EX 1

giovedì 18 giugno 2020 09:38

$$y' = \frac{e^y}{e^y - t}$$

1. Domain of local $\exists$ and uniq.

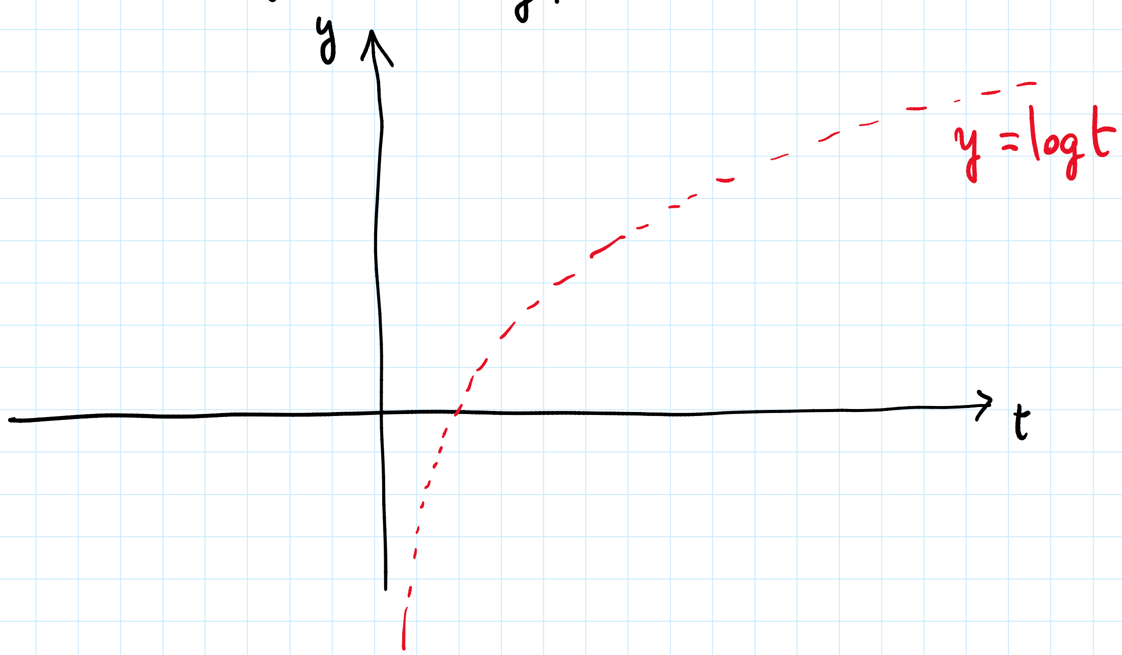
Eqn $y' = f(t,y) := \dfrac{e^y}{e^y - t}$ has domain

$$D = \{ (t,y) \in \mathbb{R}^2 : e^y - t \neq 0 \}$$

$$e^y - t = 0 \iff e^y = t \underset{t>0}{\iff} y = \log t.$$

$$\Rightarrow D = \{ (t,y) \in \mathbb{R}^2 : y \neq \log t \}$$

on D clearly $f, \partial_y f \in \mathscr{C}$



2. St sols, increasing / decr. regions

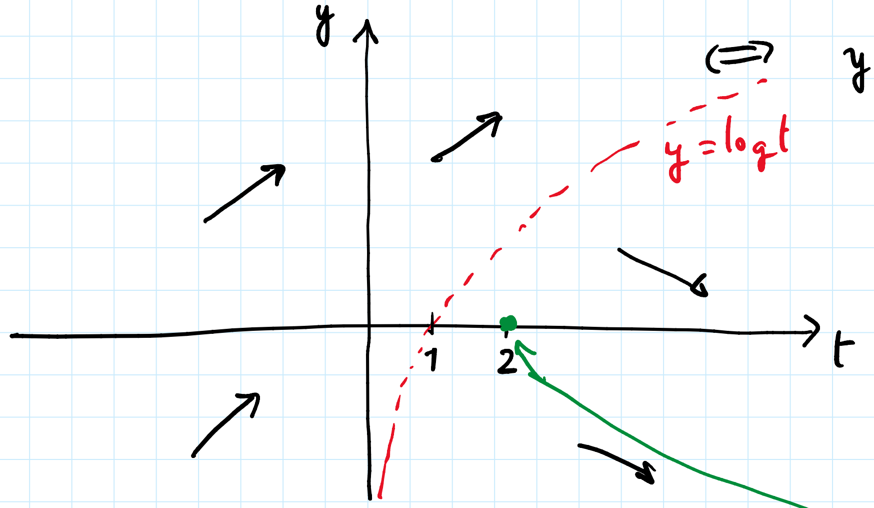
$$y \equiv C \text{ is sol } \iff 0 = \frac{e^C}{e^C - t} \quad \text{imposs}$$

$\Rightarrow$ no st. sol

$$y \nearrow \iff 0 \leq y' = \frac{e^{y} > 0}{e^{y} - t} \iff e^{y} - t > 0$$

$$\iff e^{y} > t$$

$$\iff y > \log t$$



Let now $y :]\alpha, \beta[\longrightarrow \mathbb{R}$ sol of CP $y(2) = 0$

3. y is monotone and $\alpha > 1$

Guess: $y \searrow$ This is sure once we prove

$y < \log t \quad \forall\, t \in]\alpha, \beta[$.

IF FALSE, $\exists\, \hat{t} : \quad y(\hat{t}) \geq \log \hat{t}$. But

- $y(\hat{t}) = \log \hat{t} \Rightarrow (\hat{t}, y(\hat{t})) \notin D$ impossible
- $y(\hat{t}) > \log(\hat{t})$ or $y(\hat{t}) - \log \hat{t} > 0$

  is impossible. Indeed, calling

$$g(t) = y(t) - \log t$$

then $g \in \mathcal{C}$, $g(\hat{t}) > 0$ and

$$g(2) = y(2) - \log 2$$

$$g(2) = y(2) - \log 2$$
$$= 0 - \log 2 < 0$$

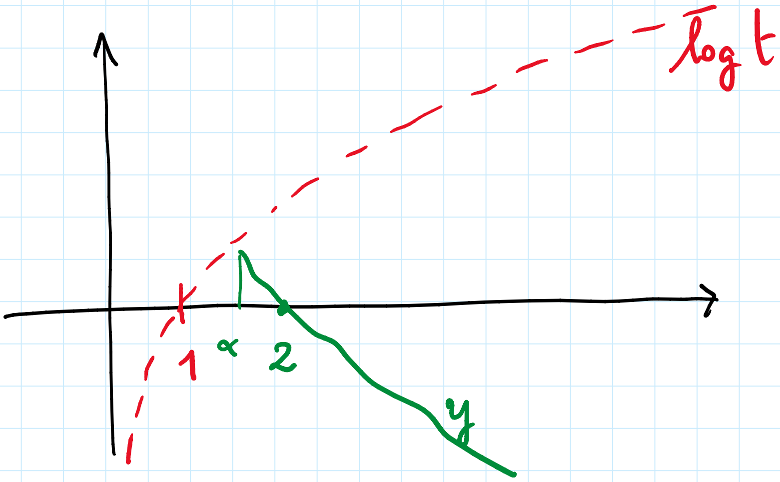
$\Rightarrow$ intermediate value thm $\Rightarrow$

