# Fundamentals of Math. An. 2

## Program

1. Differential Eqns (about 2 w)
   (Focus on Qualitative Methods to solve eqns for example: plot of the graph of sols.)

2. Integration for functs of sev. vars. (2 w)
   (mult integrals, surface int., fluxes, vector analysis)

## Differential Eqns

I) Scalar First Order Eqns. $y'(t) = f(t, y(t))$

$y : I \subset \mathbb{R} \to \mathbb{R}$

II) Systems of First Order Eqns / Higher Order Eqns.

I) Here we consider a general eqn. of type

$$\boxed{y'(t) = f(t, y(t))}$$

## Examples (known)

- $$y'(t) = \underbrace{a(t)y(t) + b(t)}_{\substack{\| \\ f(t,y(t))}} \quad (*)$$ (1st Ord Linear eqns)

Here

$$f = f(t,y) : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$$

In the case of $(*)$

$$f(t,y) = a(t)y + b(t)$$

These eqns can be solved, the gen int is

$$y(t) = e^{\int a(t)dt} \left( \int e^{-\int a(t)dt} b(t) + c \right)$$

$c \in \mathbb{R}$

- $$y'(t) = a(t)\, g(y(t))$$ (sep. vars. eqn)

$$\Updownarrow$$

$$\frac{y'(t)}{g(y(t))} = a(t)$$

$$\Updownarrow$$

$$= \int_{u=u(t)} \frac{y'(t)}{g(y(t))}\, dt = \int a(t)\, dt + c, \quad c \in \mathbb{R}$$

$$\underset{\substack{u=y(t)\\ du=y'(t)dt}}{=} \int \frac{y'(t)}{g(y(t))}\,dt = \int a(t)\,dt + c, \quad c\in\mathbb{R}$$

$$= \left(\int \frac{du}{g(u)}\right)_{u=y(t)}$$

Ex: $y'(t) = 1+y(t)^2 = f(t,y(t))$

$f(t,y) = 1+y^2$

$$\frac{y'}{1+y^2} = 1 \iff \boxed{\int \frac{y'}{1+y^2}\,dt = \int 1\,dt + c = t+c}$$

$$\int \frac{y'}{1+y^2}\,dt \underset{\substack{u=y(t)\\ du=y'(t)dt}}{=} \int \frac{1}{1+u^2}\,du = \operatorname{arctg} u$$

$$= \operatorname{arctg} y(t)$$

$$\Rightarrow \boxed{\operatorname{arctg} y(t) = t+c}$$

$$\Rightarrow \boxed{y(t) = \operatorname{tg}(t+c)} \quad c\in\mathbb{R} \qquad \square$$

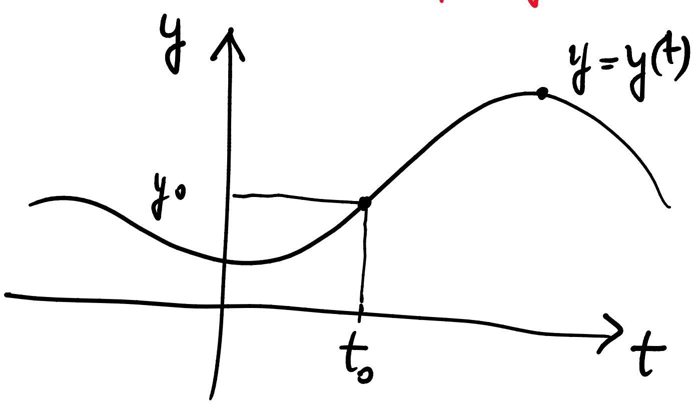
But what happens if for example we consider

$$y' = t + y^2 = t + y(t)^2 \; ?$$

A general fact emerging from solvable eqns. is that a diff eqn has infinitely many sols.

