g$ be the vector field defined by

$$\vec{F}(x,y) := \left( \frac{ax^2 + by^2}{(x^2+y^2)^2}, \frac{cxy}{(x^2+y^2)^2} \right), \quad (x,y) \in D := \mathbb{R}^2 \backslash \{0_2\}, \quad (a,b,c \in \mathbb{R}).$$

i) Find $a,b,c \in \mathbb{R}$ such that $\vec{F}$ is irrotational. ii) Find $a,b,c \in \mathbb{R}$ such that $\vec{F}$ is conservative: for such $a,b,c$ find the potentials of $\vec{F}$ (hint: . start with $\partial_y f = f_2(x,y)\dots$)

Exercise 6.4.11. Let $\vec{F}$ be the vector field defined as

$$\vec{F}(x,y) := \left( \frac{axy^2}{(x^2+y^2)^{1/2}}, \frac{bx^2y + cy^3}{(x^2+y^2)^{1/2}} \right), \quad (x,y) \in \mathbb{R}^2 \backslash \{0_2\} =: D.$$

i) Find $a,b,c \in \mathbb{R}$ such that $\vec{F}$ be irrotational. ii) For the values found at i), say if $\vec{F}$ is conservative on $\mathbb{R}^2 \backslash \{(0,y) : y \geqslant 0\}$ and on $D$. iii) For the values $a,b,c$ such that $\vec{F}$ is conservative on $D$ find the potentials of $\vec{F}$.

Exercise 6.4.12. Let $a,b,\alpha,\beta \neq 0$ and $\vec{F} \in C^1(D)$ be the vector field

$$\vec{F}(x,y) := \left( \frac{ax}{(x^2+y^2)^{\alpha}}, \frac{by}{(x^2+y^2)^{\beta}} \right), \quad (x,y) \in \mathbb{R}^2 \backslash \{0_2\} =: D.$$

i) Find $a,b,\alpha,\beta \in \mathbb{R}\backslash\{0\}$ such that $\vec{F}$ be irrotational on $D$. ii) For the values found in i) compute $\int_{\gamma} \vec{F}$ where $\gamma$ is the poligonal connecting $(2,0)$, $(0,1)$ and $(-2,0)$. iii) Find the values $a,b,\alpha,\beta$ such that $\vec{F}$ be conservative on $D$ and compute the eventual potentials.

Exercise 6.4.13. Consider the vector field

$$\vec{F}(x,y) := \left( \frac{x}{\sqrt{x+y}}, \frac{ax+b}{\sqrt{x+y}} \right), \quad (x,y) \in D := \left\{ (x,y) \in \mathbb{R}^2 \ : \ x+y > 0 \right\}.$$

i) Find values $a,b \in \mathbb{R}$ such that $\vec{F}$ be irrotational. For such values may you say, without computing the potential, if $\vec{F}$ is also conservative? ii) For values $a,b \in \mathbb{R}$ such that $\vec{F}$ be conservative, find all its potentials.

Exercise 6.4.14. Let

$$\vec{F}(x,y,z) := \left( \frac{1}{x} + \frac{y^{\alpha}}{1+x^2y^2}, \frac{1}{y} + \frac{x}{1+x^2y^2}, \frac{1}{z} \right), \ (x,y,z) \in ]0,+\infty[^3.$$

i) Find all the possible $\alpha > 0$ such that $\vec{F}$ be irrotational. ii) For the values $\alpha$ found in i), say if $\vec{F}$ is also conservative and compute all the potentials.

Exercise 6.4.15. Let $\alpha \in \mathbb{R}$ and consider the vector field

$$\vec{F}(x,y,z) := \left( \frac{x}{\sqrt{1+x^2+y^2}}, \frac{x+\alpha z}{\sqrt{1+x^2+y^2}} + e^{y+z}, e^{y+z} \right), \ (x,y,z) \in \mathbb{R}^3.$$

i) Find all the possible $\alpha > 0$ such that $\vec{F}$ be irrotational. ii) For the values $\alpha$ found in i), say if $\vec{F}$ is also conservative and compute all the potentials.

Exercise 6.4.16. Consider the vector field $\vec{F}(x,y) := \left( \frac{x}{x^2+y^2}, u(x,y) \right)$ on $D = \mathbb{R}^2 \backslash \{0_2\}$, where $u \in \mathscr{C}^1(D)$. Find all the possible $u$ in order that $\vec{F}$ be conservative.

Exercise 6.4.17. Find all the possible functions $u = u(x,y)$ belonging to $\mathscr{C}^1(\mathbb{R}^2)$ such that the vector field $\vec{F}(x,y,z) := (2xz, yz, u(x,y))$ be conservative in $D = \mathbb{R}^3$.

CHAPTER 7

# Differential Equations

*Differential equations* is an extremely important topic with many applications to Physics, Engineering, Biology, Chemistry, etc. In previous parts of the course, some classes of simple equations were introduced. In these cases the solutions can be found through calculus of primitives. In general, however, a differential equation cannot be solved explicitly and even the existence of solutions is not a trivial fact. This motivates the need at once of *qualitative methods* as well as *numerical methods* to study solutions of a differential equation. Along this Chapter we will concentrate on the firsts, referring to a course of *Numerical Calculus* for the seconds.

For pedagogical reasons, despite general results can be given directly in the most wide setting including all the important examples, we will proceed gradually. In this Chapter, we will focus first on *scalar equations*, in the next one on *vectorial equations* or *systems of differential equations*.

## 7.1. Cauchy Problem

The *Cauchy problem for first order scalar equations* consists in finding a solution of a differential equation fulfilling a *passage* or *initial value* condition. Formally,

$$\mathsf{CP}(t_0, y_0) \;:\; \begin{cases} y' = f(t,y), \quad t \in I, \\ \\ y(t_0) = y_0. \end{cases} \qquad \text{where } f = f(t,y) : D \subset \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}.$$

A $y \in \mathscr{C}^1(I)$, $I \subset \mathbb{R}$ interval is a solution if it fulfills the equation and the passage/initial condition. Notice that, in particular:

$$i)\ t_0 \in I,\ ii)\ (t, y(t)) \in D,\ \forall t \in I.$$

In other words, *the graph of the solution must be contained in the domain $D$ of $f$*. In particular, because $(t_0, y(t_0)) = (t_0, y_0)$, the passage point $(t_0, y_0) \in D$.

The first problem is: *under which conditions on $f$ there exists a unique solution to the Cauchy problem?* In the first Analysis course, we studied the particular case of *linear equations*

$$y' = a(t)y + b(t) =: f(t,y).$$

In particular, for this type of equations we obtained in general existence and uniqueness of the solution to the Cauchy Problem. In general, however, this may be false.

Example 7.1.1. *The Cauchy problem*

$$\begin{cases} y'(t) = y(t)^{1/3}, \\ \\ y(0) = 0, \end{cases}$$

*has infinitely many solutions.*

Sol. — It is clear that $y(t) \equiv 0$ is a solution. There're, however, non trivial solutions. To find them suppose for a while that $y \neq 0$ be a solution. Separating variables we get

$$\frac{y'}{y^{1/3}} = 1, \iff y^{-1/3}y' = 1, \iff \left(\frac{3}{2}y^{2/3}\right)' = 1, \iff \frac{3}{2}y^{2/3} = t + c, \iff y(t) = \left(\frac{2}{3}(t+c)\right)^{3/2}.$$

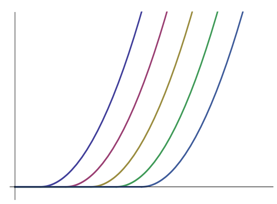
Notice that these are defined for $t \geqslant -c$ and $y(-c) = 0$. For instance, taking $c = 0$ we can easily see that

$$y_0(t) = \begin{cases} 0, & t \leqslant 0, \\ \left(\frac{2}{3}t\right)^{3/2}, & t > 0, \end{cases}$$

is a solution of the $\mathsf{CP}(0,0)$. More in general setting for $c \leqslant 0$

$$y_c(t) := \begin{cases} 0, & t \leqslant -c, \\ \left(\frac{2}{3}(t+c)\right)^{3/2}, & t > -c, \end{cases}$$

we have a solution of $\mathsf{CP}(0,0)$. Here the plots of the solutions $y_C$. ■



In the previous Example

$$y' = f(t,y), \text{ with } f(t,y) = y^{1/3} \in \mathscr{C}(\mathbb{R} \times \mathbb{R}).$$

The message delivered is that the simple continuity of $f$ is not sufficient to obtain uniqueness. Uniqueness of solutions is a very important issue because it entails the possibility to use solutions to do predictions on the future behavior of a real system. It is therefore natural to look for conditions on $f$ such that this be true.

## 7.2. Existence and Uniqueness

There're several results that prove, under different assumptions on $f$, existence and uniqueness for solutions of the Cauchy Problem. Basically they can be divided into two categories: *global* and *local* results. The main difference is that the firsts holds with stronger assumptions and give a precise information about the domain of definition of the solution, while the seconds are weaker, they holds under less restrictive assumptions but they provide much less precise informations on the solutions.

Theorem 7.2.1 (Global Cauchy–Lipschitz). *Let $f : D = [a,b] \times \mathbb{R} \longrightarrow \mathbb{R}$ be such that*

i) $f \in \mathscr{C}([a,b] \times \mathbb{R})$;

ii) $\partial_y f$ *be bounded on* $[a,b]\times\mathbb{R}$*, that is*

$$\exists L>0\ :\ |\partial_y f(t,y)|\leqslant L,\ \forall(t,y)\in[a,b]\times\mathbb{R}.$$

*Then, for every* $(t_0,y_0)\in[a,b]\times\mathbb{R}$ *there exists a unique solution to* $\mathsf{CP}(t_0,y_0)$*.*

Example 7.2.2. *Check the global Cauchy–Lipschitz conditions for the equation*

$$y'=\frac{\sin(ty)}{1+y^2}.$$

*on any strip* $[a,b]\times\mathbb{R}$*.*

Sol. — Here $f(t,y)=\frac{\sin(ty)}{1+y^2}$ is clearly defined on $(t,y)\in\mathbb{R}\times\mathbb{R}$ and $f\in\mathscr{C}$. Moreover

$$\partial_y f(t,y)=\frac{(1+y^2)t\cos(ty)-2y\sin(ty)}{(1+y^2)^2}\in\mathscr{C}(\mathbb{R}\times\mathbb{R}).$$

Therefore

$$|\partial_y f(t,y)|=\left|\frac{(1+y^2)t\cos(ty)-2y\sin(ty)}{(1+y^2)^2}\right|\leqslant\frac{(1+y^2)|t|+2|y|}{(1+y^2)^2}=\frac{|t|}{1+y^2}+\frac{2|y|}{(1+y^2)^2}.$$

Clearly $\frac{1}{1+y^2}\leqslant 1$ and because $2ab\leqslant a^2+b^2$, $\frac{2|y|}{(1+y^2)^2}\leqslant\frac{1+y^2}{(1+y^2)^2}=\frac{1}{1+y^2}\leqslant 1$, we have

$$|\partial_y f(t,y)|\leqslant|t|+1\leqslant\max\{|a|,|b|\}+1=:L,\ \forall(t,y)\in[a,b]\times\mathbb{R}.\quad\blacksquare$$

The two main restriction of the Global Cauchy–Lipschitz Theorem are: first, *the domain that must be a strip* $[a,b]\times\mathbb{R}$; second, *the* $\partial_y f$ *that must be bounded.* If one or both these conditions are not fulfilled by $f$ (we consider the continuity of $f$ a minimal assumption), we cannot apply this result. The conclusions of the Global CL Theorem, can be false.