$\exists\, \hat{\hat{t}} : g(\hat{\hat{t}}) = 0 \Rightarrow y(\hat{\hat{t}}) = \log \hat{\hat{t}}$

$\Rightarrow$ previous case : impossible!

Since we obtain a contradiction, $y(t) < \log t$ $\forall t \in]\alpha, \beta[\Rightarrow y \searrow$.

α : figure suggests $\alpha > 1$



Indeed : If $\alpha \leqslant 1$, since $y \searrow$ $\exists\, l = \lim_{t \to \alpha} y(t)$

Yet, since $y \searrow$, $l > 0$ and because $\log \alpha \leq \log 1 \leq 0$ $\Rightarrow$

$$0 < l \xleftarrow[t \to \alpha]{} y(t) < \log t \xrightarrow[t \to \alpha]{} \log \alpha \leqslant 0$$
$$\Downarrow$$
$$0 < l \leqslant 0 \quad \text{impossible.}$$

4. Concavity

4. Concavity

Let's compute y'':

$$y'' = (y')' = \left(\frac{e^y}{e^y - t}\right)'$$

(WARNING: never forget $y = y(t)$)

$$= \frac{e^y \cdot y'(e^y - t) - e^y(e^y y' - 1)}{(e^y - t)^2}$$

(don't panic!)

$$y \uparrow \text{ (convex)} \iff y'' \geq 0 \overset{\oplus}{\iff}$$

$$\cancel{e^y}\Big[\underbrace{\underbrace{y'}_{\ominus}\underbrace{(e^y - t)}_{\ominus\ (\text{by } 3.)}}_{\oplus} \underbrace{-\underbrace{(\underbrace{e^y}_{\oplus}\underbrace{y'}_{\ominus}\underbrace{-1}_{-1})}_{\ominus\ \text{by } 3}}_{\oplus}\Big] \geq 0$$

$\Downarrow$

y convex.

In alternative:

$$y'' = \frac{e^y y'(e^y - t) - e^y(e^y y' - 1)}{(e^y - t)^2}$$

$$= \frac{e^y}{(e^y - t)^2}\left[\frac{e^y}{\cancel{e^y - t}}\cancel{(e^y - t)} - \left(e^y \frac{e^y}{e^y - t} - 1\right)\right]$$

$$= \frac{e^y}{(e^y-t)^2}\left[\frac{e^y}{\cancel{e^y-t}}\cancel{(e^y-t)} - \left(e^y\frac{e^y}{e^y-t} - 1\right)\right]$$

$$= \frac{e^y}{(e^y-t)^2}\left[e^y - \frac{e^{2y} - (e^y-t)}{e^y-t}\right]$$

$$= \frac{e^y}{(e^y-t)^2}\,\frac{\cancel{e^{2y}} - te^y - \cancel{e^{2y}} + e^y - t}{e^y-t}$$

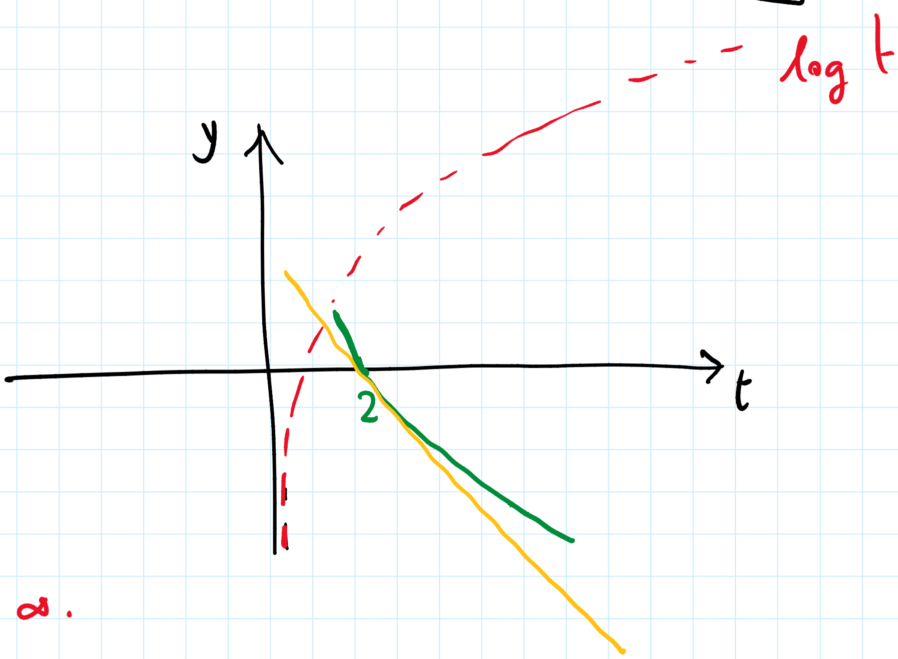
$\ominus$ $\ominus$ by 3

$$= \frac{e^y \oplus}{(e^y-t)^2}\,\frac{-te^y + (e^y-t)}{e^y-t}$$

$\oplus$ $t>0$ (because $t>\alpha>1$) $\ominus$ (by 3)

$\downarrow$ $\oplus$ $\downarrow$ $\frac{\ominus}{\ominus} = \oplus$

4. β?



We know

$y \searrow$

y convex

$\Downarrow$

guess: $\beta = +\infty$.

