martedì 13 ottobre 2020   20:00

Ex 1.

Consider

$$\begin{cases} x' = x^2 + xy \\ y' = -xy - y^2 \end{cases}$$

i) Equilibriums.

Equilib. are const sols $(x,y) \equiv (a,b) \iff$

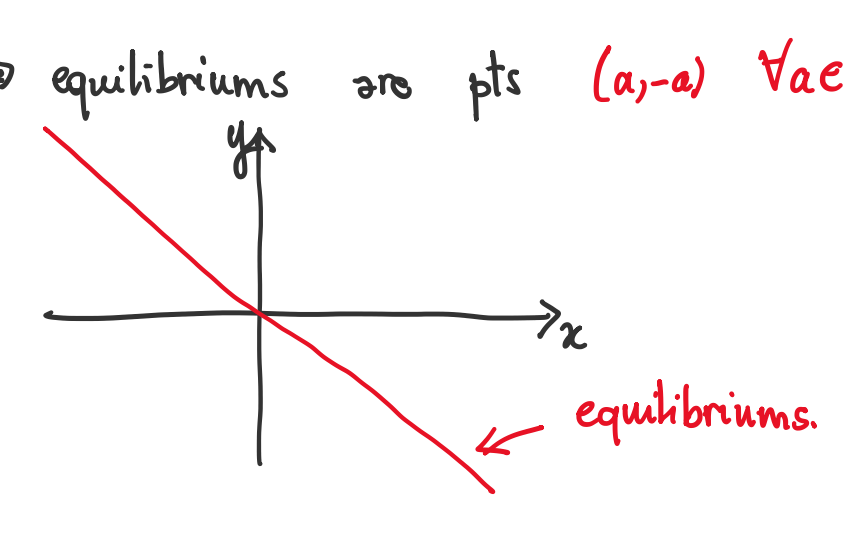
$$\begin{cases} 0 = a^2 + ab \\ 0 = -ab - b^2 \end{cases} \iff \begin{cases} a(a+b) = 0 \\ b(a+b) = 0 \end{cases}$$

$$\iff \begin{cases} a = 0 \\ b^2 = 0 \end{cases} \vee \begin{cases} a+b = 0 \\ 0 = 0 \end{cases}$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$(0,0) \qquad (a,-a) \quad a \in \mathbb{R}$$

$\Rightarrow$ equilibriums are pts $(a,-a)$ $\forall a \in \mathbb{R}$.



ii) Prime integral

The total eqn is

$$\frac{dy}{dx} = \frac{-xy - y^2}{x^2 + xy} = -\frac{y\cancel{(x+y)}}{x\cancel{(x+y)}} = -\frac{y}{x}$$

which is a sep vars eqn:

$$\iff \frac{dy}{y} = -\frac{dx}{x}$$

$$\iff \int \frac{dy}{y} = -\int \frac{dx}{x} + C$$

$$\iff \log|y| = -\log|x| + C$$

$$\iff \log|xy| \equiv C$$

Thus $\log|xy|$ is a prime int (also xy it is).

iii) Phase portrait.

We've to plot lines $E(x,y) = C$

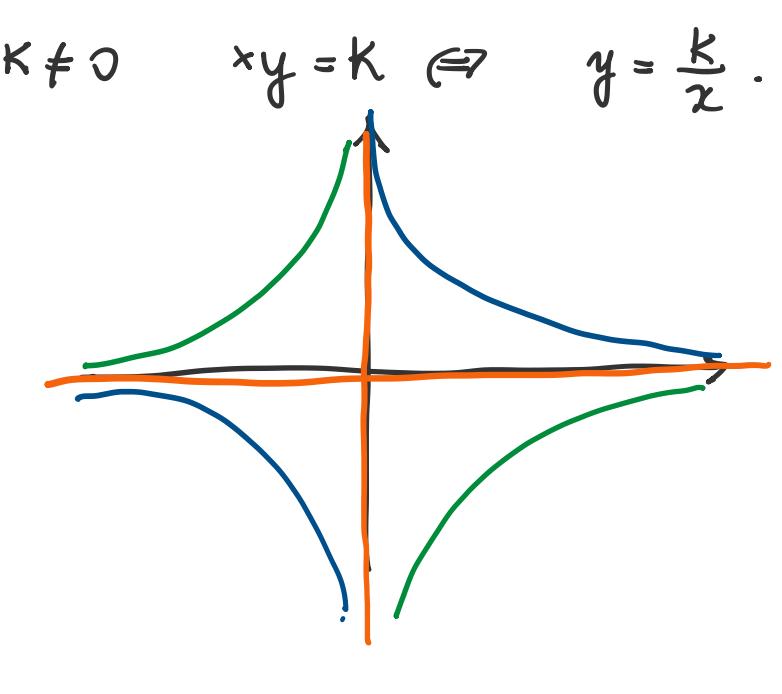
$$\iff \log|xy| = C$$

$$\iff |xy| = K \qquad (K > 0)$$

$$\iff xy = k \qquad (k \in \mathbb{R})$$

$K = 0 \quad xy = 0 \iff x = 0 \vee y = 0$ (axes)

$K \neq 0 \quad xy = K \iff y = \frac{k}{x}$.