Example 7.2.3. *Solve the Cauchy problem*

$$\begin{cases} y'=y^2,\\[2ex] y(t_0)=y_0.\end{cases}$$

Sol. — Let $f(t,y)=y^2$. Clearly $f\in\mathscr{C}(\mathbb{R}\times\mathbb{R})$. However, $\partial_y f(t,y)=2y$ is unbounded on every strip $[a,b]\times\mathbb{R}$, so the Global CL Thm doesn't apply. We can however solve the equation by separation of variables: noticed that $y\equiv 0$ is a solution, if $y\not\equiv 0$ we have

$$y'(t)=y(t)^2,\iff\frac{y'(t)}{y(t)^2}=1,\iff\left(-\frac{1}{y(t)}\right)'=1,\iff-\frac{1}{y(t)}=t+C,\iff y(t)=-\frac{1}{t+C}.$$

Notice that this solution is never $=0$ therefore, the previous argument shows that if $y\neq 0$ somewhere, $y\neq 0$ everywhere. Let's pass to the solution of the Cauchy problem. If $y_0=0$ then $y\equiv 0$ is the unique possible solution. If $y_0\neq 0$ then the solution can be only of the form $y(t)=-\frac{1}{t+C}$. By imposing the passage condition

$$y(t_0)=y_0,\iff-\frac{1}{t_0+C}=y_0,\iff C=-t_0-\frac{1}{y_0}.$$

There's a unique possible $C$ hence a unique possible solution given by

$$y(t) = -\frac{1}{t - t_0 - \frac{1}{y_0}}.$$

Notice that this, as function, would be defined in $\mathbb{R}\backslash\{t_0 + \frac{1}{y_0}\} = ]-\infty, t_0 + \frac{1}{y_0}[\cup]t_0 + \frac{1}{y_0}, +\infty[$. However, only one among these two intervals contains $t_0$. As $y_0 > 0$ it is $]-\infty, t_0 + \frac{1}{y_0}[$, while as $y_0 < 0$ it is $]t_0 + \frac{1}{y_0}, +\infty[$ (because $t_0 + \frac{1}{y_0} < t_0$). ■

This example shows some interesting facts. First, the *life interval* of the solution depends by the initial condition $(t_0, y_0)$: the interval might be bounded or unbounded, even the entire $]-\infty, +\infty[$. However, unless the solution is known, it is impossible to say *a priori* what is the case. These are general features, caught by the

Theorem 7.2.4 (Local Cauchy–Lipschitz). *Let $f : D \subset \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ be such that*

i) $f \in \mathscr{C}(D)$;

ii) $\partial_y f \in \mathscr{C}(D)$.

*Then, for every $(t_0, y_0) \in D$ there exists a unique solution $y : ]\alpha, \beta[ \longrightarrow \mathbb{R}$ of $\mathsf{CP}(t_0, y_0)$, defined on a suitable $]\alpha, \beta[\ni t_0$, with the following property: $y$ cannot be extended to any $J \supsetneq ]\alpha, \beta[$. The solution $y$ is called* **maximal solution** *of $\mathsf{CP}(t_0, y_0)$.*

Remark 7.2.5 (very important!). An important consequence of uniqueness is the following: *two different maximal solutions can never cross.* Indeed: if $y \neq \widehat{y}$ are maximal solutions such that $y(t_0) = \widehat{y}(t_0) =: y_0$, for some $t_0$, then they are both solutions of $\mathsf{CP}(t_0, y_0)$. But there's a unique solution for such CP, hence $y \equiv \widetilde{y}$. ■

In general, it is not possible to know what are $\alpha, \beta$ and, of course, they'll depend by $(t_0, y_0)$. The interval $]\alpha, \beta[$ is, by definition, the *life interval* for the maximal solution.

Example 7.2.6. *Show that the equation*

$$y' = \frac{e^{ty}}{1 + y^2}$$

*fulfills the local Cauchy–Lipschitz condition on $\mathbb{R}\times\mathbb{R}$ but* **not** *the global condition on any strip $[a, b]\times\mathbb{R}$.*

Sol. — Here $f(t, y) = \frac{e^{ty}}{1+y^2}$ is defined on $D = \mathbb{R}\times\mathbb{R}$. Clearly $f$ is continuous and

$$\partial_y f(t, y) = \frac{te^{ty}(1 + y^2) - 2ye^{ty}}{(1 + y^2)^2} = e^{ty}\frac{t(1 + y^2) - 2y}{(1 + y^2)^2} \in \mathscr{C}(\mathbb{R}^2).$$

This shows that the local Cauchy–Lipschitz condition holds. However, $\partial_y f(t, y)$ is clearly unbounded on any strip $[a, b]\times\mathbb{R}$. Indeed: if $t \neq 0$ is fixed, $|\partial_y f(t, y)| \longrightarrow +\infty$ as $y \longrightarrow +\infty$. ■

In general, the explicit determination of the life time interval for a maximal solution is difficult if not impossible. We might expect that when the solution "expires" (in the past, as $t \longrightarrow \alpha+$. in the future, as $t \longrightarrow \beta-$) something "dramatic" should happens. Imagine in fact that the "final" point $(\beta, y(\beta-)) \in D$.

Then, applying local existence and uniqueness, the $\mathsf{CP}(\beta, y(\beta-))$ would have a solution $\widehat{y} : ]\widehat{\alpha}, \widehat{\beta}[ \longrightarrow \mathbb{R}$, with $\beta \in ]\widehat{\alpha}, \widehat{\beta}[$. In particular, we could extend $y$ some time after $\beta$ by posing

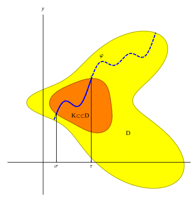
$$Y(t) := \begin{cases} y(t), & t < \beta, \\ \\ \widehat{y}(t), & \beta \leqslant t < \widehat{\beta}. \end{cases}$$

But then, $Y$ would be a solution of the differential equation, it would be an extension of $y$, contradicting the maximality of this last. The conclusion would be then that $(\beta, y(\beta-)) \notin D$, or, in other words, *when a maximal solution expires it must get out of the domain $D$*. This argument is not a proof, because there're many delicate points that should be clarified. However, it suggest a general property that turns out to be true by fixing the argument shown here. This is a fundamental property of maximal solutions:

Theorem 7.2.7 (exit by compact sets). *Let $f : D \subset \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be fulfilling Local CL Thm on $D$ open. If $y$ is a maximal solution then $y$ must get out of every compact set $K \subset D$. Formally there exists $\sigma_K, \tau_K$ exit times in the past and in the future from $K$ in the sense that*

$$(t, y(t)) \notin K, \ \forall t < \sigma_K, \ t > \tau_K.$$

*To write $K$ compact included in $D$ we will write shortly $K \Subset D$.*




## 7.3. Qualitative Study of Scalar Equations

The problem of studying qualitatively the solution of a certain CP is complex and can reach a great detail by using in a refined way the relatively elementary properties of derivatives together with key properties of maximal solutions. Instead to present general results, we show some of the possible techniques applied in specific problems.

Example 7.3.1. *Consider the Cauchy problem*

$$\begin{cases} y' = \dfrac{\tan y}{1 + y^2}, \\ \\ y(0) = y_0, \end{cases}$$

*Show that local existence and uniqueness hold. Find constant solutions and regions of $D$ where the solutions are increasing/decreasing. Let now $y : ]\alpha, \beta[ \longrightarrow \mathbb{R}$ the maximal solution with $y_0 \in ]0, \frac{\pi}{2}[$. Show that $y$ is monotone and deduce that $\alpha = -\infty$, computing also $y(-\infty)$. Show that $\beta < +\infty$ and compute $y(\beta-)$. Show that $y \in \mathscr{C}^2$ and find the concavity of $y$. Use this to show again that $\beta < +\infty$ and to deduce an estimate of $\beta$. With the previous informations plot a graph of $y$.*

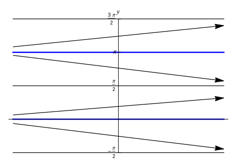
Sol. — Let $f(t,y) = \frac{\tan y}{1+y^2}$ is defined on $D := ]-\infty,+\infty[\times \left(\mathbb{R}\setminus\left\{\frac{\pi}{2}+k\pi\ :\ k\in\mathbb{Z}\right\}\right)$. On $D$, $f, \partial_y f$ are clearly continuous: therefore, local existence and uniqueness are fulfilled. Let $y \equiv C$ be a constant solution. To be a solution we need that

$$0 = y' = \frac{\tan y}{1+y^2} = \frac{\tan C}{1+C^2}, \iff \tan C = 0, \iff C = k\pi,\ k\in\mathbb{Z}.$$

Let $y$ be a solution. Then

$$y\nearrow,\ \text{on } I \iff y' = \frac{\tan y}{1+y^2} \geqslant 0,\ \text{on } I, \iff k\pi \leqslant y < k\pi + \frac{\pi}{2},\ k\in\mathbb{Z}.$$

By this we deduce the picture of plane regions where solutions are increasing/decreasing.



Let now $y : ]a,b[\longrightarrow \mathbb{R}$ be the maximal solution of $\mathsf{CP}(0,y_0)$ with $y_0 \in ]0,\frac{\pi}{2}[$. We want to prove that

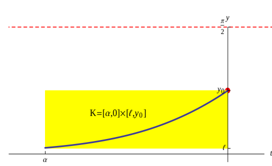
$$0 < y(t) < \frac{\pi}{2},\ \forall t\in ]a,b[, \implies \varphi\nearrow.$$

If this is true then $y\nearrow$. Suppose, by contradiction, that there exists $\exists t_1 \in ]a,b[$ such that $y(t_1)\leqslant 0$. Then, by intermediate values thm, $y(\widehat{t}) = 0$ for some $\widehat{t}$. But then $y$ must intersect a costant solution at $t=\widehat{t}$. By uniqueness, therefore, $y\equiv 0$, but this is impossible being $y(0) = y_0 > 0$. Similarly, it is impossible that there exists $t_1$ such that $y(t_1)\geqslant \frac{\pi}{2}$ otherwise there should be $\widehat{t}$ such that $y(\widehat{t}) = \frac{\pi}{2}$. In this case $(\widehat{t}, y(\widehat{t}))\notin D$ and this is impossible for a solution.

We know now that $y\nearrow$. Therefore, by properties of monotone functions, $\exists \lim_{t\to a+} y(t) = \ell$. Moreover, because $0 < y(t) < \frac{\pi}{2}$, we have $0\leqslant \ell \leqslant \frac{\pi}{2}$ (actually $<\frac{\pi}{2}$). Suppose then that $\alpha > -\infty$. We should have then

$$(t, y(t)) \in [\alpha, 0]\times[\ell, y_0] =: K,\ \forall t < 0,$$

so that the solution wouldn't get out (in the past) by the compact $K$, and this contradicts the fugue by compacts.




The unique possibility is therefore $\alpha = -\infty$. About $\ell$ we have two alternatives: either $\ell = 0$ or $\ell > 0$. The second is impossible: taking the equation and passing to the limit as $t\longrightarrow -\infty$ we would have

$$y' \longrightarrow \frac{\tan \ell}{1+\ell^2}.$$

But being $y\longrightarrow \ell$ as $t\longrightarrow -\infty$ **it cannot be** $\frac{\tan\ell}{1+\ell^2}\neq 0$. We can motivate this by a little general fact consequence of Hôpital rules:

Proposition 7.3.2. *Let $\varphi \in \mathscr{C}^1$ be such that*

$$\lim_{t\to\pm\infty} \varphi(t) = \ell, \quad \lim_{t\to\pm\infty} \varphi'(t) = \ell', \quad \textit{with } \ell, \ell' \in \mathbb{R}^d, \quad \Longrightarrow \quad \ell' = 0.$$

Proof — Indeed

$$0 = \lim_{t\to\pm\infty} \frac{\varphi(t)}{t} \stackrel{(H)}{=} \lim_{t\to\pm\infty} \varphi'(t) = \ell'. \quad \blacksquare$$

Therefore $\frac{\tan\ell}{1+\ell^2} = 0$, iff $\tan\ell = 0$, iff $\ell = k\pi$, and being $0 \leqslant \ell < y_0 < \frac{\pi}{2}$ necessarily $\ell = 0$.