Indeed: since y is convex

$$y(t) \geq y(2) + y'(2)(t-2) \qquad (\text{tg at } t=2)$$

$$= 0 + \frac{1}{1-2}(t-2)$$

$$= -(t-2)$$

$$y'(2) = \frac{e^{y(2)}}{e^{y(2)}-2} = \frac{e^0}{e^0-2}$$

$$\Rightarrow \quad y(t) \geq -(t-2)$$

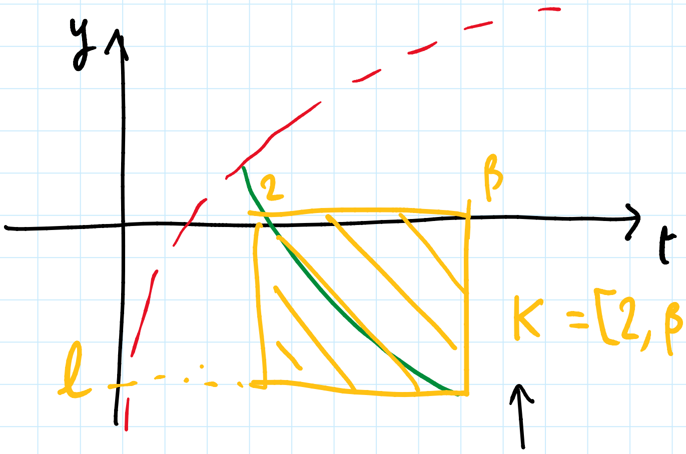
IF $\beta < +\infty \;\rightarrow\; y(t) \geq -(t-2)$

$$\Big\downarrow\, t \to \beta$$

$$\ell = \lim_{t\to\beta} y(t) \geq -(\beta-2) > -\infty$$

(exists because $y \downarrow$)

but then



$K = [2,\beta] \times [\ell, 0] \subset D$

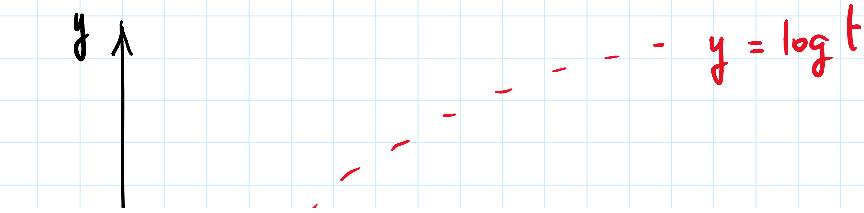
$\uparrow$

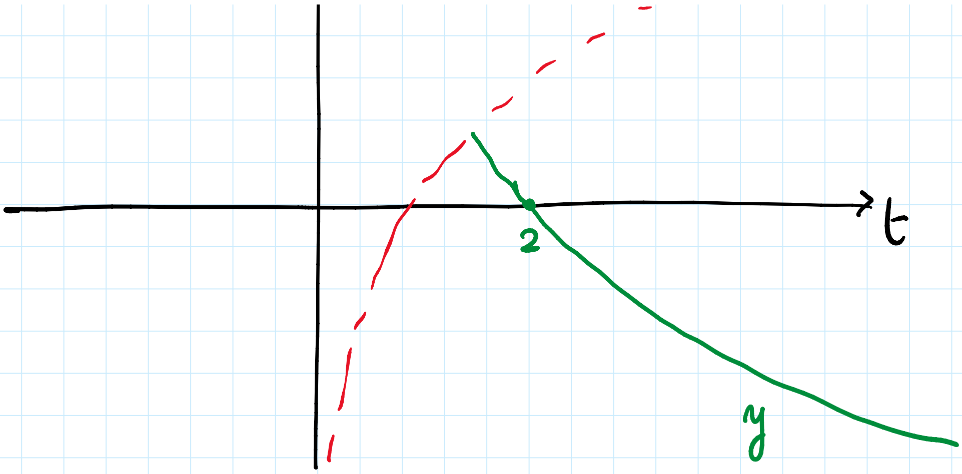
$\underbrace{\text{closed \& bdd}}_{\text{compact}} \subset D$

solution wouldn't leave D in the future! impossible

$$\Rightarrow \quad \beta = +\infty.$$

5. Plot




t
2
y

$$(S) \qquad \begin{cases} x' = y\,(x^2+y^2-2) \\ y' = x\,(x^2+y^2-2) \end{cases}$$

1. St sols:

$(x,y) \equiv (a,b)$ solves (S) $\Leftrightarrow$

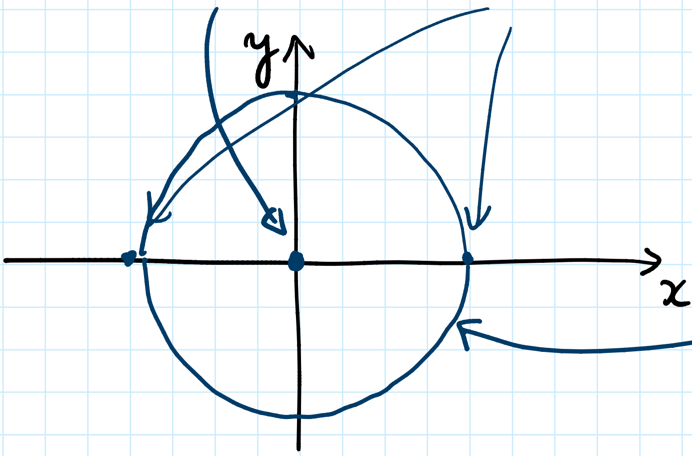
$$\begin{cases} \boxed{0 = b\,(a^2+b^2-2)} \\ 0 = a\,(a^2+b^2-2) \end{cases} \Leftrightarrow \begin{cases} b=0 \\ a(a^2-2)=0 \end{cases} \vee \begin{cases} a^2+b^2-2=0 \\ 0=0 \end{cases}$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$\begin{cases} b=0 \\ a=0 \end{cases} \vee \begin{cases} b=0 \\ a^2=2 \end{cases} \vee \boxed{a^2+b^2=2}$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$(0,0) \qquad (\pm\sqrt{2},0)$$



2. First/Prime Int.

To det a non const first int we look at the total eqn:

of the total eqn:

$$\frac{dy}{dx} = \frac{x\,(x^2+\cancel{y^2}-2)}{y\,(x^2+\cancel{y^2}-2)} = \frac{x}{y} \quad \left(\begin{array}{c}\text{sep. var.}\\ \text{eqn}\end{array}\right)$$

$$\Leftrightarrow \quad y\,dy = x\,dx$$

$$\Leftrightarrow \quad \frac{y^2}{2} = \frac{x^2}{2} + C \quad \Leftrightarrow \quad y^2 - x^2 \equiv C$$