Orientation: $x \nearrow \iff x' \geq 0 \iff$

$$x(x+y) \geq 0$$

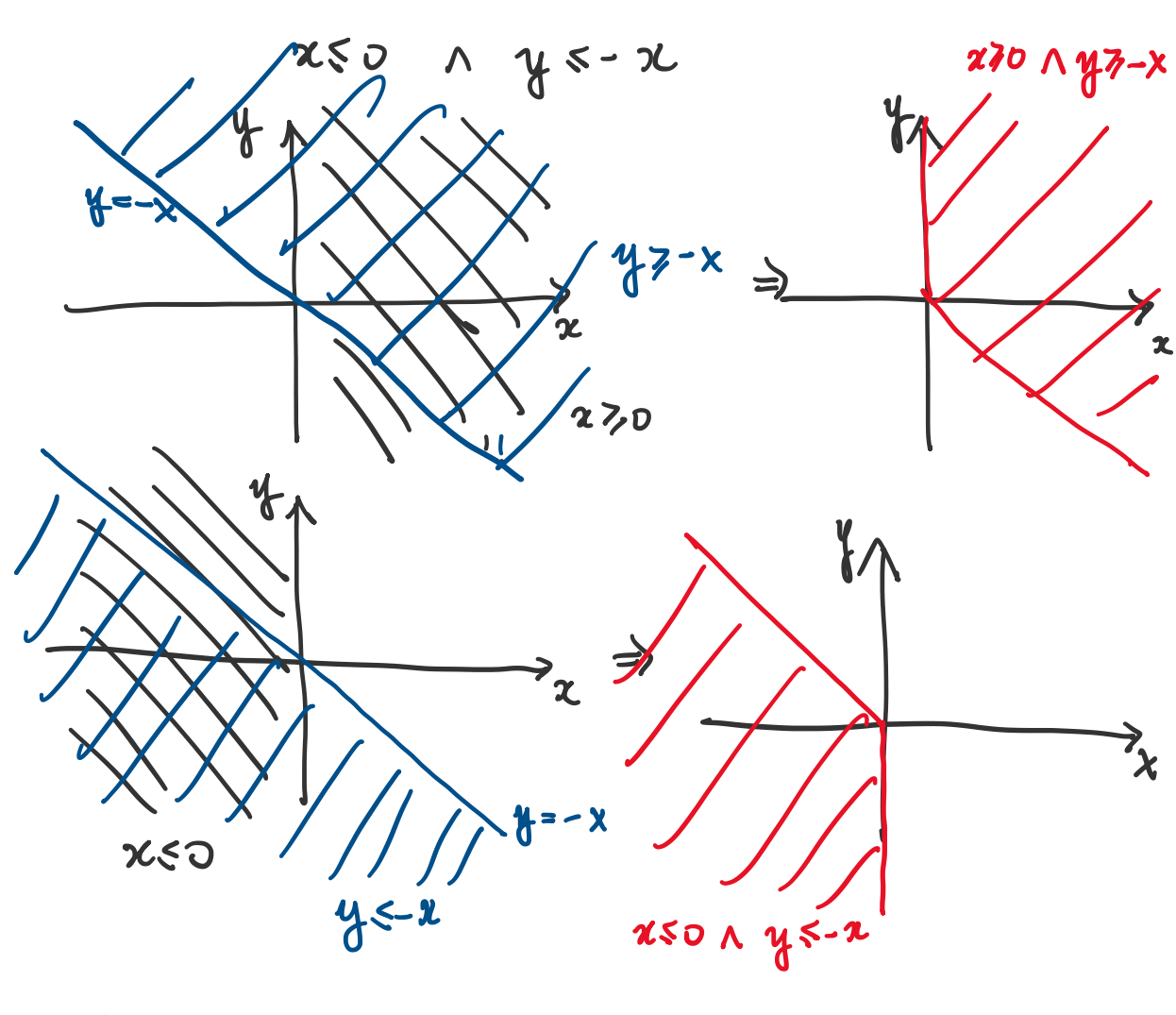
$\iff x \geq 0 \wedge x + y \geq 0$

or

$x \leq 0 \wedge x + y \leq 0$

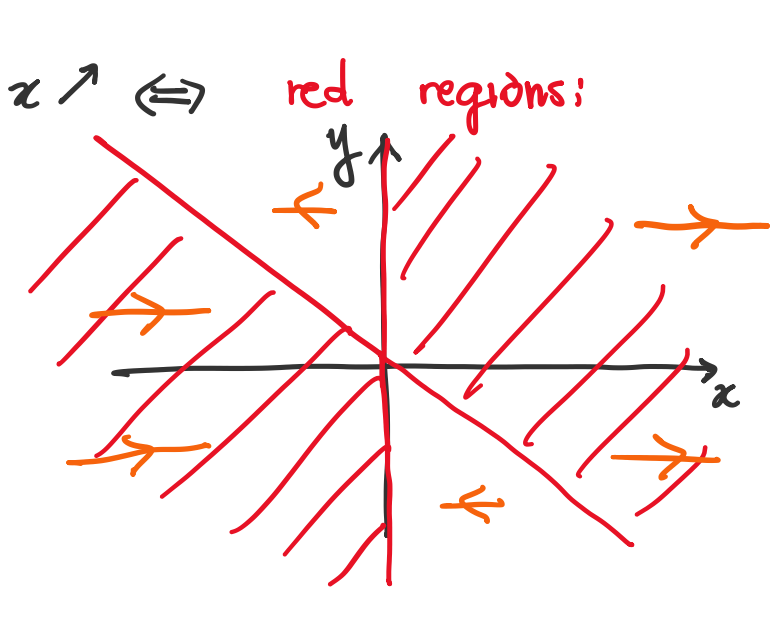
$\iff x \geq 0 \wedge y \geq -x$ or

$x \leq 0 \wedge y \leq -x$

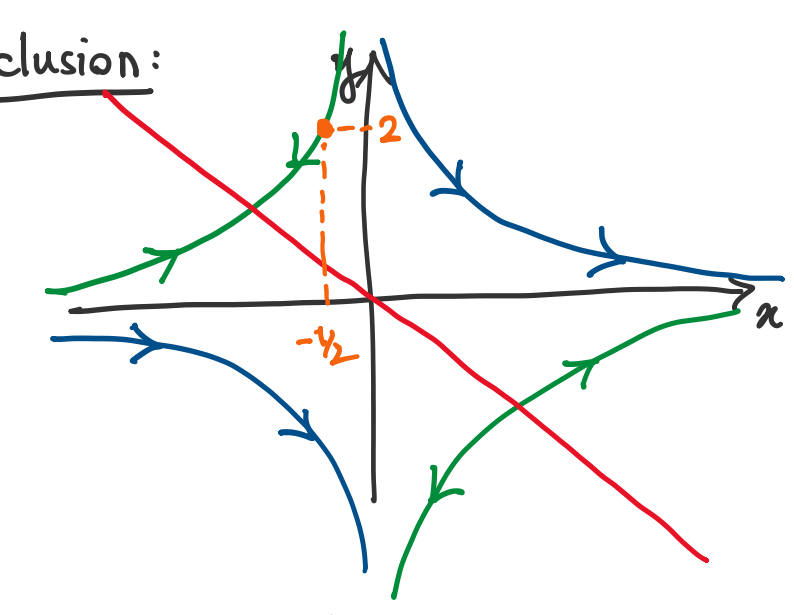


Thus:

$x \nearrow \iff$ red regions:



Conclusion:



No periodic sols.

iv) Solve CP with $x(0) = -\frac{1}{2}$, $y(0) = 2$

We know $xy \equiv C$. In part., at $t = 0$

$$-\tfrac{1}{2} \cdot 2 = C \Rightarrow C = -1$$

$$\Rightarrow xy \equiv -1 \Rightarrow y = -\tfrac{1}{x}.$$

Plugging this into the eqn for x

$$x' = x^2 - 1 \iff \frac{dx}{x^2 - 1} = 1$$

$$\underset{u = x(t)}{\iff} \int \frac{du}{u^2 - 1} = \int 1\, dt + K = t + K$$

$$\frac{1}{2}\left(\frac{1}{u-1} - \frac{1}{u+1}\right)$$

$$\iff \cancel{\tfrac{1}{2}}\left(\log|u-1| - \log|u+1|\right) = 2t + K$$

$$\iff \log\left|\frac{x(t)-1}{x(t)+1}\right| = 2t + K$$

If $t = 0$: $\log\left|\frac{-\frac{1}{2}-1}{-\frac{1}{2}+1}\right| = K \iff K = \log 3$.