Let's discuss now what happens at $\beta$. We know that $y \nearrow$ so $\exists \lim_{t\to\beta-} y(t) =: \ell$. Because $0 < y(t) < \frac{\pi}{2}$ forcely $0 \leqslant \ell \leqslant \frac{\pi}{2}$ (actually $\ell \geqslant y_0$). Let's deduce by this that $\beta < +\infty$. If $\beta = +\infty$ we would have

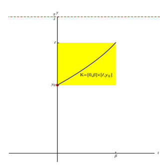
$$y'(t) \longrightarrow \frac{\tan\ell}{1+\ell^2} =: \ell'.$$

Now: either $\ell < \frac{\pi}{2}$ or $\ell = \frac{\pi}{2}$. In the first case $\ell' \in \mathbb{R}$ but then, by previous proposition, $\ell' = 0$, that is $\frac{\tan\ell}{1+\ell^2} = 0$, iff $\ell = k\pi$. But this is impossible because $y_0 < \ell \leqslant \frac{\pi}{2}$. Therefore $\ell = \frac{\pi}{2}$, hence $\ell' = +\infty$. But also this is impossible: if $\ell' = +\infty$ then $\varphi' \geqslant 1$ definitively, therefore $y(t) \longrightarrow +\infty$ so it cannot be $y(t) \longrightarrow \frac{\pi}{2}$.

Moral: $\beta < +\infty$. Let's see that $\ell = \frac{\pi}{2}$. If $\ell < \frac{\pi}{2}$ we would have that

$$(t, y(t)) \in [0,\beta] \times [y_0, \ell] \Subset D, \ \forall t > 0,$$

contradicting the fugue by compacts.



Let's pass now to the discussion of concavity. Because $y' = \frac{\tan y}{1+y^2}$, by deriving this respect to $t$ (do not forget that $y = y(t)$) we get

$$\begin{aligned} y'' &= \frac{(\tan y)'(1+y^2) - (\tan y)(1+y^2)'}{(1+y^2)^2} = \frac{(1+(\tan y)^2)y'(1+y^2) - (\tan y)2yy'}{(1+y^2)^2} \\ &= y'\frac{1+(\tan y)^2 + y^2 + y^2(\tan y)^2 - 2y\tan y}{(1+y^2)^2} = y'\frac{1+y^2(\tan y)^2 + (y-\tan y)^2}{(1+y^2)^2}. \end{aligned}$$

By this it is evident that $y'' \geqslant 0$ iff $y' \geqslant 0$, and because this is true by previous discussion, $y$ is convex. Taking the tangent at $t = 0$ of equation

$$y = y_0 + \varphi'(0)t = y_0 + \frac{\tan y_0}{1+y_0^2}t,$$

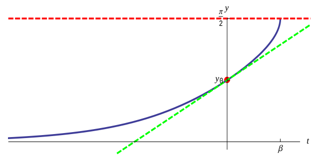
by properties of convex functions it follows that

$$y(t) \geqslant y_0 + \frac{\tan y_0}{1+y_0^2}t.$$

And because

$$y_0 + \frac{\tan y_0}{1+y_0^2}t \leqslant \frac{\pi}{2}, \quad \Longleftrightarrow \quad t \leqslant \left(\frac{\pi}{2} - y_0\right)\frac{1+y_0^2}{\tan y_0}$$

we deduce necessarily that $\beta < \left(\frac{\pi}{2} - y_0\right) \frac{1+y_0^2}{\tan y_0}$. ■




Example 7.3.3. *Consider the differential equation*

$$y' = \frac{1}{t^2 + y^2 - 1}.$$

*Determine the domain of local existence and uniqueness. Let now $y$ be the solution of the* $\mathsf{CP}(0,0)$*. Show that $y$ is odd increasing and find its concavity. With these informations estimate the lifetime of $y$.*

Sol. — Let $f(t,y) = \frac{1}{t^2+y^2-1}$. Clearly $f \in \mathscr{C}(D)$ where $D = \{(t,y) \in \mathbb{R}^2 \ : \ t^2 + y^2 \neq 1\}$ (the plane minus the circle centered in $(0,0)$ with radius 1. Moreover

$$\partial_y f = -\frac{2y}{(t^2 + y^2 - 1)^2} \in \mathscr{C}(D).$$

Let now $y : ]\alpha, \beta[ \longrightarrow \mathbb{R}$ be the maximal solution of $\mathsf{CP}(0,0)$. To show that $y$ is odd we need to prove that

$$y(-t) = -y(t),\ \forall t \iff y(t) = -y(-t),\ \forall t$$

We will use the following standard argument: let $z$ be defined as $z(t) := -y(-t)$. Notice that $z$ is still a solution of the same Cauchy problem solved by $y$. Indeed: $z(0) = -y(-0) = -y(0) = -0 = 0$ and

$$z'(t) = (-y(-t))' = y'(-t) = \frac{1}{(-t)^2 + (y(-t))^2 - 1} = \frac{1}{t^2 + z(t)^2 - 1},$$

By uniqueness $z = y$, that is $-y(-t) = y(t)$ for every $t$.

Let's pass to the monotonicity. We have

$$y \searrow, \iff y' \leqslant 0,\ \forall t, \iff t^2 + y^2 < 1,\ \forall t.$$

Now, as $t = 0$ this is true (because $y(0) = 0$). If for some $t$ it would be $t^2 + y^2 \geqslant 1$ then, either $t^2 + y(t)^2 = 1$ (but this means $(t, y(t)) \notin D$, impossible) or $t^2 + y^2 > 1$. By continuity, then, there should exists another time $s$ where $s^2 + y^2 = 1$, impossible. Therefore $t^2 + y^2 < 1$ always.

For the concavity let's compute $y''$. Deriving the equation

$$y'' = -\frac{\left(t^2 + y^2 - 1\right)'}{\left(t^2 + y^2 - 1\right)^2} = -\frac{2t + 2yy'}{(t^2 + y^2 - 1)^2}$$

Therefore

$$y'' \geqslant 0, \iff t + yy' \leqslant 0.$$

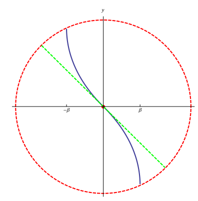
We know that $y' < 0$ always. Being $y(0) = 0$ we deduce that

$$y \geqslant 0, \iff t \leqslant 0.$$

Therefore: as $t \leqslant 0$, $y \geqslant 0$, $y' < 0$ hence $t + yy' \leqslant 0$; as $t \geqslant 0$, $y \leqslant 0$, $y' < 0$ hence $t + yy' \geqslant 0$. In conclusion: $y$ is convex as $t \leqslant 0$, concave as $t \geqslant 0$. In particular $y$ is below of its tangent for $t = 0$,

$$y(t) \leqslant y(0) + y'(0)t = -t,$$

and the straight line $y = -t$ crosses $t^2 + y^2 = 1$ at $t = \frac{\sqrt{2}}{2}$. It follows that $\beta < \frac{\sqrt{2}}{2}$. ■



## 7.4. Exercises

EXERCISE 7.4.1. Consider the equation

$$y' = y\,(e^y - 1)\,.$$

Check that local existence and uniqueness hypotheses are fulfilled. Find stationaries solutions and draw in the plane $(t, y)$ regions where solutions are increasing/decreasing. Since now, let $y$ be the maximal solution with $y(0) = y_0 > 0$. Discuss the monotonicity and concavity of $y$. Find the maximal interval and the limits at the extremes of it. Plot a graph of $y$.

EXERCISE 7.4.2. Consider the equation

$$y' = \frac{\log y}{1+y}.$$

Find the maximum domain where local existence and uniqueness hypotheses are fulfilled. Find stationaries solutions and region where solutions are increasing/decreasing. Since now let $y : ]\alpha, \beta[ \longrightarrow \mathbb{R}$ be the maximal solution with $y(0) = y_0 \in ]0, 1[$. Show that $\varphi$ is monotone and find its concavity. Show that $\alpha = -\infty$ and compute $\lim_{t\to-\infty} y(t)$. Say if $\beta < +\infty$ and compute $\lim_{t\to\beta-} y(t)$ and $\lim_{t\to\beta-} y'(t)$. Plot a graph of the solution.

EXERCISE 7.4.3. Consider the Cauchy problem

$$\begin{cases} y' = t(y^2 - 1) \arctan y, \\ \\ y(0) = y_0. \end{cases}$$

Check that local existence and uniqueness holds. Find stationaries solutions and regions in the plane $(t, y)$ where solutions are increasing/decreasing. Since now, let $y$ be the maximal solution with $y(0) = y_0 \in ]0, 1[$. Show that $y$ is even and discuss the nature of $t = 0$ for $y$. Find the maximal interval, the limits at its extremes and plot a graph of the solution.

EXERCISE 7.4.4. Let $y : ]\alpha, \beta[ \longrightarrow \mathbb{R}$ be the maximal solution of

$$\begin{cases} y' = \dfrac{1}{\log y + t}, \\ \\ y(1) = 1. \end{cases}$$

We accept that it exists. Show that $y$ is increasing, find its concavity, show that $\beta = +\infty$ and compute $y(+\infty)$. Show that $\alpha > -\infty$ and compute $\lim_{t\to\alpha+} y'(t)$. Plot a graph of the solution.

Exercise 7.4.5. Let $y : ]\alpha, \beta[ \longrightarrow \mathbb{R}$ be the maximal solution of

$$\begin{cases} y' = \dfrac{1}{t - \log y}, \\ \\ y(0) = e. \end{cases}$$

We accept that it exists. Of course, the graph of $\varphi$ is contained into $D_> := \{(t, y) \in \mathbb{R} \times \mathbb{R} \; : \; y > e^t\}$: why? Deduce that $y$ is increasing. Study the concavity of $y$. Show that if $\alpha > -\infty$ then $y(\alpha+) = +\infty$: deduce that $\alpha = \ldots$. Show that $\beta < +\infty$, and compute $\lim_{t \to \beta-} y'(t)$. Plot a graph of the solution. Show that $y(-\infty) = +\infty$.

Exercise 7.4.6. Consider the differential equation

$$y' = \frac{t}{y^2 - t^2}.$$

Find the maximum domain $D \subset \mathbb{R} \times \mathbb{R}$ on which local existence and uniqueness are fulfilled. Since now, let $y : ]\alpha, \beta[ \longrightarrow \mathbb{R}$ be a maximal solution of the Cauchy problem with $y(0) = 1$. Show that: $y$ is even, $y \nearrow$ on $[0, \beta[$ and deduce the nature of $t = 0$ for $y$. Show that $y(\beta-) = +\infty$. Plot a graph of the solution.

Exercise 7.4.7. Consider the differential equation

$$y' = \frac{\sin(ty)}{1 + y^2}.$$

Find the maximum domain $D \subset \mathbb{R} \times \mathbb{R}$ on which local existence and uniqueness are fulfilled. Do global existence and uniqueness holds? Find stationaries solutions and regions of the plane $(t, y)$ where solutions are increasing/decreasing. Since now, let $y : \mathbb{R} \longrightarrow \mathbb{R}$ be the solution of $\mathsf{CP}(0, 1)$. show that $\varphi > 0$ everywhere and that $y$ is even. Compute $y''(0)$ and deduce the nature of $t = 0$ for $y$. Show that necessarily there exists $\tau > 0$ such that $y$ has a maximum at $\tau$. Compute $y(\pm\infty)$. Plot a graph of the solution.

CHAPTER 8

# Systems of Differential Equations

In this Chapter, we extend the discussion on differential equations to the case of *systems of differential equations* or, equivalently, to *vectorial equations*. For simplicity we will limit our discussion to $2 \times 2$ *autonomous systems*, that is to systems of the form

$$\begin{cases} x' = f(x, y), \\ \\ y' = g(x, y), \end{cases} \quad \text{where } f, g : \Omega \subset \mathbb{R}^2 \longrightarrow \mathbb{R}. \tag{8.0.1}$$

After a quick view at existence and uniqueness results for the Cauchy Problem associated to this type of systems, we will move to our main goal, that is *to develop qualitative methods to study the behavior of solutions of the system*. Of course, we expect that to study solutions of a system will be more complex than studying solutions of one scalar equation. In this Chapter we will illustrate several possible methods that allow a discussion for a certain number of situations, including remarkable applications.