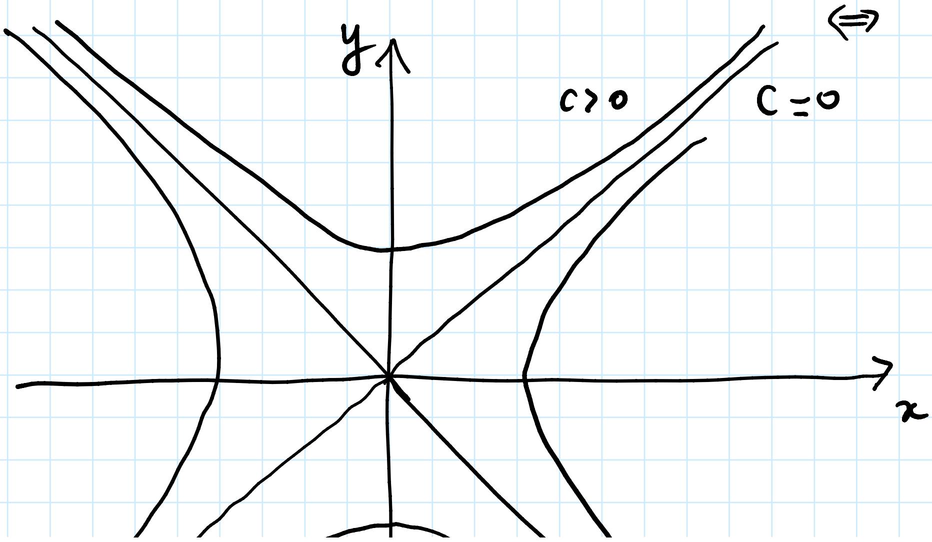
$$\Rightarrow \quad E(x,y) = y^2 - x^2.$$

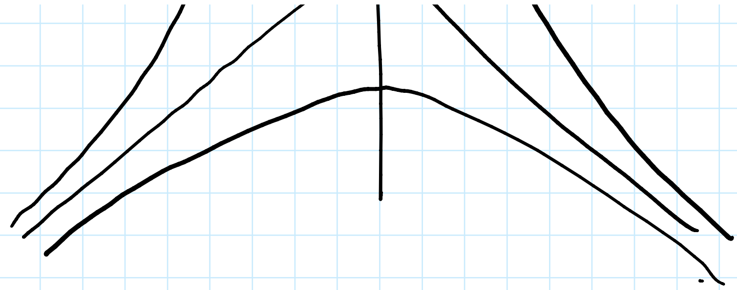
3. Phase portrait

We start by determing level sets of E that is

$$\{E(x,y) = C\} = \{y^2 - x^2 = C\}.$$

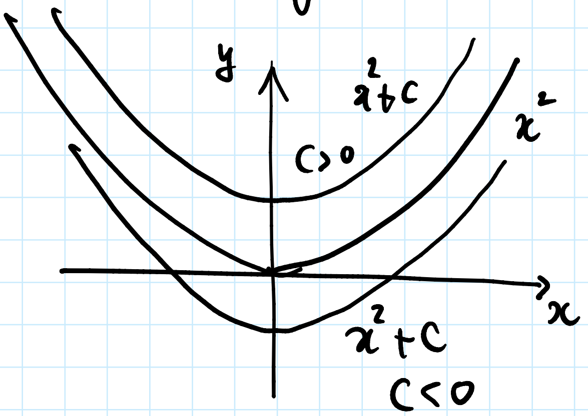
These are hyperbolas (apart for $C=0$, in which case $y^2 - x^2 = 0$ $\Leftrightarrow$ $y = \pm x$)



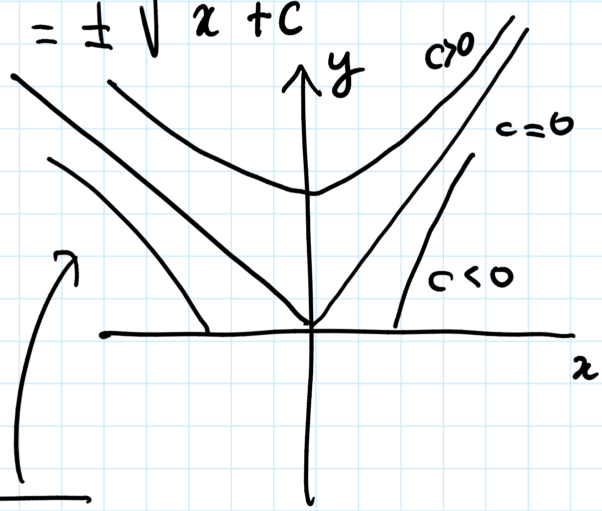


If you don't recognize hyperbolas never mind:

$$y^2 = x^2 + C \iff y = \pm\sqrt{x^2+C}$$

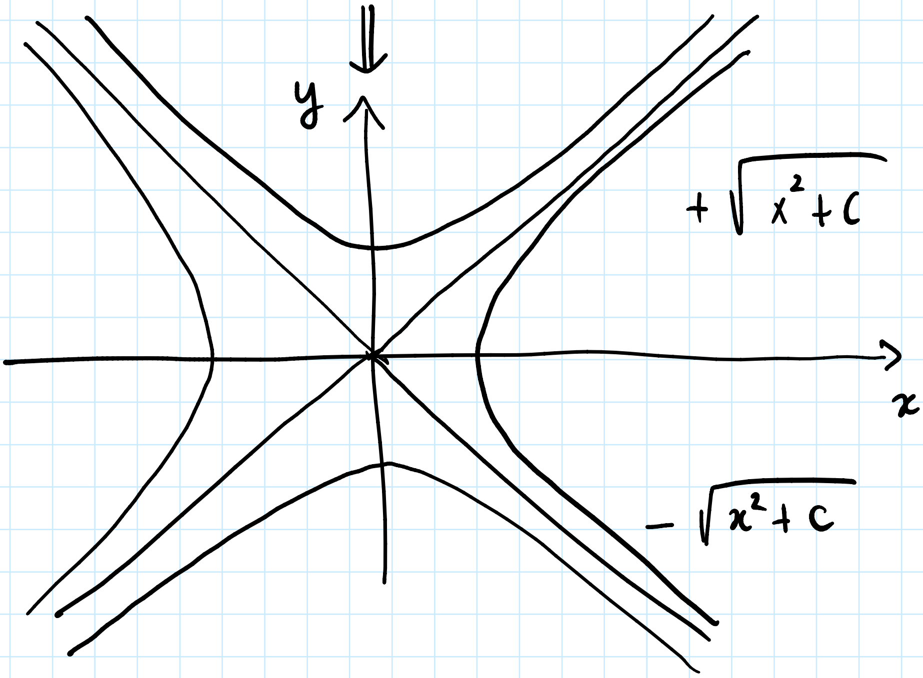


$\Longrightarrow$

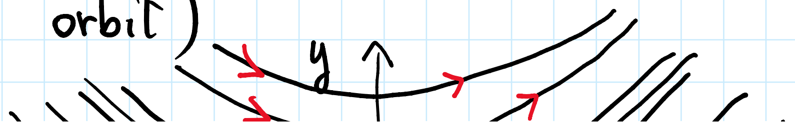


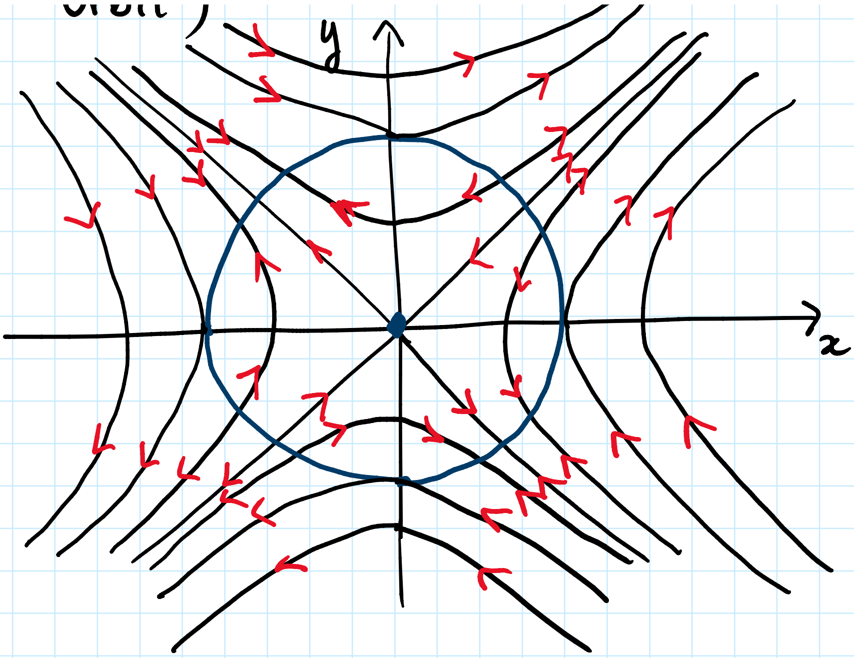
$+\sqrt{x^2+C}$

$\Downarrow$



We add st. points (each of these corresponds to an orbit)





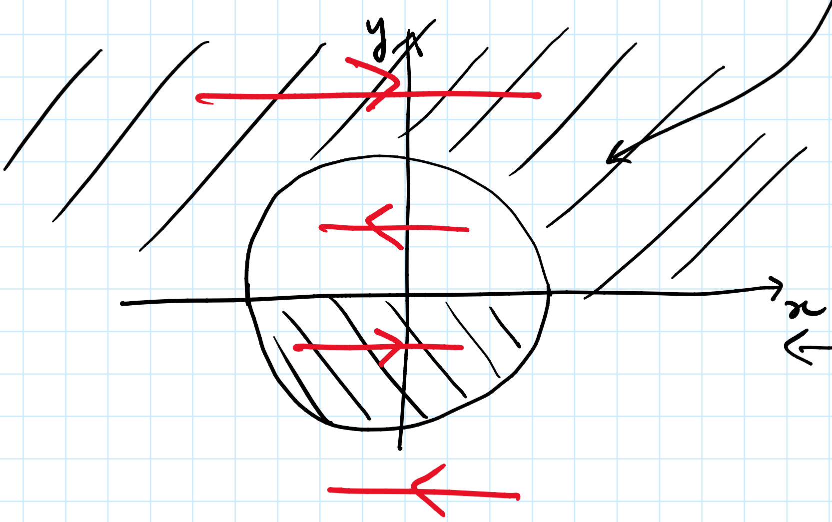
(some more line has been added)

Orientation: $x \nearrow \iff x' \geq 0 \iff y\,(x^2+y^2-2) \geq 0$

$\iff \boxed{y \geq 0 \ \wedge \ x^2+y^2-2 \geq 0} \iff \boxed{y \geq 0 \ \wedge \ x^2+y^2 \geq 2}$

OR

$\boxed{y \leq 0 \ \wedge \ x^2+y^2-2 \leq 0} \iff \boxed{y \leq 0 \ \wedge \ x^2+y^2 \leq 2}$



Periodic sols: no cycles $\Rightarrow$ no periodic sols

(WARNING: $x^2+y^2=2$ is NOT a cycle:

(WARNING: $x^2+y^2=2$ is NOT a cycle: remember that each point on this circle is just an orbit)

Global sols: apart obvious const sols, all sols whose orbits are in $\{x^2+y^2<2\}$ are global (this because orbit $\subset$ compact set).

4 Sol of CP $x(0) = 1/\sqrt{2}$, $y(0) = 1/\sqrt{2}$.

Let's first determine

$$E(x(0), y(0)) = \left(\frac{1}{\sqrt{2}}\right)^2 - \left(\frac{1}{\sqrt{2}}\right)^2 = 0$$