$$\Rightarrow \left|\frac{x-1}{x+1}\right| = e^{2t+K} = 3e^{2t}$$

$$\iff \frac{x-1}{x+1} = \pm 3e^{2t}$$

(at $t = 0$ $\quad \frac{-\frac{1}{2}-1}{-\frac{1}{2}+1} = \pm 3 \iff -3 = \pm 3 \Rightarrow \oplus$)

$$\Rightarrow \frac{x-1}{x+1} = 3e^{2t}$$

$$\iff x - 1 = 3e^{2t}(x+1)$$

$$\iff x(1 - 3e^{2t}) = 1 + 3e^{2t}$$

$$\iff x = \frac{1 + 3e^{2t}}{1 - 3e^{2t}}.$$

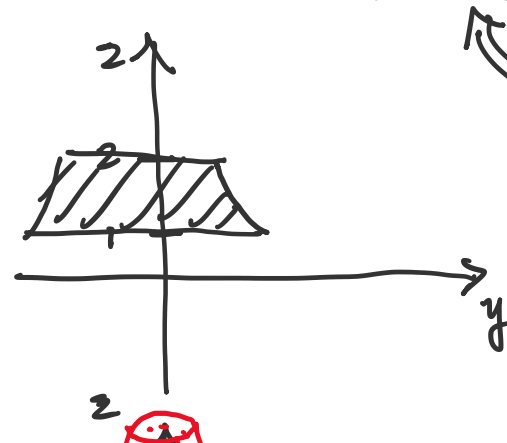
$$y = -\tfrac{1}{x}. \qquad \blacksquare$$

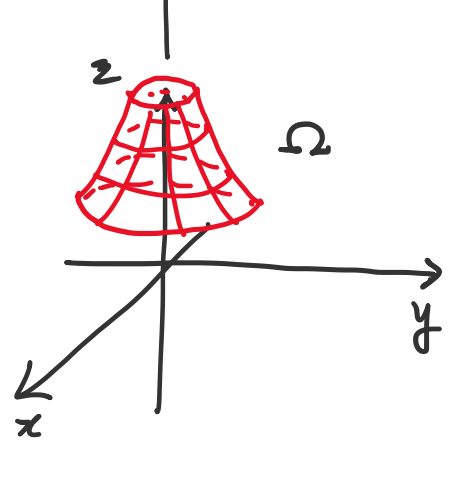
Ex 2: $\Omega = \{1 \leq z \leq 2,\ x^2 + y^2 \leq \frac{1}{z^2}\}$

i) Draw Ω

Clearly, Ω is a rotation solid with z-axis as symmetry axis: its section on yz plane $(x = 0)$ is

$$\{1 \leq z \leq 2,\ y^2 \leq \tfrac{1}{z^2}\} \quad |y| \leq \tfrac{1}{z} \quad (z > 0)$$





ii) Vol Ω. $= \int_\Omega dx\,dy\,dz$

$$= \int_{\substack{1 \leq z \leq 2 \\ x^2+y^2 \leq \frac{1}{z^2}}} dx\,dy\,dz$$

$$\overset{RF}{=} \int_1^2 \left(\int_{x^2+y^2 \leq \frac{1}{z^2}} dx\,dy\right) dz$$

$$\pi \cdot \tfrac{1}{z^2}$$

$$= \pi \int_1^2 \frac{1}{z^2}\, dz = \pi\left[-\frac{1}{z}\right]_1^2 = \pi\left(-\tfrac{1}{2} + 1\right)$$

$$= \pi/2.$$

iii) Flux of $\vec{F} = (x^2, y^2, z)$ by $\partial\Omega$ and component on $\partial\Omega \cap \{x^2 + y^2 = \frac{1}{z^2}\}$.

According to divergence thm, the outward flux of $\vec{F}$ through Ω is

$$\int_{\partial\Omega} \vec{F} \cdot \vec{n}_e = \int_\Omega \operatorname{div} \vec{F}$$

$$= \int_\Omega \left(\partial_x x^2 + \partial_y y^2 + \partial_z z\right)$$

$$= \int_\Omega (2x + 2y + 1)$$

$$= 2\left(\int_\Omega x + \int_\Omega y\right) + \operatorname{Vol}\Omega.$$

We compute $\int_\Omega x\,dx\,dy\,dz$ the other being similar.

$$\int_\Omega x\,dx\,dy\,dz = \int_{\substack{1 \leq z \leq 2 \\ x^2+y^2 \leq \frac{1}{z^2}}} x\,dx\,dy\,dz$$