## 8.1. Cauchy Problem: Existence and Uniqueness

The Cauchy Problem for a $2 \times 2$ system can be naturally stated as

$$\mathsf{CP}(t_0, (x_0, y_0)) \; : \; \begin{cases} x' = f(x, y), \\ y' = g(x, y), \\ x(t_0) = x_0, \\ y(t_0) = y_0. \end{cases} \tag{8.1.1}$$

A solution is now a couple $x, y \in \mathscr{C}^1(I)$ fulfilling the system. In particular, $(x(t), y(t)) \in \Omega$ for all $t \in I$, hence also $(x_0, y_0) \in \Omega$. The fact that the system is *autonomous*, that is $f, g$ do not depend by $t$ explicitly, has a nice remarkable consequence on solutions:

Proposition 8.1.1. *$(x(t), y(t))$ solves $\mathsf{CP}(t_0, (x_0, y_0))$ iff $(x(t - t_0), y(t - t_0))$ solves $\mathsf{CP}(0, (x_0, y_0))$.*

In particular, for the Cauchy Problem we can conventionally consider $t_0 = 0$. Existence and uniqueness for CP (8.1.1) follows similar results as for the scalar case (be careful: here $f, g$ are not depending by $t$ explicitly). For instance we have

Theorem 8.1.2 (Global Cauchy–Lipschitz). *Let $f, g : \Omega = \mathbb{R}^2 \longrightarrow \mathbb{R}$ be such that*

i) *$f \in \mathscr{C}(\mathbb{R}^2)$;*
ii) *$\partial_x f,\ \partial_y f,\ \partial_x g,\ \partial_y g$ are bounded in $\mathbb{R}^2$, that is*

$$\exists L > 0 \; : \; |\partial_x f(x, y)|,\ |\partial_y f(x, y)|,\ |\partial_x g(x, y)|,\ |\partial_y g(x, y)| \leqslant L,\ \forall (x, y) \in \mathbb{R}^2.$$

*Then, for every $(x_0, y_0) \in \mathbb{R}^2$ there exists a unique solution to $\mathsf{CP}(0, (x_0, y_0))$.*

This Thm suffers the same critics of its homologous for scalar equations. A considerably weaker version is the

THEOREM 8.1.3 (LOCAL CAUCHY–LIPSCHITZ). *Let* $f, g : \Omega \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ *be such that*

i) $f \in \mathscr{C}(\Omega)$;
ii) $\partial_x f,\ \partial_y f,\ \partial_x g,\ \partial_y g \in \mathscr{C}(\Omega)$.

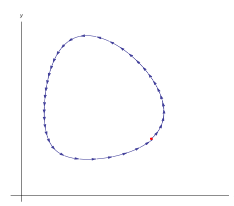
*Then, for every* $(x_0, y_0) \in \Omega$ *there exists a unique maximal solution to* $\mathsf{CP}(0, (x_0, y_0))$. *Such maximal solution is defined on a life time interval* $]\alpha, \beta[$ *with the following property: it is not possible to extend* $(x, y)$ *to* $J \supsetneq ]\alpha, \beta[$ *as solution.*

## 8.2. Orbits and Phase Portrait

Since now, we will assume that Local CL Thm Hypotheses are fulfilled. Let's consider a maximal solution $(x, y)$. Of course, to have an idea on its behavior, we could try to plot the two graphs of $x = x(t)$ and $y = y(t)$. This, however, might not be the most incisive way to describe the qualitative behavior for a system. Indeed, the most interesting qualitative information concerns the couple $(x, y)$ not just the single $x$ and $y$. The natural graph would be that of the function

$$t \longmapsto (x(t), y(t)),$$

but this is hard to be described being that of a curve in the space $t, x, y$. A good compromise is the following: we may see point $(x(t), y(t))$ as a point in movement in the plane $xy$, the *state space* of the system.



The curve $t \longmapsto (x(t), y(t))$ represents the evolution of the state of the system, so in some way its "trace" on the state space reveals a qualitative behavior. For instance, if the point $(x(t), y(t))$ is moving from left to right it means that $x \nearrow$. Let's introduce the important

DEFINITION 8.2.1. *Given a maximal solution* $(x, y)$ *defined on* $]\alpha, \beta[$, *we call* **orbit** *of the solution the set*

$$\gamma(x, y) := \{(x(t), y(t)) \ : \ t \in ]\alpha, \beta[\} \subset \Omega.$$

Among all the possible solutions, constant solutions are very important for applications: they represent states where nothing happens.

DEFINITION 8.2.2. *A point* $(x_0, y_0) \in \Omega$ *is said* **equilibrium** *if* $(x, y) \equiv (x_0, y_0)$ *is a constant solution.*

REMARK 8.2.3.

$$(x_0, y_0) \text{ equilibrium} \iff \begin{cases} f(x_0, y_0) = 0, \\ \\ g(x_0, y_0) = 0. \end{cases}$$

So, for instance, the orbit of a constant solution reduces to the singleton $\{(x_0, y_0)\}$, where $(x_0, y_0)$ is an equilibrium. In general, however, $\gamma(x, y)$ is a curve in the plane. As consequence of the existence and uniqueness, a certain number of elementary properties of orbits:

- every $(x_0, y_0) \in \Omega$ belongs precisely to a unique orbit;
- if two orbits have a common point they must coincide;
- if a solution is *periodic*, that is if there exists $T$ such that $(x(t+T), y(t+T)) = (x(t), y(t))$ for every $t$, then the orbit is a closed circuit in the plane.

A less immediate property, that actually follows by a suitable extension of the argument of exit by compact sets, is the following:

COROLLARY 8.2.4. *If an orbit $\gamma(x, y) \subset K \subset \Omega$ where $K$ is compact (that is, closed and bounded), then the maximal interval for the solution $(x, y)$ is $]-\infty, +\infty[$. In particular, if $\gamma(x, y)$ is a closed circuit, the solution $(x, y)$ is periodic.*

The way the point $(x(t), y(t))$ moves along $\gamma(x, y)$ is a fundamental qualitative information. Noticed that the tangent vector to $(x(t), y(t))$ is $(x'(t), y'(t))$, we can use this vector to define the **orientation** of the orbit. Qualitatively, we might distinguish four possible orientations:

- $\nearrow$, when $x \nearrow$ and $y \nearrow$;
- $\searrow$, when $x \nearrow$ and $y \searrow$;
- $\nwarrow$, when $x \searrow$ and $y \nearrow$;
- $\swarrow$, when $x \searrow$ and $y \searrow$.

Notice that

$$x \nearrow, \iff x' \geqslant 0, \iff f(x, y) \geqslant 0, \text{ while } y \nearrow, \iff y' \geqslant 0, \iff g(x, y) \geqslant 0$$

An orbit endowed with its orientation is called **oriented orbit**. We call **phase portrait** the plot of the typical oriented orbit of the system. Our goal will be to develop methods to plot the phase portrait (or parts of it) for a given system.

### 8.3. Autonomous Linear Systems

The global Cauchy–Lipschitz theorem applies in particular to the class of *autonomous linear system*, namely system of the form

$$\begin{cases} x' = ax + by, \\ y' = cx + dy, \end{cases} \qquad a, b, c, d \in \mathbb{R} \text{ (constants).} \tag{8.3.1}$$

It is clear that by posing $(f, g) := (ax + by, cx + dy)$, both $f, g$ fulfill the hypotheses of the Global CL Thm (8.1.2), because

$$\partial_x f \equiv a, \ \ \partial_y f \equiv b, \ \ \partial_x g \equiv c, \ \ \partial_y g \equiv d.$$

Of course, in this case we expect that some explicit solution formula should hold. Starting with equilibriums,

$$(x_0, y_0) \text{ equilibrium}, \iff \begin{cases} ax_0 + by_0 = 0, \\ cx_0 + dy_0 = 0. \end{cases}$$

Of course $(x_0, y_0) = 0_2$ is a solution, therefore it is always an equilibrium. This is the unique equilibrium if and only if

$$\det A = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0.$$

**Throughout this Section we will assume** $\det A \neq 0$. We will assume also that at least one between $b, c$ be $\neq 0$, this because if both $b = c = 0$ the system is actually a decoupled system of two first order equations

$$\begin{cases} x' = ax, \\ \\ y' = dy, \end{cases} \quad \begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{at} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{dt} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

This turns out a special case of case $\Delta > 0$ discusses below. For this reason we will assume $b \neq 0$.

To determine non constant solutions, notice that by deriving side by side the first equation we obtain

$$x'' = ax' + by' = ax' + b(cx + dy) = ax' + bcx + d(x' - ax) = (a + d)x - (ad - bc)x,$$

that is $x$ fulfills a second order linear equation with constant coefficients. The characteristic polynomial for this equation is

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0.$$

Incidentally, notice that this is the same of

$$0 = \det(\lambda \mathbb{I} - A) = \det \begin{bmatrix} \lambda - a & -b \\ -c & \lambda - d \end{bmatrix}.$$

In other words, $\lambda$ are the eigenvalues of $A$. As expected, we will have three main situations according $\Delta = (a + d)^2 - 4(ad - bc) \gtreqless 0$.

**Case** $\Delta > 0$**.** The characteristic polynomial has two real roots $\lambda_2 > \lambda_1$. Therefore,

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}, \quad c_1, c_2 \in \mathbb{R}.$$

Now, by $x' = ax + by$ it follows

$$by = x' - ax = c_1(\lambda_1 - a)e^{\lambda_1 t} + c_2(\lambda_2 - a)e^{\lambda_2 t}.$$

**If** $b \neq 0$,

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{\lambda_1 t} \begin{pmatrix} 1 \\ \frac{\lambda_1 - a}{b} \end{pmatrix} + c_2 e^{\lambda_2 t} \begin{pmatrix} 1 \\ \frac{\lambda_2 - a}{b} \end{pmatrix} \equiv c_1 e^{\lambda_1 t} \vec{v} + c_2 e^{\lambda_2 t} \vec{w}. \tag{8.3.2}$$

By this it is easy to have the phase portrait of the system. First, notice that being $\lambda_1 \neq \lambda_2$, $\vec{v}, \vec{w}$ are linearly independent, that is $(\vec{v}, \vec{w})$ form a base for $\mathbb{R}^2$. A straightforward calculation shows that $\vec{v}$ and $\vec{w}$ are eigenvectors for the coefficient matrix $A$ associated to eigenvalues $\lambda_1$ and $\lambda_2$ respectively. In such a base, the coordinates of the solution are

$$\xi = c_1 e^{\lambda_1 t}, \; \eta = c_2 e^{\lambda_2 t}.$$

Because $\lambda_1, \lambda_2$ cannot be both $= 0$, assuming for instance $\lambda_1 \neq 0$,

$$\eta = c_1 e^{\lambda_2 t} = c_1 \left(e^{\lambda_1 t}\right)^{\lambda_2/\lambda_1} = k\xi^{\lambda_2/\lambda_1},$$

that is orbits are described by lines $\eta = k\xi^{\gamma}$ with $\gamma = \frac{\lambda_2}{\lambda_1}$.

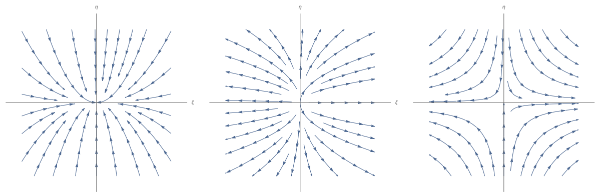


FIGURE 1. From the left to the right: $\lambda_2 > \lambda_1 > 0$, $0 > \lambda_2 > \lambda_1$, $\lambda_2 > 0 > \lambda_1$.

We notice that the orientation here may be determined by signs of $\lambda_1, \lambda_2$. Indeed, for instance: if $\lambda_1 > 0$, $\xi = c_1 e^{\lambda_1 t} \longrightarrow \infty$ as $t \longrightarrow +\infty$, that is the $\xi$ coordinate of $x$ "escapes" to infinity in the future while $\xi = c_1 e^{\lambda_1 t} \longrightarrow 0$ as $t \longrightarrow -\infty$. The same holds for $\eta$. Therefore:

- If $\lambda_2 > \lambda_1 > 0$, the solution "escapes" to infinity in the future: in this case we say that $0_2$ is an **unstable knot**;
- if $0 > \lambda_2 > \lambda_1$, the solution is "attracted" by $0_2$ in the future: in this case we way that $0_2$ is a stable a **stable knot**;
- if $\lambda_2 > 0 > \lambda_1$, the solution "escapes" in the first coordinate while it is attracted in the second: we say that $0_2$ is a **saddle**.

Finally, to have the real phase portrait we have to remind that $(\xi, \eta)$ are the coordinates of the solution $(x, y)$ in the base $\vec{v}, \vec{w}$.