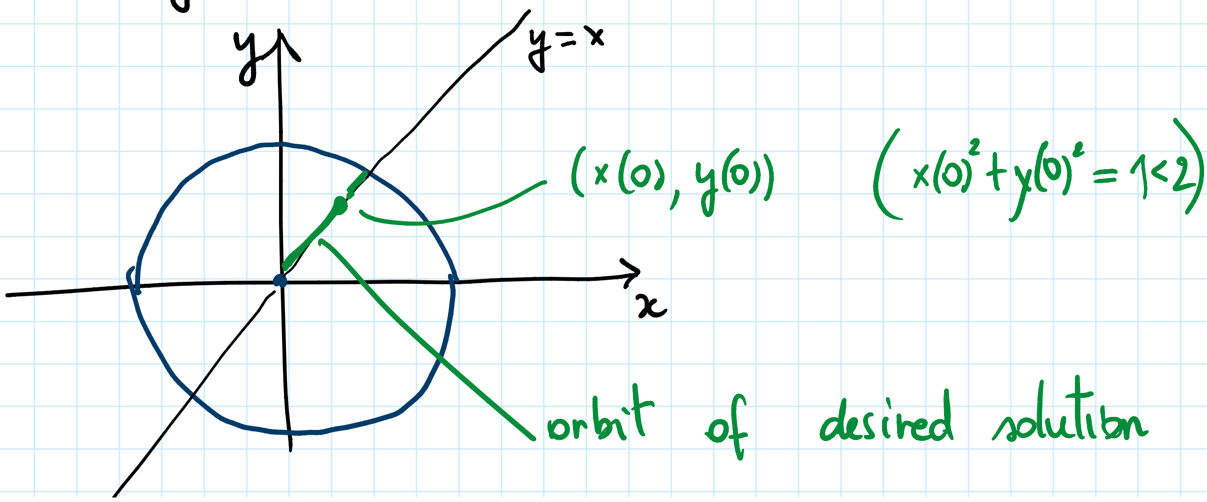
Now, since E is constant along sols,

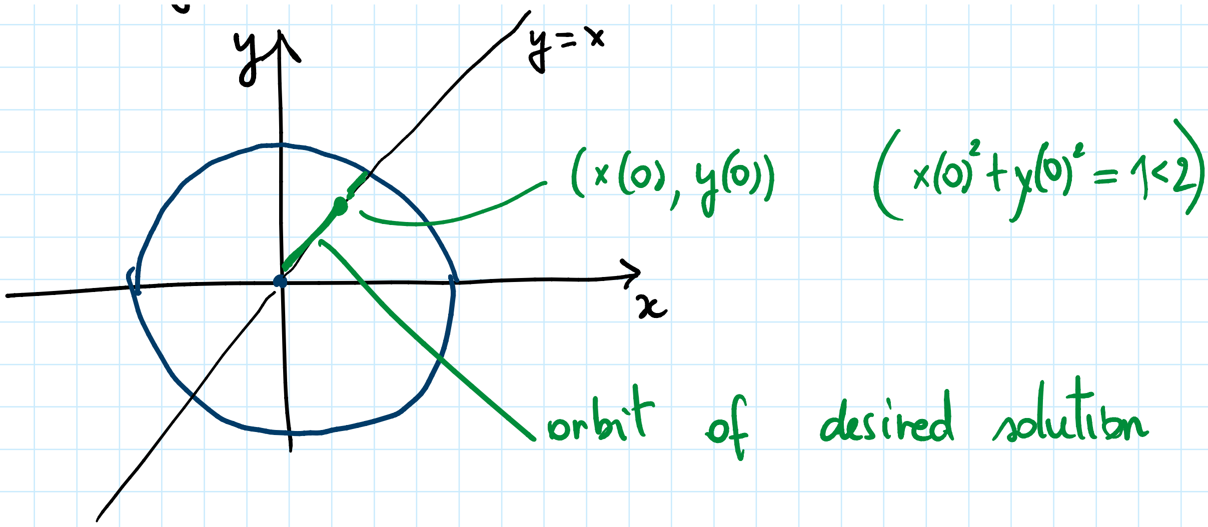
$$E(x(t), y(t)) \equiv 0$$

$$\Leftrightarrow \quad y^2 - x^2 \equiv 0 \quad \Rightarrow \quad y \equiv \pm x$$

Actually, because at $t=0$ $y(0) = x(0)$ we must have

$$y(t) \equiv x(t).$$





To compute the solution notice that by

$$x' = y\,(x^2+y^2-2) \underset{y=x}{=} x\,(2x^2-2) = 2x(x^2-1)$$

which is now a sep vars eqn:

$$\frac{dx}{dt} = 2x(x^2-1) \iff \frac{dx}{x(x^2-1)} = 2\,dt$$

$$\iff \int \frac{dx}{x(x^2-1)} = 2t + C.$$

$$\int \frac{1}{x(x^2-1)}\,dx = \int \frac{1\cancel{-x}}{x(x^2-1)} + \frac{\cancel{x}}{\cancel{x}(x^2-1)}\,dx$$

$$-(\cancel{x-1})(x+1)$$

$$= -\int \frac{1}{x(x+1)} + \int \frac{1}{(x-1)(x+1)}$$

$$\frac{1}{x} - \frac{1}{x+1} \qquad \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right)$$

$$= -\left[\log|x| - \log|x+1| \right] + \frac{1}{2}\left[\log|x-1| - \log|x+1| \right]$$

$$= -\log|x| + \tfrac{1}{2}\log|x+1| + \tfrac{1}{2}\log|x-1|$$

$$= \frac{1}{2}\log\left|\frac{x^2-1}{x^2}\right| = \frac{1}{2}\log\left|1 - 1/x^2\right|$$

$\Rightarrow$ the implicit form for sol is

$$\frac{1}{2}\log\left|1 - 1/x^2\right| = 2t + c$$

Imposing $x(0) = 1/2$ we obtain $c = \frac{1}{2}\log|1-4|$

$$= \frac{1}{2}\log 3$$

To finish we extract x

$$\log\left|1 - 1/x^2\right| = 4t + \log 3$$

$$\Rightarrow \left|1 - 1/x^2\right| = 3e^{4t}$$

$$\Rightarrow 1 - 1/x^2 = \pm 3e^{4t} \qquad \pm\,?$$

(we impose $t=0$: $1-4 = \pm 3e^0 \Rightarrow -$)

$$\Rightarrow 1 - \frac{1}{x^2} = -3e^{4t}$$

$$\Rightarrow \frac{1}{x^2} = 1 + 3e^{4t} \Rightarrow x(t) = \frac{1}{\sqrt{1+3e^{4t}}}$$

$$\Rightarrow \quad \frac{1}{x^2} = 1 + 3e \qquad \Rightarrow \qquad x(t) = \frac{\quad}{\sqrt{1+3e^{4t}}}$$

by this we see clearly that x (as well as y) is defined $\forall t \in \,]-\infty, +\infty[$. $\blacksquare$