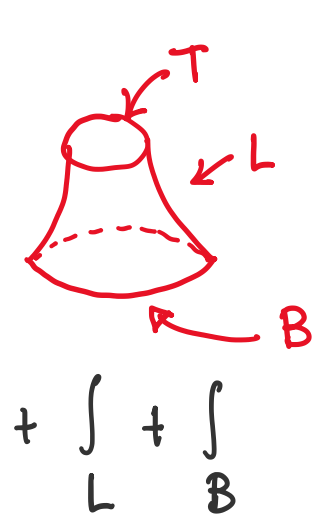
$$\overset{RF}{=} \int_1^2 \left(\int_{x^2+y^2 \leq \frac{1}{z^2}} x\,dx\,dy\right) dz$$

$$\overset{\text{pol coords}}{=} \int_1^2 \int_0^{2\pi} \left(\int_0^{\frac{1}{z^2}} \rho^2 \cos\theta\, d\rho\right) d\theta\, dz$$

$$= \underbrace{\int_0^{2\pi} \cos\theta\, d\theta}_{0} \int_1^2 \left(\int_0^{\frac{1}{z^2}} \rho^2\, d\rho\right) dz$$

$$= 0$$

Similarly $\int_\Omega y\,dx\,dy\,dz = 0 \Rightarrow \int_{\partial\Omega} \vec{F} \cdot \vec{n}_e = \operatorname{Vol}\Omega = \pi/2.$



We may decompose the outward flux in three components

$$\int_{\partial\Omega} = \int_T + \int_L + \int_B$$

We've to compute $\int_L$. We can deduce it by difference once we compute $\int_T$, $\int_B$.

$$\int_T \vec{F} \cdot \vec{n}_e = \int_{x^2+y^2 \leq \frac{1}{4}} (x^2, y^2, 2) \cdot (0,0,1)\, d\sigma$$

$$= \int_{x^2+y^2 \leq 1/4} 2\, d\sigma = 2 \operatorname{Area}(x^2 + y^2 \leq \tfrac{1}{4})$$

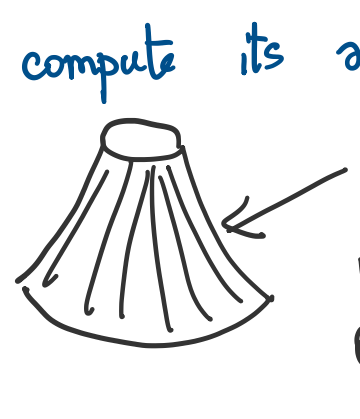
$$= 2 \cdot \pi \cdot \tfrac{1}{4} = \pi/2.$$

$$\int_B \vec{F} \cdot \vec{n}_e = \int_{x^2+y^2 \leq 1} (x^2, y^2, 1) \cdot (0,0,-1)\, d\sigma$$

$$= \int_{x^2+y^2 \leq 1} -1\, d\sigma = -\pi \cdot 1 = -\pi.$$

By this we determine $\int_L \vec{F} \cdot \vec{n}_e$.

iv) Parametrize $\partial\Omega \cap \{x^2 + y^2 = \frac{1}{z^2}\}$ and compute its area.



$\partial\Omega \cap \{x^2 + y^2 = \frac{1}{z^2}\}$

We may use two coords θ and z:

$$\phi(\theta, z) = \underbrace{\left(\tfrac{1}{z}\cos\theta,\ \tfrac{1}{z}\sin\theta\right.}_{\text{param of } x^2+y^2 = \frac{1}{z^2}}\left.,\ z\right)$$

$$\theta \in [0, 2\pi],\quad 1 \leq z \leq 2$$

Thus $\operatorname{Area} = \int_{\substack{0 \leq \theta \leq 2\pi \\ 1 \leq z \leq 2}} \|\partial_\theta \phi \wedge \partial_z \phi\|\, d\theta\, dz$

We have

$$\partial_\theta \phi \wedge \partial_z \phi = \det\begin{bmatrix} i & j & k \\ -\frac{1}{z}\sin\theta & \frac{1}{z}\cos\theta & 0 \\ -\frac{1}{z^2}\cos\theta & -\frac{1}{z^2}\sin\theta & 1 \end{bmatrix}$$

$$= \left(\tfrac{1}{z}\cos\theta,\ \tfrac{1}{z}\sin\theta,\ \tfrac{1}{z^3}\left[(\sin\theta)^2 + (\cos\theta)^2\right]\right)$$

$$= \tfrac{1}{z}\left(\cos\theta, \sin\theta, \tfrac{1}{z^2}\right)$$

$$\Rightarrow \|\partial_\theta \phi \wedge \partial_z \phi\| = \underset{(z>0)}{\tfrac{1}{z}}\sqrt{1 + \tfrac{1}{z^4}}$$

$$\Rightarrow \operatorname{Area} = \int_{\substack{1 \leq z \leq 2 \\ 0 \leq \theta \leq 2\pi}} \tfrac{1}{z}\sqrt{1 + \tfrac{1}{z^4}}\, dz$$

$$= 2\pi \int_1^2 \tfrac{1}{z}\sqrt{1 + \tfrac{1}{z^4}}\, dz$$

$$\frac{1}{z^4} = u^2$$

$$z = \frac{1}{\sqrt{u}}$$

$$dz = -\frac{1}{2u^{3/2}}\, du$$

$$= 2\pi \int_{1/4}^1 \sqrt{u} \cdot \sqrt{1+u^2} \cdot \frac{1}{2u^{3/2}}\, du$$