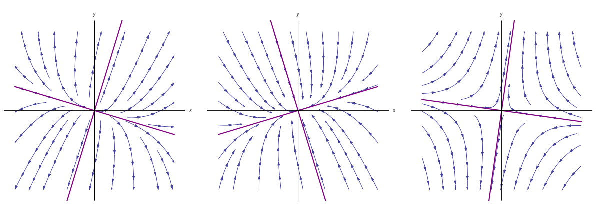


FIGURE 2. From the left: *unstable knot*, *stable knot*, *saddle*.

EXAMPLE 8.3.1. *Find the general solution and phase portrait of the system*

$$\begin{cases} x' = -2x + y, \\ \\ y' = -2y + x. \end{cases}$$

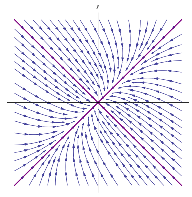
SOL. — The characteristic equation is

$$p(\lambda) = \det \begin{bmatrix} \lambda + 2 & -1 \\ -1 & \lambda + 2 \end{bmatrix} = (\lambda + 2)^2 - 1 = 0, \iff \lambda + 2 = \pm 1, \iff \lambda = -1, -3.$$

Therefore, according to (8.3.2),

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Because $\lambda_1, \lambda_2 < 0$ we have that $(0,0)$ is a stable knot. ■



**Case $\Delta = 0$.** The characteristic polynomial has a unique root $\lambda_1 \neq 0$ and

$$x(t) = c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_1 t}, \quad c_1, c_2 \in \mathbb{R}.$$

Again, by $x' = ax + by$

$$by = x' - ax = (c_1(\lambda_1 - a) + c_2)\, e^{\lambda_1 t} + c_2(\lambda_2 - a)e^{\lambda_2 t},$$

and because $b \neq 0$,

$$\begin{pmatrix} x \\ y \end{pmatrix} = (c_1 + c_2 t)e^{\lambda_1 t} \begin{pmatrix} 1 \\ \frac{\lambda_1 - a}{b} \end{pmatrix} + c_2 e^{\lambda_1 t} \begin{pmatrix} 0 \\ \frac{1}{b} \end{pmatrix} \equiv (c_1 + c_2 t)e^{\lambda_1 t}\vec{v} + c_2 e^{\lambda_1 t}\vec{w}. \tag{8.3.3}$$

Again, setting $\xi = (c_1 + c_2 t)e^{\lambda_1 t}$ and $\eta = c_2 e^{\lambda_1 t}$ we have

$$\xi = k\eta + h\eta \log \eta.$$

Also in this case the orientation can be easily deduced by the sign of $\lambda_1$.

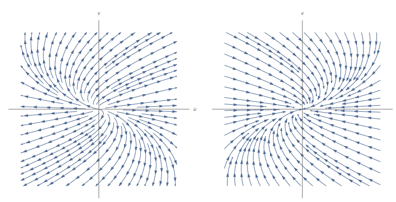


FIGURE 3. From left $\lambda_1 > 0$ (**unstable improper knot**), $\lambda_1 < 0$ (**stable improper knot**).

EXAMPLE 8.3.2. *Find the phase portrait of*

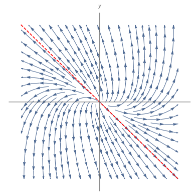
$$\begin{cases} x' = x - y, \\ y' = x + 3y. \end{cases}$$

SOL. — The characteristic polynomial is

$$\det(\lambda\mathbb{I} - A) = \det \begin{bmatrix} \lambda - 1 & 1 \\ -1 & \lambda - 3 \end{bmatrix} = (\lambda - 1)(\lambda - 3) + 1 = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2,$$

hence the unique eigenvalue is $\lambda = 2$. According to (8.3.3),

$$\begin{pmatrix} x \\ y \end{pmatrix} = (c_1 + c_2 t)e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{2t} \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$



**Case** $\Delta < 0$**.** Let $\lambda_{1,2} = \alpha \pm i\omega$, $\omega \neq 0$. Then

$$x(t) = c_1 e^{\alpha t} \cos(\omega t) + c_2 e^{\alpha t} \sin(\omega t).$$

Setting $\phi = e^{-\alpha t} x$, $\psi = e^{-\alpha t} y$, we have

$$\phi' = -c_1 \omega \sin(\omega t) + c_2 \omega \cos(\omega t),$$

and because $by = x' - ax$,

$$b\psi = be^{-\alpha t} y = be^{-\alpha t} x' - ae^{-\alpha t} x = be^{-\alpha t} x' - a\phi.$$

Now,

$$\phi' = \left(e^{-\alpha t} x\right)' = -\alpha e^{-\alpha t} x + e^{-\alpha t} x' = -\alpha\phi + e^{-\alpha t} x',$$

we obtain finally,

$$\psi = (\phi' + \alpha\phi) - \frac{a}{b}\phi = \phi' + (\alpha - \frac{a}{b})\phi.$$

Therefore

$$\begin{aligned}
\begin{pmatrix} x \\ y \end{pmatrix} &= e^{\alpha t} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = e^{\alpha t} \begin{pmatrix} c_1 \cos(\omega t) + c_2 \sin(\omega t) \\ -c_1 \omega \sin(\omega t) + c_2 \omega \cos(\omega t) + (\alpha - \frac{a}{b})(c_1 \cos(\omega t) + c_2 \sin(\omega t)) \end{pmatrix} \\
&= e^{\alpha t} \left( (c_1 \cos(\omega t) + c_2 \sin(\omega t)) \begin{pmatrix} 1 \\ \alpha - \frac{a}{b} \end{pmatrix} + (-c_1 \sin(\omega t) + c_2 \cos(\omega t)) \begin{pmatrix} 0 \\ -\omega \end{pmatrix} \right) \\
&= e^{\alpha t} \left(\xi \vec{v} + \eta \vec{w}\right), \quad \text{where } \vec{v} = \begin{pmatrix} 1 \\ \alpha - \frac{a}{b} \end{pmatrix}, \ \vec{w} = \begin{pmatrix} 0 \\ -\omega \end{pmatrix}.
\end{aligned}$$

Clearly, $\vec{v}$ and $\vec{w}$ are linearly independent. Furthermore, $\xi^2 + \eta^2 = c_1^2 + c_2^2$ constant, therefore

$$\xi \vec{v} + \eta \vec{w}$$

are points of an ellipse with axes $\vec{v}$ and $\vec{w}$. The factor $e^{\alpha t}$ leads to three sub cases:

- if $\alpha > 0$, orbits are elliptic spirals and solutions escape to infinity as $t \longrightarrow +\infty$: we say that $0_2$ is an **unstable focus**;
- if $\alpha = 0$, orbits are ellipses: we say that $0_2$ is a **center**;
- if $\alpha < 0$, orbits are elliptic spirals and solutions go to $0_2$ as $t \longrightarrow +\infty$: we say that $0_2$ is a **stable focus**.

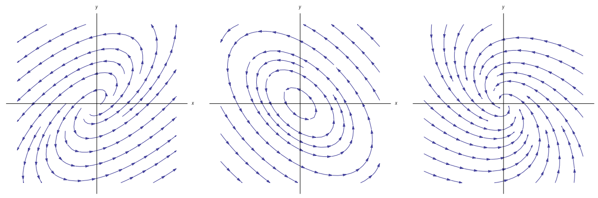


Figure 4. From left to right: *unstable focus*, *center*, *stable focus*.

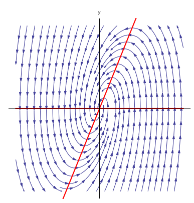
Example 8.3.3. *Find the phase portrait of the system*

$$\begin{cases} x' = -y, \\ y' = 5x - 2y. \end{cases}$$

Sol. — We have

$$A = \begin{bmatrix} 0 & -1 \\ 5 & -2 \end{bmatrix}, \quad \det(\lambda \mathbb{I} - A) = \det \begin{bmatrix} \lambda & 1 \\ -5 & \lambda + 2 \end{bmatrix} = \lambda(\lambda + 2) + 5 = \lambda^2 + 2\lambda + 5 = 0,$$

by which $\lambda_{1,2} = \frac{-2 \pm \sqrt{-16}}{2} = -1 \pm 2i$, that is $\alpha = -1$, $\omega = 2$. The point $(0, 0)$ is a stable focus then. ■



## 8.4. Non Linear Systems: First Integrals

The plot of the phase portrait for a general $2 \times 2$ system can be very difficult. Even in the case of linear systems, as shown in the previous section, the discussion is non trivial. In this Section we will see a general method to get a picture of the phase portrait. This method works in a certain number of cases, but not in general.

If we imagine orbits as curves in the plane $x, y$, we might think that, at least locally (that is at least by taking small pieces of orbit) they will looks like graph of functions. For instance, at least locally, we might expect that an orbit has the form $y = y(x)$. We have to be aware with the use of this notation because we're improperly using the same letter $y$ for two different things:

- $y = y(t)$ is the $y-$solution of the system;
- $y = y(x)$ is the ordinate of the point $(x, y)$ along the orbit having abscissa $x$.

So, accepting a bit of ambiguity, with this notation

$$y(t) = y(x(t)),$$

where it is clear that the two $y$ present in this identity have different meanings. Now, imagine we're on the orbit at the point $(x, y(x))$. At this point we will have a tangent to the curve with angular coefficient $\frac{dy}{dx}$ (we prefer here to do not use $y'$ that denotes $\frac{dy}{dt}$). The point is that the same angular coefficient can be computed in another way. Assume that at time $t$ the curve passes through $(x, y)$: as every plane curve will have a tangent vector $(x'(t), y'(t))$. The slope of this vector is

$$\frac{y'(t)}{x'(t)} = \frac{g(x(t), y(t))}{f(x(t), y(t))} = \frac{g(x, y)}{f(x, y)}.$$

But this must be the same as $\frac{dy}{dx}$. The moral is: *if $y = y(x)$ is a function whose graph is the orbit then*

$$\boxed{\frac{dy}{dx} = \frac{g(x, y)}{f(x, y)}.} \tag{8.4.1}$$

This equation is called **total equation**.

Example 8.4.1. *By using the total equation, determine the orbits of the system*

$$\begin{cases} x' = y, \\ y' = -x. \end{cases}$$

Sol. — We've a linear system. In this particular case we can compute directly the solution by noticing that

$$x'' = y' = -x, \iff x'' + x = 0.$$

This leads to $x = c_1 \cos t + c_2 \sin t$ and $y = -c_1 \sin t + c_2 \cos t$. It is easy to see that $x^2 + y^2 = c_1^2 + c_2^2$. Let's see what happens with the total equation: this is

$$\frac{dy}{dx} = -\frac{x}{y}.$$

This equation can be solved by separation of variables:

$$y\, dy = -x\, dx, \iff \frac{y^2}{2} = -\frac{x^2}{2} + k, \iff y^2 + x^2 = k. \quad \blacksquare$$

The previous example illustrates a situation that can be formalized in general:

Proposition 8.4.2. *Let the total equation be a separable variables equation*

$$\frac{dy}{dx} = \frac{g(x, y)}{f(x, y)} \equiv \frac{a(x)}{b(y)}.$$

*Then, by setting*

$$E(x, y) := \int b(y)\, dy - \int a(x)\, dx \tag{8.4.2}$$

*orbits are contained into the level sets of $E$, that is into sets $\{E \equiv c\}$ for some $c \in \mathbb{R}$.*

Proof — Consider a couple $(x(t), y(t))$ solution of the system. Then,

$$\frac{d}{dt} E(x(t), y(t)) = \partial_x E(x(t), y(t)) x'(t) + \partial_y E(x(t), y(t)) y'(t).$$

Now,

$$\partial_x E = \partial_x \left( \int b(y)\, dy - \int a(x)\, dx \right) = -a(x), \quad \partial_y E = \partial_y \left( \int b(y)\, dy - \int a(x)\, dx \right) = b(y).$$

Moreover, because of the equation $x' = f(x, y)$ and $y' = g(x, y)$. Therefore

$$\frac{d}{dt} E(x(t), y(t)) = -a(x(t)) f(x(t), y(t)) + b(y(t)) g(x(t), y(t)) \equiv 0,$$

because, by hypothesis $\frac{g(x,y)}{f(x,y)} \equiv \frac{a(x)}{b(y)}$, that is $a(x) f(x, y) \equiv b(y) g(x, y)$. ■

Definition 8.4.3. *A function* $E : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *is called* **first integral** *for (8.0.1) if*

$$E(x(t), y(t)) = \textit{costant}, \ \forall \ \textit{solution}\ (x(t), y(t)).$$

Example 8.4.4. *Consider the system*

$$\begin{cases} x' = -\sinh y, \\ \\ y' = \sinh x, \end{cases}$$

*i) Find the stationary points. ii) Find a non trivial first integral. iii) Plot the phase portrait of the system.*

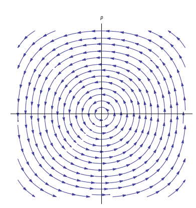
Sol. — i) $(x, y) \equiv (\xi, \eta)$ is a stationary point iff

$$\begin{cases} 0 = -\sinh \eta, \\ \\ 0 = \sinh \xi, \end{cases} \iff (\xi, \eta) = (0, 0).$$

ii) Let's write the total equation:

$$\frac{dy}{dx} = -\frac{\sinh x}{\sinh y}, \iff \sinh y\, dy = -\sinh x\, dx, \iff \cosh y = -\cosh x + c.$$

Notice that the surfaces $\{E = E_0\}$ are clearly closed (being $E$ continuous) and bounded (because it is easy to check that $\lim_{(x,y)\to\infty_2} E(x, y) = -\infty$).