A way to identify a specific sol is through the Cauchy Pb:

$$CP(t_0, y_0) \quad \begin{cases} y'(t) = f(t, y(t)) \\ \underbrace{y(t_0) = y_0}_{\text{passage cond. / inial value}} \end{cases} \qquad (t_0, y_0) \text{ known (and fixed)}$$




Example For linear eqns. CP has always a unique solution (existence and uniqueness) unique solution (existence and uniqueness)

Indeed, the general sol of

$$y' = ay + b \qquad \begin{array}{l} a = a(t) \\ b = b(t) \\ y = y(t) \end{array}$$

$$\Rightarrow y(t) = \underbrace{e^{\int a(t)dt}}_{0 < E(t)} \Big( \underbrace{\int e^{-\int a(t)dt} b(t)\,dt}_{D(t)} + c \Big) \qquad c \in \mathbb{R}$$

$$= E(t)D(t) + cE(t)$$

If we have $\begin{cases} y' = ay + b \\ \boxed{y(t_0) = y_0} \end{cases}$ pass cond

pass cond $\Leftrightarrow \underbrace{E(t_0)D(t_0) + cE(t_0)}_{y(t_0)} = y_0$

$$\Leftrightarrow \quad c = \frac{y_0 - E(t_0)D(t_0)}{\underbrace{E(t_0)}_{\neq 0}}$$

$\Rightarrow$ there's a unique value for $c$ : $y$ is sol of $CP(t_0, y_0)$.

This is not necessarily true for non lin eqns.

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Example:

$$\begin{cases} y' = \sqrt[3]{y} \equiv y^{1/3} \\ y(0) = 0 \end{cases}$$

Let's look for sols starting by solving

$$y' = y^{1/3} = \underset{\underset{a(t)}{\parallel}}{1} \cdot \underset{\underset{g(y(t))}{\parallel}}{y^{1/3}} \quad \text{sep. vars. eqn.}$$

$$y \neq 0 \quad \Updownarrow$$

$$y^{-1/3} y' = \frac{y'}{y^{1/3}} = 1$$

$$\Updownarrow$$

$$= \int \frac{y'(t)}{y(t)^{1/3}} dt = \int 1 \, dt + c = t + c$$

$$\underset{\substack{u = y(t) \\ du = y'(t) dy}}{=} \int \frac{du}{u^{1/3}} = \int u^{-1/3} du = \frac{u^{-1/3+1}}{-1/3+1} = \frac{u^{2/3}}{2/3}$$

$$= \frac{3}{2} u^{2/3}$$

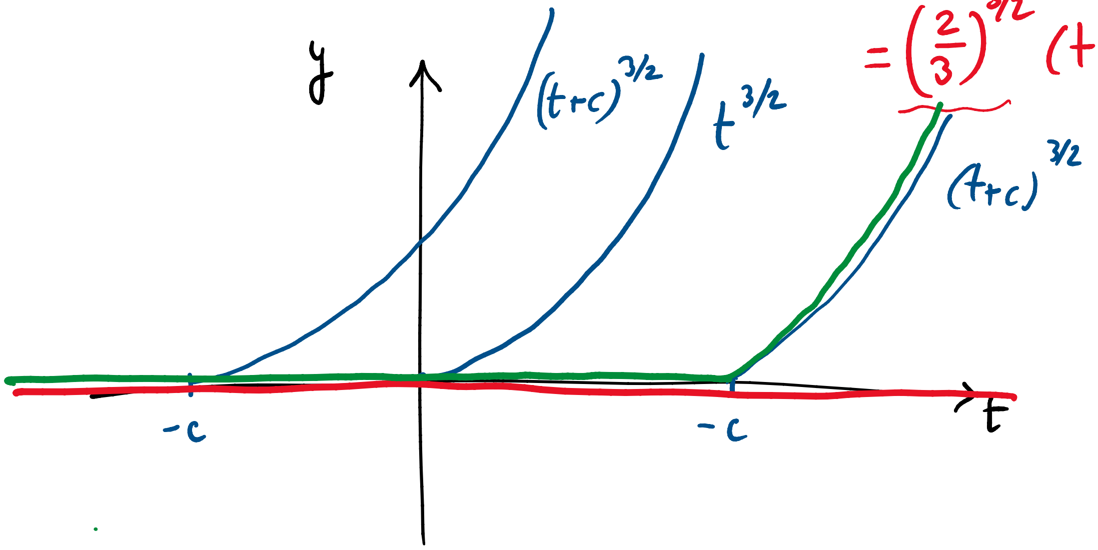
$$= \frac{3}{2} y(t)^{2/3}$$

$$\frac{3}{2} y(t)^{2/3} \quad t + c$$

$$\Rightarrow \quad \boxed{\frac{3}{2}\, y(t)^{2/3} = t + c}$$

$$\Rightarrow \quad y^{2/3} = \frac{2}{3}(t+c) \quad \Rightarrow \quad y(t) = \left(\frac{2}{3}(t+c)\right)^{3/2}$$

$$= \left(\frac{2}{3}\right)^{3/2}(t+c)^{3/2}$$




$$y(t) = \left(\frac{2}{3}\right)^{3/2} t^{3/2} \quad \text{fulfills} \quad \begin{cases} y' = y^{1/3} \\ y(0) = 0 \end{cases} \quad (CP)$$

However this is not the unique sol to this pb.