$$= 2\pi \int_{1/4}^1 \frac{1}{u}\sqrt{1+u^2}\, du$$

$u = \operatorname{Sh} v$

$du = \operatorname{Ch} v\, dv$

$$= 2\pi \int_{\operatorname{Sh}^{-1} 1/4}^{\operatorname{Sh}^{-1} 1} \frac{1}{\operatorname{Sh} v} \underbrace{\sqrt{1 + \operatorname{Sh}^2 v}}_{} \operatorname{Ch} v\, dv$$

$$|\operatorname{Ch} v| = \operatorname{Ch} v$$

$$= 2\pi \int_{\operatorname{Sh}^{-1} 1/4}^{\operatorname{Sh}^{-1} 1} \frac{1}{\operatorname{Sh} v} (\operatorname{Ch} v)^2\, dv$$

$$\overset{\operatorname{Ch}^2 = 1 + \operatorname{Sh}^2}{=} 2\pi \int_{\operatorname{Sh}^{-1} 1/4}^{\operatorname{Sh}^{-1} 1} \frac{1}{\operatorname{Sh} v} + \operatorname{Sh} v\, dv$$

$$= 2\pi\left(\left[\operatorname{Ch} v\right]_{\operatorname{Sh}^{-1} 1/4}^{\operatorname{Sh}^{-1} 1} - 2\int_{\operatorname{Sh}^{-1} 1/4}^{\operatorname{Sh}^{-1} 1} \frac{e^v}{e^{2v} - 1}\, dv\right)$$

$$= 2\pi\left(\left[\operatorname{Ch} v\right]_{\operatorname{Sh}^{-1} 1/4}^{\operatorname{Sh}^{-1} 1} + \log\left|\frac{e^v - 1}{e^v + 1}\right|_{1/4}^{1}\right)$$

$$\blacksquare$$

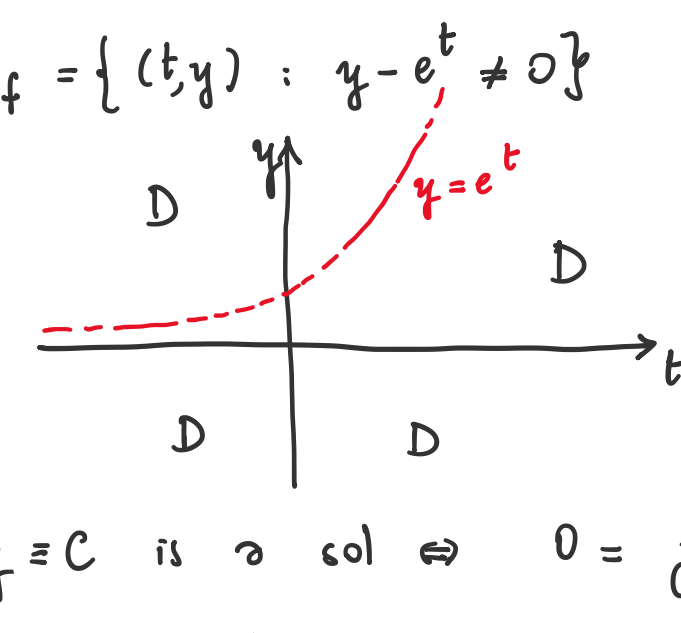
Ex 3 Consider

$$y' = \frac{y}{y - e^t}$$

i) Det D domain of loc ∃!, const. sols, regions of D where sols are ↗, ↘

Domain: $D = D_f$ where $f(t,y) = \frac{y}{y - e^t} \in \mathscr{C}^1(D_f)$

$$D_f = \{(t,y) : y - e^t \neq 0\}$$



$y \equiv C$ is a sol $\iff 0 = \frac{C}{C - e^t} \iff C = 0$

$y \nearrow \iff y' \geq 0 \iff \frac{y}{y - e^t} \geq 0$

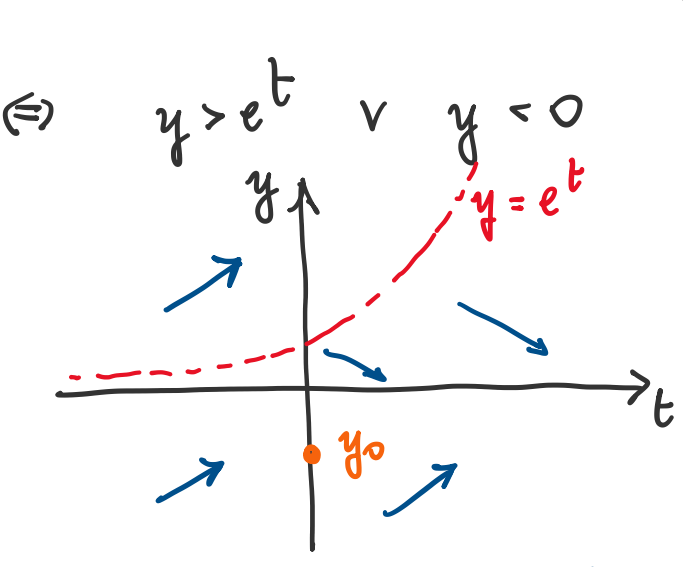
$$\iff \begin{cases} y \geq 0 \\ y - e^t > 0 \end{cases} \vee \begin{cases} y < 0 \\ y - e^t < 0 \end{cases}$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$y > e^t \qquad\qquad y < e^t$$