It is also easy to find out the orientation of the orbits: by the system we have that

- $x' > 0$, iff $-\sinh y > 0$, iff $y < 0$: therefore $x \nearrow$ when $y < 0$;
- $y' > 0$, iff $\sinh x > 0$, iff $x > 0$: therefore $y \nearrow$ when $x > 0$. ■

Notice that a *trivial* first integral always exists: just take any constant function $E$. Of course, this kind of integral is completely useless: the level sets $\{E = c\}$ are empty or $\mathbb{R}^2$ according to the value of $c$. Moreover, the first integral doesn't contain any information on the orientation of the orbits, as the following example shows.

Example 8.4.5.

$$\begin{cases} x' = y, \\ \\ y' = -x, \end{cases} \quad e \quad \begin{cases} x' = -y, \\ \\ y' = x, \end{cases}$$

Sol. — In both cases the total equation is the same:

$$\frac{dy}{dx} = \frac{x}{-y} = -\frac{x}{y}, \quad \frac{dy}{dx} = \frac{x}{-y} = -\frac{x}{y},$$

and leads easily to the first integral $E(x, y) = x^2 + y^2$. ∎

An important remark is that *in general a level set $\{E = c\}$ could not coincide with a single orbit, but it may contains several orbits.*

Example 8.4.6. *Find stationary points, non trivial first integral and phase portrait of the system*

$$\begin{cases} x' = y(x - y), \\ \\ y' = -x(x - y), \end{cases}$$

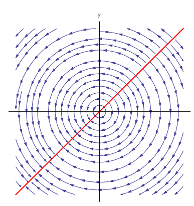
Sol. — $(x, y) \equiv (a, b)$ is stationary point iff

$$\begin{cases} 0 = b(a - b), \\ \\ 0 = -a(a - b). \end{cases} \iff (a, b) = (0, 0), \text{ or } a = b.$$

Therefore, all points (infinitely many) of the straight line $y = x$ are stationary points. The total equation is

$$\frac{dy}{dx} = \frac{-x(x - y)}{y(x - y)} = -\frac{x}{y}, \iff y\, dy = -x\, dx, \iff E(x, y) := x^2 + y^2 = c.$$

On the level line $E(x, y) = E_0$ with $E_0 > 0$ there're two stationary points (hence two orbits) as well as other two non stationary orbits (two half circles). It is indeed evident that any of the two half circles is a separate orbit because a solution $(x(t), y(t))$ cannot jump from one half circle to the other being a continuous function. It remains therefore all time in one of the two half. The orientation is easily deduced. ∎



Example 8.4.7 (prey–predator system). *Find the phase portrait in the first quarter of the system*

$$\begin{cases} x' = \mu x - \nu x y, \\ \\ y' = \lambda y - \kappa x y. \end{cases}$$

Sol. — Let $F \equiv (f, g) := (\mu x - \nu xy, \lambda y + \kappa xy)$, $f, g \in \mathscr{C}^1$. Easily local existence and uniqueness conditions are fulfilled but nor global lipschitz or sub–linear growth hypotheses are fulfilled. Let's look for a first integral. The total equation

$$\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)} = \frac{\lambda y - \kappa xy}{\mu x - \nu xy}.$$

This is a separable variables equation writing

$$\frac{dy}{dx} = \frac{y(\lambda - \kappa x)}{x(\mu - \nu y)}, \iff \frac{\mu - \nu y}{y} dy = \frac{\lambda - \kappa x}{x} dx, \iff \left(\frac{\mu}{y} - \nu\right) dy = \left(\frac{\lambda}{x} - \kappa\right) dx.$$

Integrating

$$\mu \log |y| - \nu y = \int \left(\frac{\mu}{y} - \nu\right) dy = \int \left(\frac{\lambda}{x} - \kappa\right) dx = \lambda \log |x| - \kappa x + c,$$

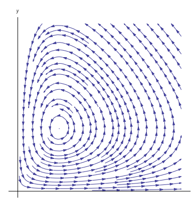
hence

$$\mu \log |y| - \nu y = \lambda \log |x| - \kappa x + c.$$

Being these the curves $y = y(x)$ along which $E(x, y)$ is constant, it follows that

$$E(x,y) = \lambda \log |x| - \kappa x - \mu \log |y| + \nu y,$$

is a first integral. The plot of the level sets of $E$ is, however, now easy: we limit here to reproduce the plot.



The picture shows clearly a cyclical behavior of the system. ■

First Integrals are also useful to find explicit solutions of the systems through a *dimensional reduction*. To show this, assume that $E$ be a non trivial first integral. Hence

$$E(x(t), y(t)) \equiv E(x(0), y(0)) =: E_0.$$

This is an algebraic equation that could be used to express one of the two functions in terms of the other, for instance

$$y(t) = \Phi(x(t), E_0). \tag{8.4.3}$$

Then, by the first equation of the system (8.0.1) we have

$$x'(t) = f(x(t), \Phi(x(t), E_0)),$$

which is an ODE involving only $x$. This can be studied and in some cases even solved finding the $x$ of the couple solution. By (8.4.3) the $y$ can be deduced.

Example 8.4.8. *Consider the system*

$$\begin{cases} x' = xy, \\ \\ y' = -x^2 + 2x^4. \end{cases}$$

*i) Find stationary points. ii) Find a non trivial first integral. iii) Plot the phase portrait of the system: are there periodic or global solutions? iv) Find the x solution of the Cauchy problem $x(0) = 2,\ y(0) = 2\sqrt{3}$.*

Sol. — i) A point $(a, b)$ is a stationary solution iff

$$\begin{cases} ab = 0, \\ \\ 2a^4 - a^2 = 0, \end{cases} \iff a = 0,\ \forall b,\ \vee\ b = 0,\ a^2(2a^2 - 1) = 0, \iff a = 0,\ \forall b,\ \vee\ b = 0,\ a = \pm\frac{1}{\sqrt{2}}.$$

All points $(0, b)$ $(b \in \mathbb{R})$ and $\left(\pm\frac{1}{\sqrt{2}}, 0\right)$ are stationary points.

ii) The total equation is

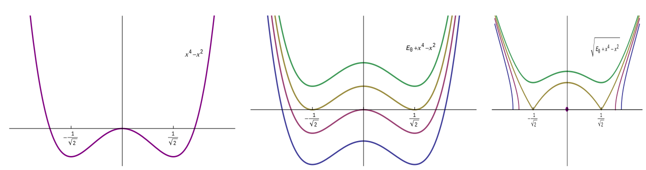
$$xy\ dy = (-x^2 + 2x^4)\ dx, \iff y\ dy = (-x + 2x^3)\ dx, \iff \frac{y^2}{2} = -\frac{x^2}{2} + \frac{x^4}{2} + C,$$

by which $E(x, y) = y^2 + x^2 - x^4$ is a non trivial first integral.

iii) We have to plot lines $E \equiv E_0$, that is

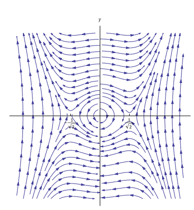
$$y^2 + x^2 - x^4 \equiv E_0, \iff y = \pm\sqrt{E_0 + x^4 - x^2}.$$

It is easy to plot $x^4 - x^2$: it is even, $(x^4 - x^2)' = 4x^3 - 2x = 2x(2x^2 - 1) = 0$ as $x = 0, \pm\frac{1}{\sqrt{2}}$. Easily we deduce the monotonicity and we see that $x = 0$ is a local max, $x = \pm\frac{1}{\sqrt{2}}$ are global mins. To plot the level lines we have to translate up and down this graph, hence take $\pm$ the root, reminding that the root follows the behavior of its argument ($\sqrt{0} = 0$, $\sqrt{\diamond}$ has a minimum/increase/decrease when $\diamond$ has a minimum/increase/decrease). The result is the following:



By this it is easy to plot the level sets $E = E_0$. Finally the orientation is

$$x \nearrow, \iff x' = xy > 0, \iff (x, y) \in \text{first and third quarter.}$$



It is evident that, apart the constant solutions, there're periodic solutions on energy levels $0 < E_0 < \frac{1}{2}$ on the component of the level sets with $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$. These solutions are also global. The case $E_0 = \frac{1}{2}$ produces also two non constant non periodic globally defined solutions.
iv) Our solution fulfills $E(x, y) = E(x(0), y(0)) = E(2, 2\sqrt{3}) = (2\sqrt{3})^2 + 2^2 - 2^4 = 12 + 4 - 16 = 0$, hence it belongs to the level set $E = 0$. Then $y^2 = x^4 - x^2$. Because $x(0), y(0) > 0$, at least in a neighborhood of $t = 0$ we must have $y = \sqrt{x^4 - x^2} = x\sqrt{x^2 - 1}$. Replacing in the first equation we obtain

$$x' = xy = x \cdot x\sqrt{x^2 - 1} = x^2\sqrt{x^2 - 1}, \iff \frac{x'}{x^2\sqrt{x^2 - 1}} = 1.$$

This is a separable variable equation: integrating and setting $u = x(t)$

$$t + C = \int \frac{1}{u^2\sqrt{u^2 - 1}}\, du = \int \frac{1}{u^2\sqrt{u^2 - 1}}\, du \overset{u=\cosh v,\ du=\sinh v\ dv}{=} \int \frac{1}{(\cosh v)^2\sqrt{(\cosh v)^2 - 1}} \sinh v\, dv.$$

Now $\sqrt{(\cosh v)^2 - 1} = \sqrt{(\sinh v)^2} = |\sinh v| = \sinh v$ if $v \geqslant 0$, hence

$$t + C = \int \frac{1}{(\cosh v)^2}\, dv = \tanh v, \iff v = \tanh^{-1}(t + C), \iff x(t) = \cosh^{-1}\left(\tanh^{-1}(t + C)\right).$$

By imposing the initial condition we find $C$. ■

## 8.5. Conservative systems

The method presented in the previous Section can be applied to important second order differential equations of the form

$$y'' = \partial_y V(y), \text{ where } V : D \subset \mathbb{R} \longrightarrow \mathbb{R}. \tag{8.5.1}$$

This type of equation is very important because it represents the paradigm for *Newtonian conservative systems*. The reason why these equations are called *conservative* is because there's an important mechanical quantity conserved along the solutions:

Proposition 8.5.1. *If y is a solution of (8.5.1) then the* mechanical energy

$$E(y, y') := \frac{1}{2}y'^2 - V(y),$$

*is constant.*

Proof — This is easy to check:

$$\frac{d}{dt}E(y, y') = \frac{d}{dt}\left(\frac{1}{2}y'^2 - V(y)\right) = \frac{1}{2}2y'y'' - \partial_y V(y)y' = y'\left(y'' - \partial_y V(y)\right) \equiv 0. \quad ■$$

