Fundamentals of Mathematical Analysis II — ENSTP

Exam Simulation (Solution by Wed 17th of June)

Exercise 1. Consider the differential equation

$$y' = \frac{e^y}{e^y - t}.$$

i) Find the domain D of local existence and uniqueness.
ii) Find eventual stationary solutions and plot regions of D where solutions are increasing/decreasing.

Let now $\varphi :]\alpha, \beta[\longrightarrow \mathbb{R}$ be the maximal solution of the Cauchy problem $\varphi(2) = 0$.

iii) Show that φ is monotone and deduce that $\alpha > 1$.
iv) Find the concavity of φ and discuss if $\beta = +\infty$ or less.
v) With all the previous informations plot a qualitative graph of φ.

Exercise 2. Consider the system

$$\begin{cases} x' = y(x^2 + y^2 - 2), \\ \\ y' = x(x^2 + y^2 - 2). \end{cases}$$

i) Find stationary solutions (if any).
ii) Determine a non constant first/prime integral for the system.
iii) Plot the phase portrait of the system. Are there periodic solutions? Are there non constant global solutions? Justify your answers.
iv) Find explicitly the $x-$solution of the Cauchy problem with initial conditions $x(0) = 1/\sqrt{2}$ and $y(0) = 1/\sqrt{2}$.

Exercise 3. Let

$$\Omega := \{(x, y, z) \in \mathbb{R}^3 \ : \ z \geqslant 1 - (x^2 + y^2), \ x^2 + y^2 + z^2 \leqslant 1\}.$$

i) Draw Ω.
ii) Compute the volume of Ω.
iii) Compute the outward flux by Ω of $\vec{F} := -(x, y, z)$. Determine also its components on $x^2 + y^2 + z^2 = 1$ and on $z = 1 - (x^2 + y^2)$.
iv) Parametrize $\partial\Omega \cap \{z = 1 - (x^2 + y^2)\}$ and determine its area.

Time: 4h.