$$\iff \begin{cases} y \geq 0 & \text{redundant} \\ y > e^t \end{cases} \vee \begin{cases} y < 0 \\ y < e^t & \text{redundant} \end{cases}$$

$$\iff y > e^t \vee y < 0$$



Consider now the sol $\overset{y :]\alpha, \beta[\to \mathbb{R}}{\text{of}}$ CP $y(0) = y_0 < 0$

ii) y monot and $\beta = +\infty$.

Guess: since $y(0) = y_0 < 0 \Rightarrow$ initial cond belongs to $\{(t,y) : y < 0\}$

We claim $y(t) < 0 \quad \forall t \in]\alpha, \beta[$

If this happens, by prev analysis $y \nearrow$.

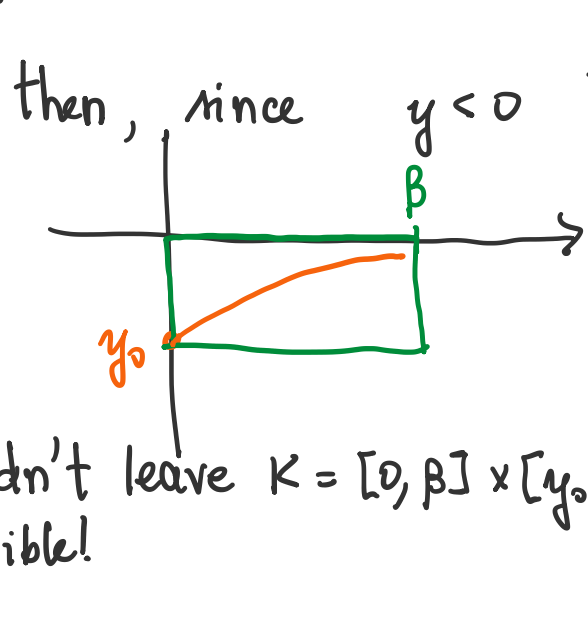
To prove the claim, assume be false: $\exists \hat{t} : y(\hat{t}) \geq 0$. But:

- if $y(\hat{t}) = 0 \Rightarrow y$ would cross another sol (constant $\equiv 0$)
  $\Rightarrow$ by uniqueness $y \equiv 0$
  $\Rightarrow \underset{y_0 < 0}{y(0)} = 0$ impossible!
- if $y(\hat{t}) > 0$ then, by continuity $\Rightarrow \exists \hat{\hat{t}} : y(\hat{\hat{t}}) = 0$ (because $y(0) = y_0 < 0$)
  $\Rightarrow$ prev case $\Rightarrow$ imposs.

Conclusion: $y \nearrow$

$\beta = +\infty$.

If $\beta < +\infty$ then, since $y < 0 \ \forall t$,



y wouldn't leave $K = [0, \beta] \times [y_0, 0] \subset\subset D$

impossible!

$\Rightarrow \beta = +\infty$.

iii) Concavity

We need y'':

$$y'' = \left(\frac{y}{y - e^t}\right)'$$

$$= \frac{y'(\cancel{y} - e^t) - y(\cancel{y'} - e^t)}{(y - e^t)^2}$$

$$= \frac{e^t(-y' + y)}{(y - e^t)^2}$$

thus $y'' \geq 0 \iff y - y' \geq 0$

Since we know

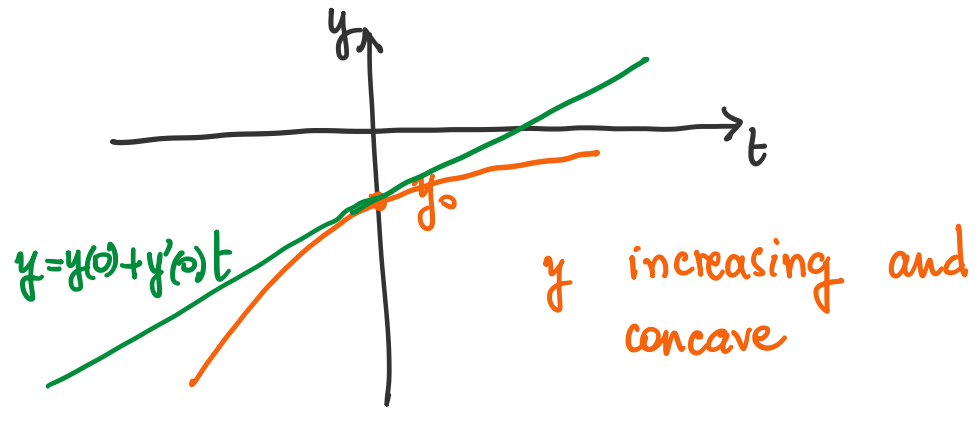
$y < 0 \ \forall t$ and $y' \geq 0 \ \forall t$