Indeed:

- $y \equiv 0$ is a sol of (CP).

- but also

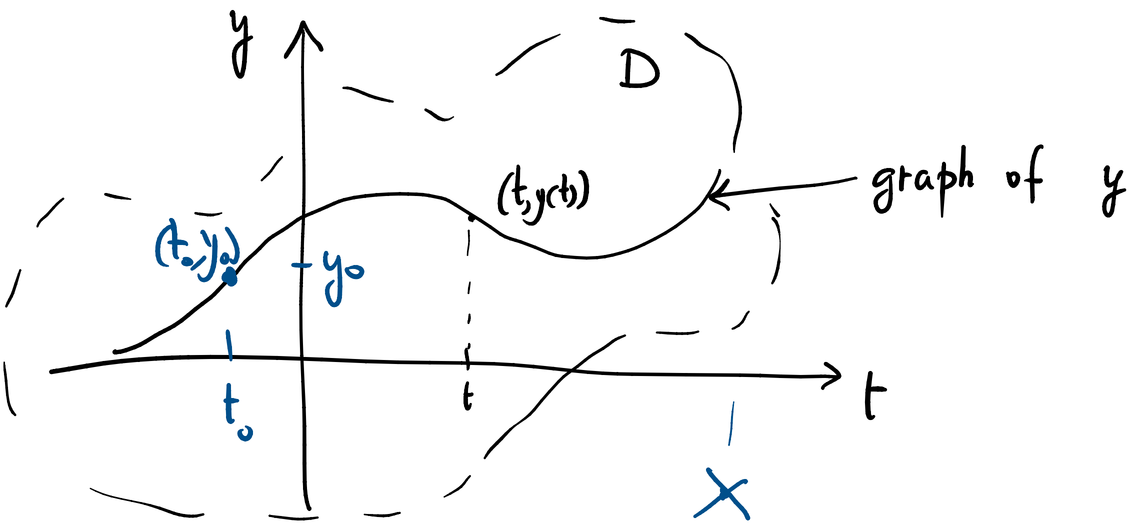
$$y(t) = \begin{cases} 0 & t \leq -c \\ \left(\frac{2}{3}\right)^{3/2}(t+c)^{3/2} & t > -c \end{cases}$$

is a sol of (CP). for $c<0$.

## Some general Def

Consider a function

$$f = f(t,y) : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$$




**Def:** A function $y : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ is a solution of the Cauchy Pb

$$CP(t_0,y_0) \quad \begin{cases} y'(t) = f(t,y(t)) \\ y(t_0) = y_0 \end{cases} \qquad \left(\text{shortly } y' = f(t,y)\right)$$

if:

- $(t,y(t)) \in D \quad \forall t \in I$
- $y \in \mathcal{C}^1(I) \quad$ (that is $y \in \mathcal{C}$, $\exists y'$, $y' \in \mathcal{C}$)
- $y(t_0) = y_0$