What is the connection with the discussion done in the previous section? We can always transform an equation like (8.5.1) into an equivalent $2 \times 2$ system: indeed, setting $q = y$, $p = y'$ (we adopt here the canonical mechanical letters), then

$$\begin{cases} q' = p =: f(q, p), \\ \\ p' = \partial_q V(q) =: g(q, p). \end{cases} \tag{8.5.2}$$

Mechanical energy turns out to be a first integral for this system. Indeed, the total equation leads to

$$\frac{dp}{dq} = \frac{\partial_q V(q)}{p}, \iff p\,dp = \partial_q V(q)\,dq, \iff \frac{p^2}{2} = V(q) + c, \iff \frac{1}{2}p^2 - V(q) = c,$$

and of course $E(q,p) = \frac{1}{2}p^2 - V(q)$ is nothing but the mechanical energy written with other letters. This correspondance makes natural to adopt the language and the methods developed in the previous section but be careful:

- an orbit is a curve in the space $(q,p) = (y,y')$ (space×velocity/momentum) which is called *phase space*;
- the orientation of orbits follows a simple rule: by the first equation of the system (8.5.2)

$$q \nearrow \iff q' = p \geqslant 0, \iff (q,p) \text{ belongs in the upper half plane of the phase space.}$$

Moreover the equivalent system is not always really needed. For instance in this case *the conservation of the energy offers a non trivial and powerful reduction of the order of the equation*. Let's see in general why: writing the conservation of the energy we have

$$\frac{1}{2}(y')^2 - V(y) = E_0. \tag{8.5.3}$$

This is a *first order equation not in normal form*. In this case it is however easy to extract $y'$ reducing effectively the order of the (8.5.1), because

$$y' = \pm\sqrt{2(E_0 - V(y))}.$$

A slightly delicate question concern the sign. Because these are two equations, which one should be considered really? Unless you're in the delicate case when $y'(0) = 0$, if for instance $y'(0) > 0$ it is clear that at least for a neighborhood of $t = 0$ $y' > 0$ hence the correct equation must be that one with $+$.

Example 8.5.2 (pendulum without friction). *Find energy and trace of orbits in phase space for the pendulum without friction*

$$m\ell\theta''(t) = -mg\sin(\theta(t)).$$

Sol. — We assume $m = 1$. First notice that the equation may be rewritten as

$$\theta'' = -\frac{g}{\ell}\sin\theta = \frac{g}{\ell}\partial_\theta\cos\theta = \partial_\theta\left(\frac{g}{\ell}\cos\theta\right)$$

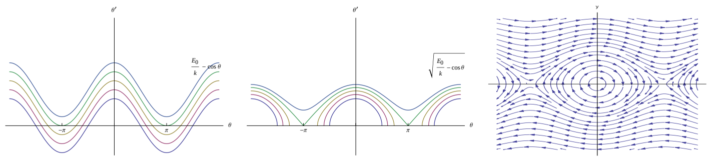
that is the equation is conservative. The mechanical energy is

$$E(\theta,\theta') = \frac{1}{2}\theta'^2 - \frac{g}{l}\cos\theta.$$

Now

$$E(q,p) = E_0, \iff \frac{\theta'^2}{2} - \frac{g}{l}\cos\theta = E_0, \iff \theta'^2 = 2\left(E_0 + \frac{g}{l}\cos\theta\right), \;\; \theta' = \pm\sqrt{2\left(E_0 + \frac{g}{l}\cos\theta\right)}.$$

It is not very difficult to plot the surfaces $E = E_0$. The orientation being standard we obtain the following picture: By looking at the picture there're some interesting remarks. We see the classical periodic oscillatory motion (the closed cycles). The limit case when the cycle closes on an equilibrium correspond to a non periodic motion reaching in an infinite time in the future/past the equilibrium position $\theta = \pi$. Finally the not closed orbits in the upper and lower half plane corresponds to rotations: when the mass receive initially more than certain minimum energy



the mass rotate infinitely many times. This is visible being $\theta$ an increasing/decreasing according to the direction anti/clockwise of the motion. ■

Example 8.5.3. *Consider the differential equation*

$$y'' = y^2 - y, \quad (\star).$$

*It is easy to check that local existence and uniqueness are fulfilled. Find stationary solutions. Show that if $\varphi : I \longrightarrow \mathbb{R}$ is a solution of $(\star)$ then also $\psi : -I \longrightarrow \mathbb{R}$, $\psi(t) = \varphi(-t)$ is a solution. Let now $\varphi$ be a maximal solution of the Cauchy problem with initial conditions $\varphi(0) = a$, $\varphi'(0) = 0$. Show that $\varphi$ is even. Find the energy of the system and use this to find explicitly the solution of the Cauchy problem $\varphi(0) = 3$, $\varphi'(0) = \sqrt{3}$.*

Sol. — We have that $\varphi(t) \equiv C$ is a solution iff $0 = C^2 - C = C(C-1)$, that is for $C = 0$, $C = 1$.

Let $\varphi : I \longrightarrow \mathbb{R}$ be a solution and let $\psi : -I \longrightarrow \mathbb{R}$, be such that $\psi(t) := \varphi(-t)$. Therefore $\psi'(t) = -\varphi'(-t)$, $\psi''(t) = \varphi''(-t)$ hence

$$\psi''(t) = \varphi''(-t) = \varphi(-t)^2 - \varphi(-t) = \psi(t)^2 - \psi(t),$$

that is $\psi$ is a solution.

Let $\varphi : ]\alpha, \beta[ \longrightarrow \mathbb{R}$. We have seen that $\psi : ]-\beta, -\alpha[ \longrightarrow \mathbb{R}$, $\psi(t) := \varphi(-t)$ is a solution. Moreover $\psi(0) = \varphi(0) = a$ and $\psi'(0) = -\varphi'(0) = 0$. Therefore $\psi$ solves the same Cauchy problem solved by $\varphi$: by uniqueness the two are equal where both defined. Easily it follows that $\alpha = -\beta$ (by maximality) hence $\varphi(t) = \varphi(-t)$ for every $t \in ]-\beta, \beta[$.

We may see $(\star)$ as a Newton equation (mass $m = 1$) with force

$$F(y) = y^2 - y = \nabla_y \left( \frac{y^3}{3} - \frac{y^2}{2} \right).$$

This means that $V(y) = \frac{y^3}{3} - \frac{y^2}{2}$ is the potential so the energy in the phase space $q = y$, $p = my' = y'$ is

$$E(q,p) := \frac{p^2}{2} - \left( \frac{q^3}{3} - \frac{q^2}{2} \right).$$

Let $\varphi$ then the maximal solution of the Cauhcy problem with $\varphi(0) = 3$, $\varphi'(0) = 3$. Because $E(3, \sqrt{3}) = \frac{9}{2} - \left( \frac{27}{3} - \frac{9}{2} \right) = 9 - 9 = 0$, the solution must fulfills

$$E(\varphi, \varphi') \equiv 0, \iff \frac{\varphi'^2}{2} - \frac{\varphi^2}{2}\left( \frac{2}{3}\varphi - 1 \right) \equiv 0, \iff \varphi'^2 \equiv \varphi^2 \left( \frac{2}{3}\varphi - 1 \right), \iff \varphi' = \pm|\varphi|\sqrt{\frac{2}{3}\varphi - 1}.$$

Now $\varphi'(0) = 3$ while $\varphi(0) = 3$, so, at least in a neighborhood of initial time $t = 0$ we have

$$\varphi' = \varphi\sqrt{\frac{2}{3}\varphi - 1}, \iff \frac{\varphi'}{\varphi\sqrt{\frac{2}{3}\varphi - 1}} = 1, \iff \int \frac{\varphi'}{\varphi\sqrt{\frac{2}{3}\varphi - 1}} = t + c.$$

Now

$$\int \frac{\varphi'}{\varphi\sqrt{\frac{2}{3}\varphi - 1}} \overset{u=\varphi(t)}{=} \int \frac{1}{u\sqrt{\frac{2}{3}u - 1}}\, du \overset{v=\sqrt{\frac{2}{3}u-1}, u=\frac{3}{2}(v^2+1),\ du=3\ dv}{=} \int \frac{1}{\frac{3}{2}(v^2+1)v} 3v\ dv = 2\arctan v$$

$$= 2\arctan\sqrt{\frac{2}{3}u - 1} = 2\arctan\sqrt{\frac{2}{3}\varphi - 1},$$

hence

$$2\arctan\sqrt{\frac{2}{3}\varphi(t) - 1} - \frac{\pi}{2} = t + c.$$

Setting $t = 0$ we find $2\arctan 1 = c$, that is $c = \frac{\pi}{2}$, so the solution is $\varphi(t) = \frac{3}{2} + \frac{3}{3}\left(\tan\frac{2t+\pi}{4}\right)^2$. ■

## 8.6. Stability

Stationary points are very important state for a system: they express *equilibriums*. An important question is: *what happens to a system when we perturb the state from an equilibrium?* Precisely: assume that $(x_0, y_0)$ be a stationary point for a system. Imagine that we consider a solution of the system with $(x(0), y(0)) \approx (x_0, y_0)$. We wonder if

- does the solution remain close to $(x_0, y_0)$ for all future times?
- if yes, is it true that $(x(t), y(t)) \longrightarrow (x_0, y_0)$ as $t \longrightarrow +\infty$?

Let's introduce two formal definitions:

Definition 8.6.1. *We say that an equilibrium $(x_0, y_0)$ is*

- **stable** *if:*

$$\forall U_{(x_0,y_0)},\ \exists V_{(x_0,y_0)}\ :\ \forall (x(0), y(0)) \in V_{(x_0,y_0)} \implies (x(t), y(t)) \in U_{(x_0,y_0)},\ \forall t \geqslant 0. \tag{8.6.1}$$

- **asimptotically stable** *if*

$$\exists V_{(x_0,y_0)}\ :\ \forall (x(0), y(0)) \in V_{(x_0,y_0)}, \implies (x(t), y(t)) \longrightarrow (x_0, y_0). \tag{8.6.2}$$

For a linear system, the nature of the equilibrium $(0, 0)$ follows quickly by the detailed discussion done at beginning of the Chapter. Easily we obtain that

Theorem 8.6.2. *We have that*

- *a (proper or improper)* **stable knot** *and a* **stable focus** *are stable and asymptotically stable;*
- *a* **center** *is stable but not asymptotically stable;*
- *a (proper or improper)* **unstable knot** *and an* **unstable focus** *are neither stable and asymptotically stable.*

For a general non linear system, if a phase portrait is available it is in general possible by this to deduce the local nature of equilibriums. If, however, a phase portrait is not available, it is possible to compare the local behavior of the non linear system to that of a suitable linear system. This system is called *linearized system* of the initial one. Let's sketch the key idea on which this reduction is based.