we deduce $y - y' \leq 0 \ \forall t \Rightarrow y'' \leq 0 \ \forall t$

$\Rightarrow y$ concave.

iv) $\alpha = -\infty$?

This is more difficult:



Let's see if something wrong happens assuming $\alpha = -\infty$.

Since $y \nearrow \Rightarrow \exists \lim_{t \to -\infty} y(t) =: \ell$.

Since $y \downarrow$ concave $\ell = -\infty$ (indeed

$$y(t) \leq y(0) + \underset{0}{\underset{\vee}{y'(0)}}\, t \underset{t \to -\infty}{\longrightarrow} -\infty$$

If $\alpha > -\infty$, same reasons $\exists \lim_{t \to \alpha} y(t) = \ell$

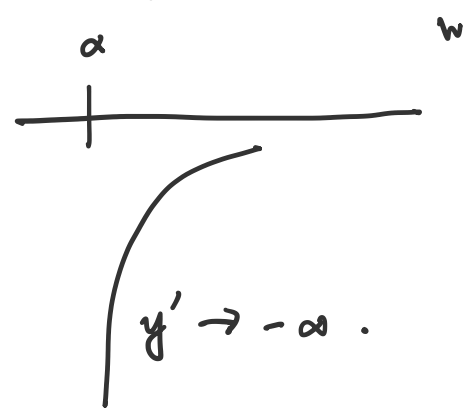
It must be $= -\infty$

(otherwise we have a contradiction with argument of compact sets)

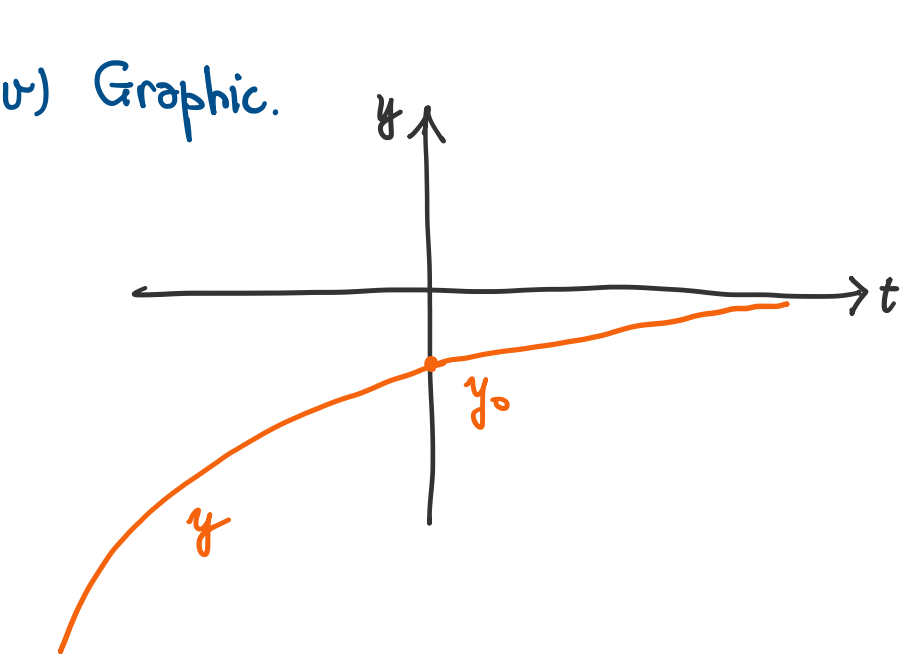
But

$$y' = \frac{\overset{\nearrow -\infty}{y}}{\underset{\swarrow -\infty}{y} - \underset{\downarrow e^\alpha}{e^t}} \underset{t \to \alpha}{\longrightarrow} \frac{\cancel{y}}{\cancel{y}(1 - \underset{\swarrow 0}{\frac{e^t}{y}})_{\searrow 1}} \longrightarrow 1$$

and $y' \not\to 1$ for $t \to \alpha$ finite with $y \to -\infty$



$$\Rightarrow \boxed{\alpha = -\infty} \quad (\text{and limit} = -\infty)$$

v) Graphic.