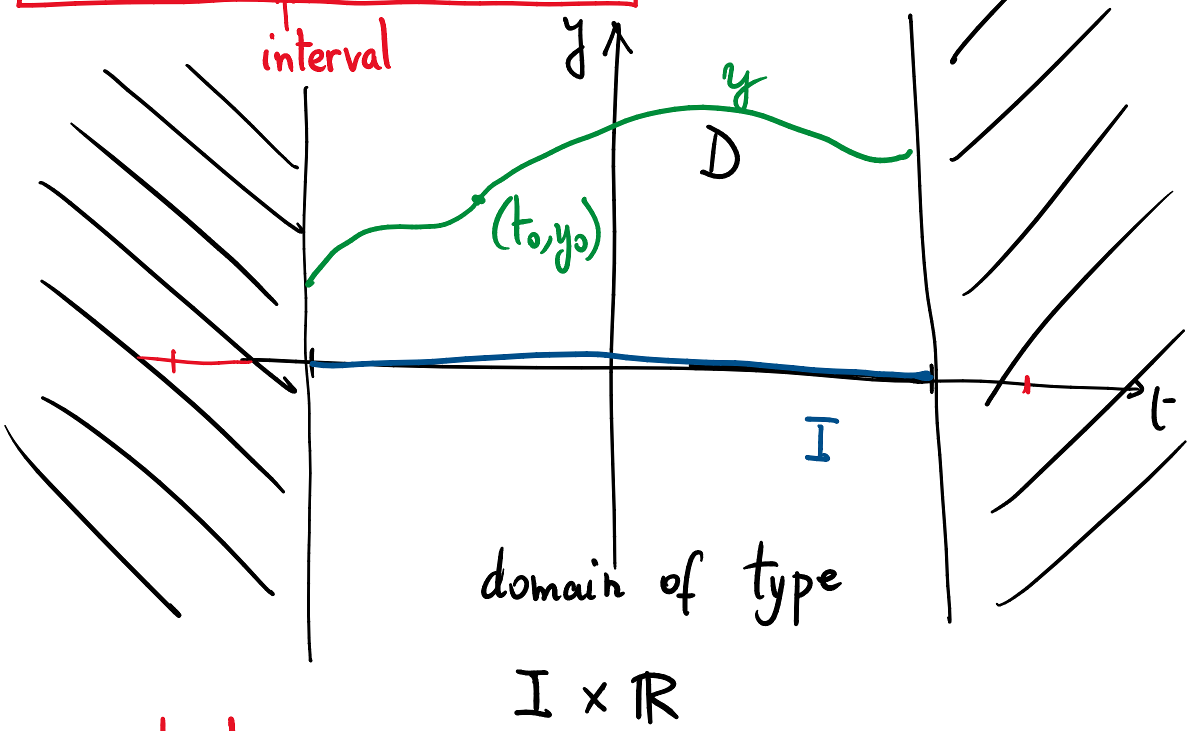
Rmk: In part also $(t_0, y_0) \in D$.

Pb: Under which conds on $f$ a $CP(t_0,y_0)$ has a unique sol?

Thm 1 (Global Existence and Uniqueness)

Let $\boxed{f : D = I \times \mathbb{R} \longrightarrow \mathbb{R}}$

interval




be such that

1. $f \in \mathcal{C}(D)$
2. $\partial_y f$ be bounded on $D$

(that is $\exists M : \; |\partial_y f(t,y)| \leq M \quad \forall (t,y) \in D$)

Then $\forall (t_0,y_0) \in D \; \exists! \; y : I \longrightarrow \mathbb{R}$ sol of $CP(t_0,y_0)$.

(exists and unique)

## Examples:

- $y' = ty + e^t = a(t)\,y + b(t)$
  $= f(t,y)$

$f(t,y) = ty + e^t \qquad D = \{(t,y) : t \in \mathbb{R},\ y \in \mathbb{R}\}$
$$= \mathbb{R} \times \mathbb{R}$$
$$\| $$
$$I$$

$f \in \mathcal{C}(D)$

$\partial_y f = t \quad$ : is $\partial_y f$ bdd on $\mathbb{R} \times \mathbb{R}$?

(if yes $\quad \exists M : \quad |\partial_y f(t,y)| \le M \quad \forall (t,y) \in \mathbb{R}\times\mathbb{R}$.
$$\| $$
$$|t| \le M \qquad \forall (t,y) \in \mathbb{R}\times\mathbb{R}$$
NO! )

- $y' = \sqrt{1-t^2}\,\sin y$

Here $\quad f(t,y) = \sqrt{1-t^2}\,\sin y \quad$ is dfd on

$D = \{ 1-t^2 \geq 0 \quad y \in \mathbb{R} \} = \{ -1 \le t \le 1,\ \ y \in \mathbb{R} \}$