EX 3

giovedì 18 giugno 2020 09:39

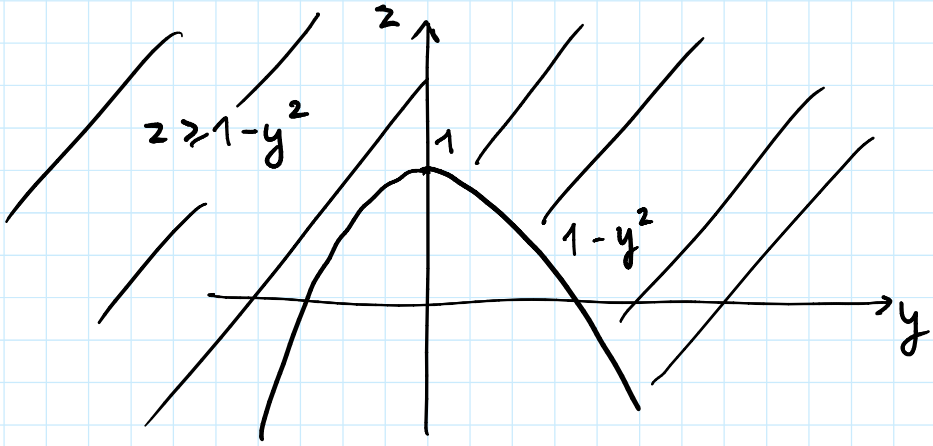
$$\Omega = \{ z \geq 1 - (x^2+y^2) \, , \, x^2+y^2+z^2 \leq 1 \}$$

1. Draw Ω.

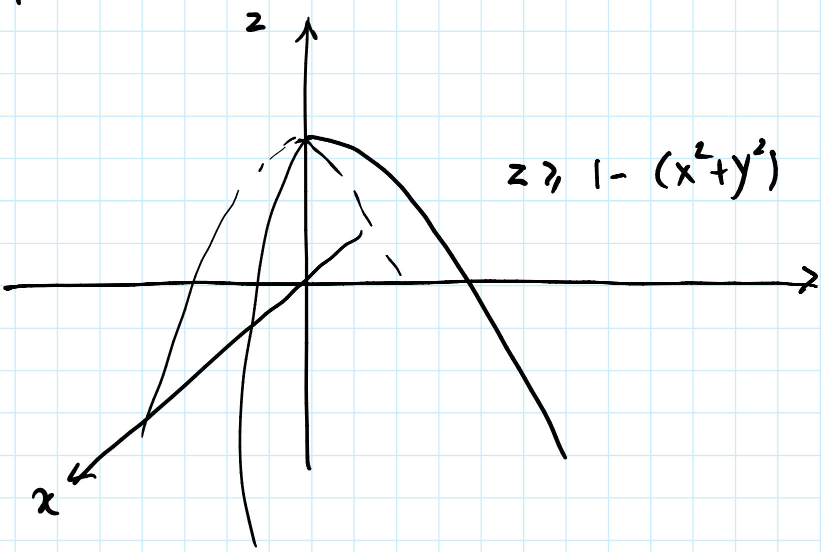
$\{x^2+y^2+z^2 \leq 1\}$ = ball centred at $(0,0,0)$ radius $=1$

$\{z \geq 1-(x^2+y^2)\}$ ← rotation solid (x^2+y^2 is invariant by rot around z-axis)

$\{z \geq 1-(x^2+y^2)\} \cap \{x=0\} = \{ z \geq 1-y^2 \}$



in $\mathbb{R}^3$

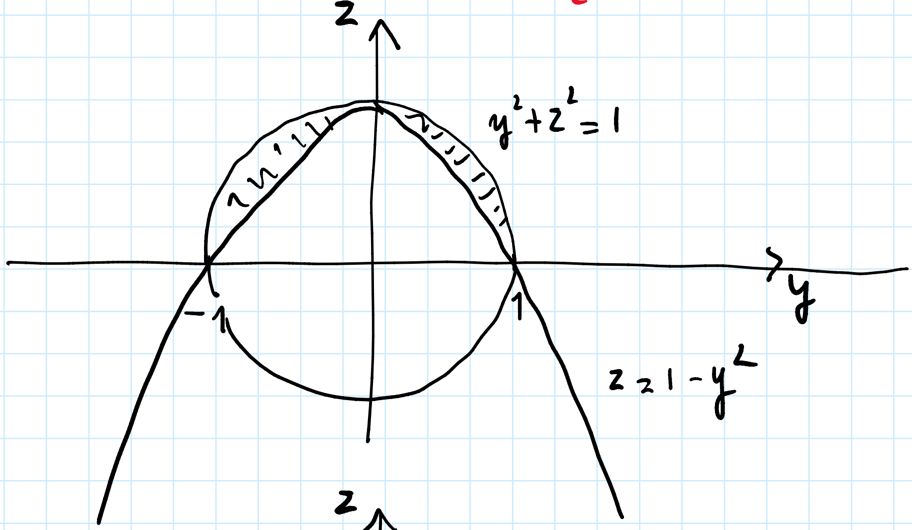


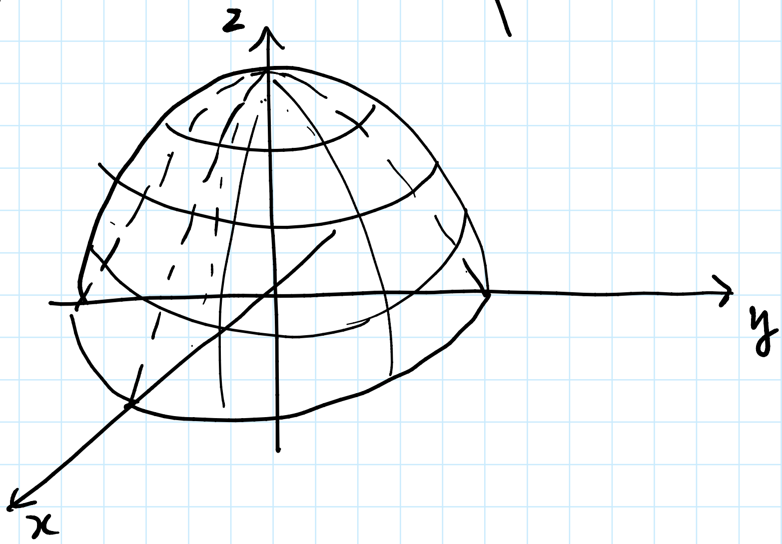
(part of space above paraboloid)