Suppose that $(x_0, y_0)$ is a stationary point for the system. We recall that

$$(x_0, y_0) \text{ stationary point} \iff \begin{cases} f(x_0, y_0) = 0, \\ g(x_0, y_0) = 0. \end{cases} \tag{8.6.3}$$

To study the perturbation from the equilibrium let's introduce the gaps

$$u := x - x_0, \ \ v := y - y_0,$$

in such a way that

$$\begin{cases} u' = x' = f(x, y) = f(x_0 + u, y_0 + v) = f(x_0, y_0) + \nabla f(x_0, y_0) \cdot (u, v) + o(u, v), \\ v' = y' = g(x, y) = g(x_0 + u, y_0 + v) = g(x_0, y_0) + \nabla g(x_0, y_0) \cdot (u, v) + o(u, v). \end{cases}$$

Because of (8.6.3) we deduce that

$$\begin{cases} u' = \nabla f(x_0, y_0) \cdot (u, v) + o(u, v), \\ v' = \nabla g(x_0, y_0) \cdot (u, v) + o(u, v). \end{cases} \tag{8.6.4}$$

This is just a way to rewrite the initial system. Now, $(x, y) \approx (x_0, y_0)$ iff $(u, v) \approx (0, 0)$ so we could say that $(u, v)$ should look like the solution of the following system

$$\begin{cases} u' = \nabla f(x_0, y_0) \cdot (u, v) = \partial_x f(x_0, y_0)u + \partial_y f(x_0, y_0)v, \\ v' = \nabla g(x_0, y_0) \cdot (u, v) = \partial_x g(x_0, y_0)u + \partial_y g(x_0, y_0)v. \end{cases} \tag{8.6.5}$$

System (8.6.5) is called **linearized system around** $(x_0, y_0)$. We have the

Theorem 8.6.3 (linearization). *Let $(x_0, y_0)$ a stationary point for the system (8.0.1). If $(0, 0)$ is a knot, a focus or a saddle for the linearized system (8.6.5) then the same holds for the original system. In particular: if $(0, 0)$ is asymptotically stable (unstable) for the linearized system, $(x_0, y_0)$ is asymptotically stable (unstable) for the original system.*

Example 8.6.4. *Study equilibriums for the system*

$$\begin{cases} x' = -x + e^{-y} - 1, \\ y' = e^{x-y} - 1. \end{cases}$$

Sol. — An equilibrium $(x, y)$ fulfills

$$\begin{cases} -x + e^{-y} - 1 = 0, \\ e^{x-y} - 1 = 0. \end{cases} \iff \begin{cases} x = y, \\ -x + e^{-x} - 1 = 0, \iff e^{-x} = 1 + x. \end{cases}$$

Easily we see that $e^{-x} = 1 + x$ iff $x = 0$, by which $y = 0$, that is $(0, 0)$ is the unique equilibrium. Now, if $F = (f, g) = (-x + e^{-y} - 1, e^{x-y} - 1)$, the jacobian matrix of $F$ is

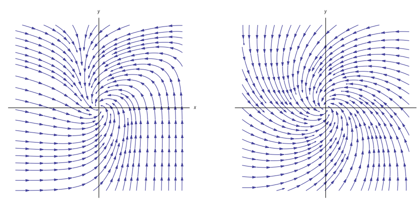
$$F(x, y) = (-x + e^{-y} - 1, e^{x-y} - 1), \ \ F'(x, y) = \begin{bmatrix} -1 & -e^{-y} \\ e^{x-y} & -e^{x-y} \end{bmatrix}, \implies F'(0, 0) = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}.$$

Therefore the linearized system is

$$\begin{cases} u' = -u - v, \\ v' = u - v. \end{cases}$$

The eigenvalues are

$$\det(\lambda\mathbb{I} - F'(0,0)) = \det\begin{bmatrix} \lambda+1 & 1 \\ -1 & \lambda+1 \end{bmatrix} = (\lambda+1)^2 + 1 = 0, \iff (\lambda+1)^2 = -1, \iff \lambda = -1 \pm i.$$



Because Re $\lambda_{1,2} = -1$, $(0,0)$ is a stable focus for the linearized system hence, by the linearization thm, $(0,0)$ is a stable focus for the original system. The orbit of the linearized system and those of the original one are similar around the equilibrium, but very different far from it (look at the picture). ■

Example 8.6.5 (pendulum wit friction). *Discuss stability of equilibriums for*

$$\ell\theta'' = -\nu\ell\theta' - g\sin\theta.$$

Sol. — We will use the "mechanical letters" $q = \theta$ for the angle and $p := \ell\theta'$ for the impulse. The equivalent system is

$$\begin{cases} q' = \theta' = \frac{p}{\ell}, \\ p' = \ell\theta'' = -\nu\ell\theta' - g\sin\theta = -\nu p - g\sin q. \end{cases} \iff \begin{cases} q' = \frac{p}{\ell}, \\ p' = -\nu p - g\sin q. \end{cases}$$

An equilibrium $(\widehat{q}, \widehat{p})$ fulfills

$$\begin{cases} \frac{\widehat{p}}{\ell} = 0, \\ -\nu\widehat{p} - g\sin\widehat{q} = 0, \end{cases} \iff \begin{cases} \widehat{p} = 0, \\ \sin\widehat{q} = 0. \end{cases} \iff (\widehat{q}, \widehat{p}) = (k\pi, 0),\ k \in \mathbb{Z}.$$

We may expect that as $k = 0, 2, 4, \dots$ the equilibrium is stable, as $k = 1, 3, 5, \dots$ is unstable. Let's compute the linearized systems for these points:

$$F(q,p) = \left(\frac{p}{\ell}, -\nu p - g\sin q\right),$$

so

$$F'(q,p) = \begin{bmatrix} 0 & -\frac{1}{\ell} \\ -g\cos q & -\nu \end{bmatrix}, \implies F'(k\pi, 0) = \begin{bmatrix} 0 & -\frac{1}{\ell} \\ -(-1)^k g & -\nu \end{bmatrix}.$$

The eigenvalues of $A$ are

$$0 = \det(\lambda\mathbb{I} - A) = \det\begin{bmatrix} \lambda & \frac{1}{\ell} \\ (-1)^k g & \lambda+\nu \end{bmatrix} = \lambda(\lambda+\nu) - (-1)^k\frac{g}{\ell} = \lambda^2 + \nu\lambda - (-1)^k\frac{g}{\ell}.$$

that is

$$\lambda_{1,2} = \frac{-\nu \pm \sqrt{\nu^2 + 4(-1)^k \frac{g}{\ell}}}{2}.$$

Therefore:

- if $k$ is even, $\lambda_{1,2} = \frac{-\nu \pm \sqrt{\nu^2 + 4\frac{g}{\ell}}}{2}$ are reals and because $\sqrt{\nu^2 + 4\frac{g}{\ell}} > \nu$, one of them is positive the other is negative. The equilibrium is a saddle (hence unstable).
- if $k$ is odd $\lambda_{1,2} = \frac{-\nu \pm \sqrt{\nu^2 - 4\frac{g}{\ell}}}{2}$ are reals iff $\nu^2 - 4\frac{g}{\ell} \geqslant 0$. Hence
  - if $\nu^2 - 4\frac{g}{\ell} > 0$ (that is if $\nu > 2\sqrt{\frac{g}{\ell}}$ "big friction"), being $\sqrt{\nu^2 - 4\frac{g}{\ell}} < \nu$, the two eigenvalues are negative: we have a *stable knot*;
  - if $\nu^2 - 4\frac{g}{\ell} = 0$, the two eigenvalues are equal to $\lambda_1 = -\nu < 0$: the equilibrium is an *stable node*;
  - if $\nu^2 - 4\frac{g}{\ell} < 0$ (that is if $\nu < 2\sqrt{\frac{g}{\ell}}$ "small friction"), the equilibrium is a *stable focus* being $\mathrm{Re}\,\lambda_{1,2} = -\nu$.

Finally, notice that

- $q \nearrow$ iff $q' > 0$ iff $p > 0$;
- $p \nearrow$ iff $p' > 0$ iff $-\nu p + g \sin q > 0$, that is $p < \frac{g}{\nu} \sin q$.

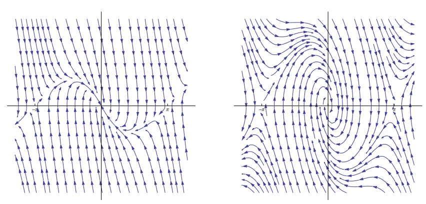


FIGURE 5. Big friction (at left) and small friction (right) pictures.

## 8.7. Exercises

EXERCISE 8.7.1. Solve the following Cauchy problems:

$$1. \begin{cases} x' = -2x + \frac{y}{2}, \\ y' = 2x - 2y, \\ x(0) = 0, \\ y(0) = 1. \end{cases} \quad 2. \begin{cases} x' = 5x + 4y, \\ y' = x + 2y, \\ x(0) = 2, \\ y(0) = 3. \end{cases} \quad 3. \begin{cases} x' = x - y, \\ y' = 5x - y, \\ x(0) = 1, \\ y(0) = 1. \end{cases} \quad 4. \begin{cases} x' = 9x - 4y, \\ y' = 12x - 5y, \\ x(0) = 1, \\ y(0) = 0. \end{cases}$$

EXERCISE 8.7.2. For each of the following systems: check that local existence and uniqueness holds; find stationaries solutions; show that there exists a non trivial first integral (hint: look at the total equation). Plot a graph of orbits finding also the orientation. Discuss for which initial conditions the solutions are globally defined.

$$1. \begin{cases} x' = y, \\ y' = -x + x^3. \end{cases} \quad 2. \begin{cases} x' = 2y(y - 2x), \\ y' = (1 - x)(y - 2x). \end{cases} \quad 3. \begin{cases} x' = x(1 + y), \\ y' = -y(1 + x). \end{cases} \quad 4. \begin{cases} x' = 2x^2 y, \\ y' = y^2 x + x. \end{cases}$$

EXERCISE 8.7.3. Consider the equation
$$y'' = y^3 - y.$$
Check that local existence and uniqueness holds. Find stationaries solutions and a first integral. Use this to find explicitly the solution of the Cauchy problem $y(0) = 2$, $y'(0) = 2$.

EXERCISE 8.7.4. Consider the equation
$$y'' = \sin(2y).$$
Check that global existence and uniqueness holds. Find stationaries solutions and a first integral. Use this to find explicitly the solution of the Cauchy problem $y(0) = \frac{\pi}{4}$, $y'(0) = -1$.

EXERCISE 8.7.5. Consider the equation
$$y'' = \cosh y \sinh y.$$
Check that local existence and uniqueness holds. Find stationaries solutions. Show that if $\varphi$ is a solution, then $\varphi(-t)$ and $-\varphi(t)$ are still solutions. For which values $a, b \in \mathbb{R}$ the solutions of the Caucy problem with $y(0) = a$, $y(0) = b$ are even? odd? Find a first integral and use it to solve the Cauchy problem $y(0) = \log(1 + \sqrt{2})$, $y'(0) = 1$.

EXERCISE 8.7.6. Consider the Cauchy problem
$$\begin{cases} y'' = -y^2, \\ y(0) = y_0, \\ y'(0) = y_0', \end{cases}$$
Show that local existence and uniqueness holds. Find stationaries solutions and find explicitly the non constant solutions.

EXERCISE 8.7.7. Discuss the nature of the equilibrium $(0,0)$ for the following systems, giving also a qualitative plot of orbits:
$$1. \begin{cases} x' = 3x + y, \\ \\ y = -x + y. \end{cases} \quad 2. \begin{cases} x' = 4x - y, \\ \\ y = -2x + 3y. \end{cases} \quad 3. \begin{cases} x' = -x - 2y, \\ \\ y = -3y. \end{cases} \quad 4. \begin{cases} x' = 2x - 2y, \\ \\ y = 3x - y. \end{cases}$$

EXERCISE 8.7.8. For each of the following systems: find their equilibrium points and classify their nature.
$$1. \begin{cases} x' = y, \\ \\ y = -y - \sin x. \end{cases} \quad 2. \begin{cases} x' = -x + y^2, \\ \\ y = y^2 - 2x. \end{cases} \quad 3. \begin{cases} x' = 2x - y^2, \\ \\ y = -y + xy. \end{cases}$$

EXERCISE 8.7.9. The position $x(t)$ of a particle follows the Newton equation
$$x'' = -x + x^2.$$
Find the energy of the system and orbits in phase space. For which initial conditions $x(0)$, $x'(0)$ it is impossible that the motion be oscillatory?

EXERCISE 8.7.10 (SIR DISEASE). A model for the diffusion of a disease in a population of $N$ individuals works as follows. Let $x(t)$ be the number of healthy individuals, $y(t)$ the infected one. We assume that a) any individual may be healthy, infected or deceased; b) a certain fraction of infected individuals die; c) the diffusion of the disease is proportional to the number of healthy and infected individuals (the contagion take place by contact between infected individuals with healthy ones). We may translate these prescriptions in the model
$$\begin{cases} x' = -axy, \\ \\ y' = axy - by, \end{cases}$$

with $a, b > 0$. Find the equilibriums and discuss their stability. Find a non trivial first integral and plot the orbits, discussing the behavior of the system as much as you can.

Exercise 8.7.11 ($\star$). Two species are in competition. In absence of interaction, each one of the two follows a logistic model $x' = ax(b - x)$, $y' = cy(d - y)$. The interaction works to reduce both proportionally to their size in such a way that a model could be described by the equations

$$\begin{cases} x' = ax(b - x) - \alpha xy, \\ \\ y' = cy(d - y) - \beta xy, \end{cases}$$

with $a, b, c, d, \alpha, \beta > 0$. Discuss the behavior of the system as better as you can.