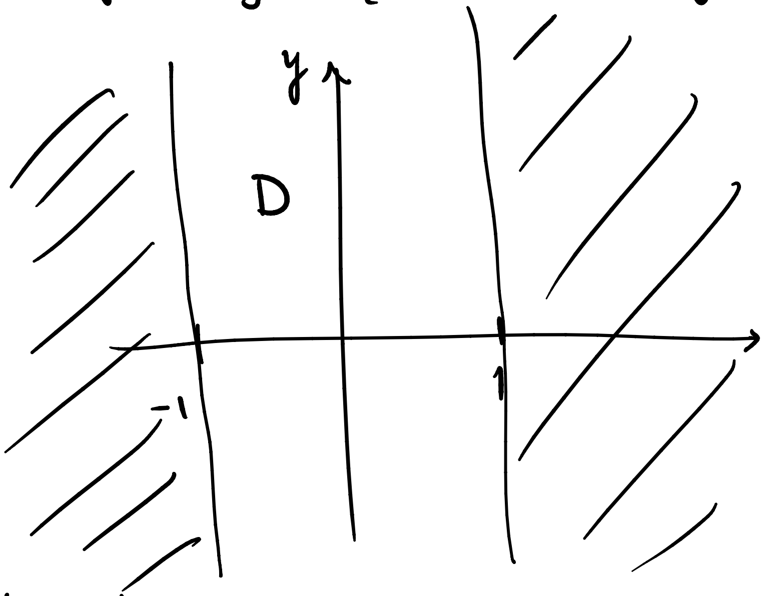
$$D = \{ 1-t^2 \geq 0,\ y \in \mathbb{R} \} = \{-1 \leq t \leq 1,\ y \in \mathbb{R}\}$$

$$\Updownarrow$$

$$t^2 \leq 1$$

$$\Updownarrow$$

$$|t| \leq 1$$




$$\Downarrow$$

$$D = [-1,1] \times \mathbb{R}$$

So domain is ok. Then

1. $f \in \mathcal{C}(D)$ (obvious)

2. $\partial_y f = \sqrt{1-t^2}\cos y$ is bded on $[-1,1]\times\mathbb{R}$?

$$|\partial_y f(t,y)| = \left|\sqrt{1-t^2}\cos y\right|$$

$$\leq 1 \cdot \sqrt{1-t^2} \leq 1 \quad \forall (t,y) \in [-1,1]\times\mathbb{R}$$

$\Longrightarrow$ yes in this case the hypotheses of G.l. $\exists!$ are fulfilled. $\boxtimes$

- $y' = y^{1/3} = f(t,y)$ $\quad D = \{t \in \mathbb{R},\ y \in \mathbb{R}\} = \mathbb{R}\times\mathbb{R}$

vertical strip.

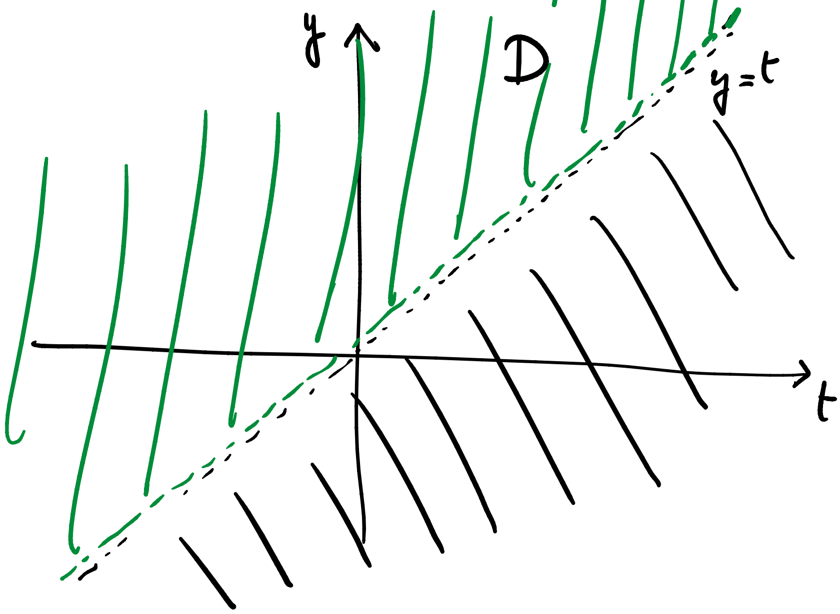
- $f \in \mathcal{C}(D)$
- $\partial_y f = \frac{1}{3} y^{-2/3} = \frac{1}{\ y^{2/3}}$ unbounded

- $\partial_y f = \frac{1}{3} y^{-2/3} = \frac{1}{3y^{2/3}}$ unbounded

We cannot apply GI $\exists!$

- $y' = \log(y-t) = f(t,y)$

$D = \{(t,y) : y-t>0\} = \{y>t\}$




domain in green