We now draw Ω. Since also $\{x^2+y^2+z^2 \leq 1\}$ is invariant by rot around z axis, we start plotting

plotting

$$\Omega \cap \{x=0\} = \{z \geq 1-y^2, \ y^2+z^2 \leq 1\}$$





2. Vol Ω.

$$\mathrm{Vol}\,\Omega = \int_{\Omega} 1 \, dx\, dy\, dz$$

$$\underset{\text{cyl coords}}{=} \int_{\substack{\rho \geq 0, \ \theta \in [0,2\pi], \ z \in \mathbb{R} \\ z \geq 1-\rho^2, \ \rho^2+z^2 \leq 1}} \rho \, d\rho \, d\theta \, dz$$

$x = \rho\cos\theta$

$y = \rho\sin\theta$

$z = z$

$$\overset{RF}{=} 2\pi \int_{\substack{\underbrace{z \geq 1-\rho^2, \quad \rho^2+z^2 \leq 1}}} \rho \, d\rho \, dz$$

$$\Downarrow$$

$$\underbrace{\rho^2 \geq 1-z \quad , \quad \rho^2 \leq 1-z^2} \quad (z^2-1 \geq 0 \Leftrightarrow |z| \leq 1)$$

$$\underbrace{1-z \leq \rho^2 \leq 1-z^2}$$

$$1-z \leq 1-z^2 \Leftrightarrow \underset{z(z-1)}{z^2-z} \leq 0$$

$$\Updownarrow$$

$$0 \leq z \leq 1$$

$$= 2\pi \int_{\substack{0 \leq z \leq 1 \\ 1-z \leq \rho^2 \leq 1-z^2}} \rho \, d\rho \, dz$$

$$\overset{RF}{=} 2\pi \int_0^1 \left(\int_{1-z}^{1-z^2} \rho \, d\rho \right) dz$$

$$= \cancel{2}\pi \int_0^1 \frac{\rho^2}{\cancel{2}} \Big|_{1-z}^{1-z^2} dz$$

$$= \pi \int_0^1 (1-z^2)^2 - (1-z)^2 \, dz$$

$$= \pi \int_0^1 \cancel{1} + z^4 \underbrace{- 2z^2 - \cancel{1} - z^2}_{-3z^2} + 2z \, dz$$

$$\left[\frac{z^5}{5} \Big|^1 \quad -z^3 \Big|^1 \quad z^2 \Big|^1 \right]$$

$$= \pi \left[\frac{z^5}{5}\Big|_0^1 - \cancel{3}\,\frac{z^3}{\cancel{3}}\Big|_0^1 + z^2\Big|_0^1 \right] \quad (-3z^2)$$

$$= \pi \left[\frac{1}{5} - 1 + 1 \right] = \pi/5.$$

3. $\vec{F} = -(x,y,z)$

Outward flux. We apply the div. thm:

$$\int_{\partial\Omega} \vec{F}\cdot\vec{n}_e = \int_{\Omega} \operatorname{div}\vec{F}$$

where $\operatorname{div}\vec{F} = \partial_x(-x) + \partial_y(-y) + \partial_z(-z)$

$$= -3$$

$$= -3 \int_\Omega 1 \, dx\,dy\,dz = -3\cdot\frac{\pi}{5}.$$

Components on $\partial\Omega \cap \{x^2+y^2+z^2=1\}$ and $\partial\Omega \cap \{z = 1-(x^2+y^2)\}$.

We need to det just one of the two, the other is det. by difference. We choose to compute the first one. This because

$$\vec{n}_e = (x,y,z)$$

$$\Rightarrow \int_{\partial\Omega\cap\{x^2+y^2+z^2=1\}} \vec{F}\cdot\vec{n}_e \qquad \vec{F}\cdot\vec{n}_e = -\|(x,y,z)\|^2 = -1$$

$$\partial\Omega \cap \{x^2+y^2+z^2=1\}$$

$$= \int_{\partial\Omega \cap \{x^2+y^2+z^2=1\}} -\|(x,y,z)\|^2$$

$$-\|(x,y,z)\|^2 = -(x,y,z)\cdot(x,y,z)$$

$$= - \int_{\partial\Omega \cap \{x^2+y^2+z^2=1} 1$$

$$= -\,\text{Area half spere radius} = 1$$

$$\text{Area half spere radius } 1 = \frac{1}{2}\, 4\pi \cdot 1^2$$

$$= -2\pi$$

3. Area $\partial\Omega \cap \{z = 1 - (x^2+y^2)\}$

Recall that if $\phi = \phi(u,v)$ is a param. of $\partial\Omega \cap \{z = 1 - (x^2+y^2)\}$ we have

$$\text{Area} = \int_D \| \partial_u \phi \wedge \partial_v \phi \| \, du\, dv$$

D = domain of param. In our case

$$\partial\Omega \cap \{z = 1-(x^2+y^2)\} = \text{Graph } f$$

where $f(x,y) = 1 - (x^2+y^2)$ $\quad (x,y) \in \{x^2+y^2 \leq 1\}$

thus we may use the special formula

thus we may use the special formula

$$\text{Area} = \int_D \sqrt{1 + \|\nabla f\|^2} \; dx\,dy$$

$$= \int_{x^2+y^2 \leq 1} \sqrt{1 + \|(-2x, -2y)\|^2} \; dx\,dy$$

$$= \int_{x^2+y^2 \leq 1} \sqrt{1 + 4(x^2+y^2)} \; dx\,dy$$

$$\underset{\text{pol coords}}{=} \int_{\substack{0 \leq \rho \leq 1 \\ 0 \leq \theta \leq 2\pi}} \sqrt{1 + 4\rho^2} \; \rho \, d\rho \, d\theta$$

$$\overset{RF}{=} 2\pi \int_0^1 \underbrace{(1 + 4\rho^2)^{1/2} \rho}_{} \; d\rho$$

$$\left[(1+4\rho^2)^{3/2}\right]' = \frac{3}{\cancel{2}}(1+4\rho^2)^{1/2} \cdot 4 \cdot \cancel{2}\rho$$

$$= 12 \, (1+4\rho^2)^{1/2} \rho$$

$$= \frac{2}{12}\pi \left[(1+4\rho^2)^{3/2}\right]_0^1$$

$$= \frac{\pi}{6}\left(5^{3/2} - 1\right) \quad \boxtimes$$