sabato 29 febbraio 2020 10:44

**Exercise**

$$(S)\quad \begin{cases} x' = y(x^2+y^2) \\ y' = x(x^2+y^2) \end{cases}$$

i) St sols, ii) First int iii) Phase portrait, global sols periodic sols if any

iv) Solve CP $x(0)=0$, $y(0)=1$

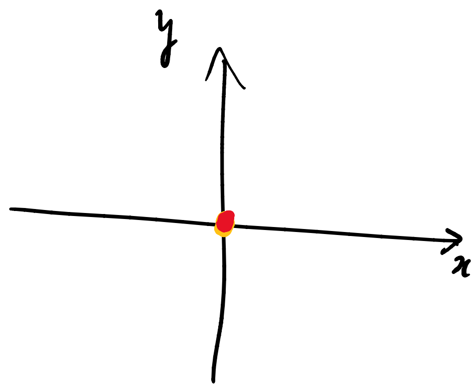
Sol: i) $(x,y)\equiv(a,b)$ is a sol of $(S) \iff$

$$\begin{cases} \boxed{0 = b(a^2+b^2)} \\ 0 = a(a^2+b^2) \end{cases}$$

$$\iff \begin{cases} b=0 \\ a^3=0 \end{cases} \quad \vee \quad \begin{cases} a^2+b^2=0 \\ 0=0 \end{cases}$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$(0,0) \qquad\qquad (0,0)$$



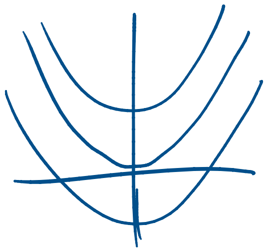

there's a unique st sol.

ii) To det a non trivial first int we write the total eqn

$$\frac{dy}{dx} = \frac{x\cancel{(x^2+y^2)}}{y\cancel{(x^2+y^2)}} = \frac{x}{y}$$

$\Longleftrightarrow \quad y\,dy = x\,dx \quad \Longleftrightarrow \quad \frac{y^2}{2} = \frac{x^2}{2} + C$

$\Longleftrightarrow \quad \underbrace{y^2 - x^2}_{E(x,y)} = C$



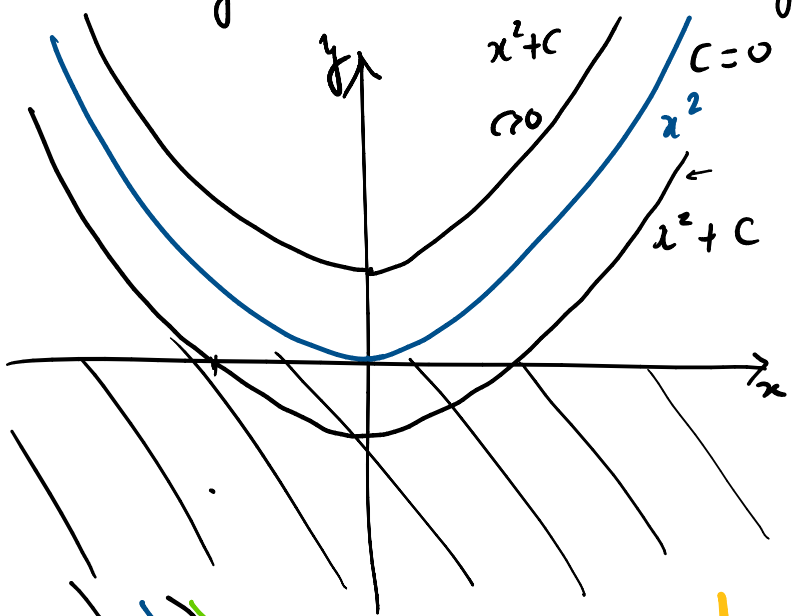
$\boxed{y = x^2 + C}$

$x^2 + y^2 = x + c$

iii) Phase portrait: we have to plot typical orbits. We know that for orbits the following rel. holds:

$$E(x,y) \equiv C \quad \Longleftrightarrow \quad y^2 - x^2 = C$$

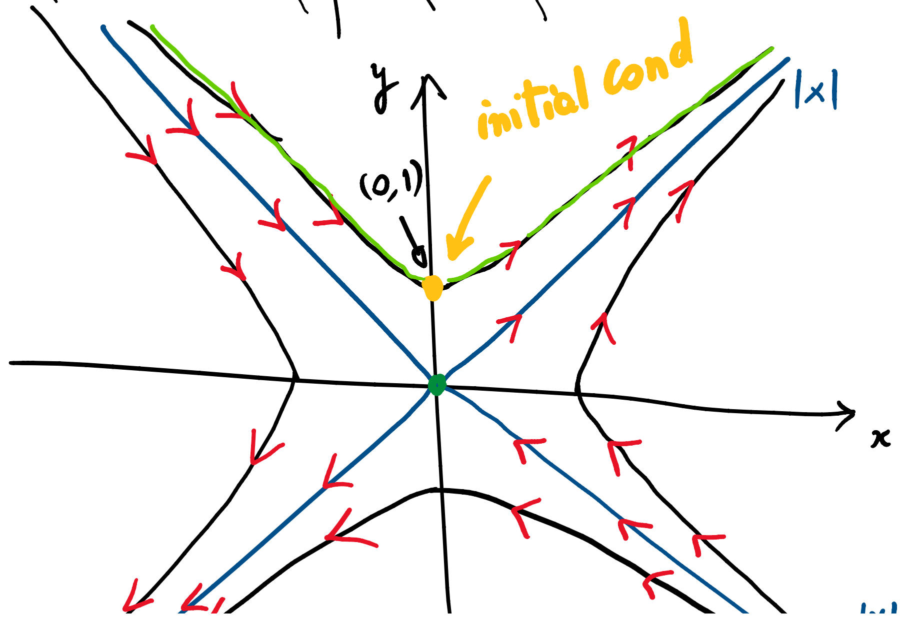
$$\Longleftrightarrow \quad y^2 = x^2 + C \quad \Longleftrightarrow \quad y = \pm\sqrt{x^2 + C}$$

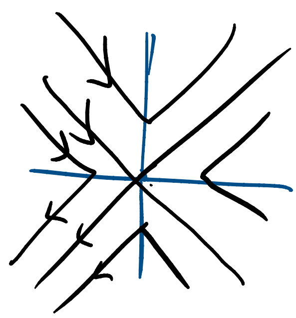


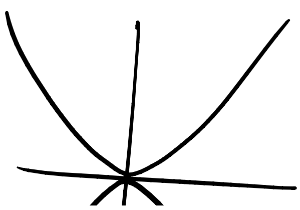

$C < 0$

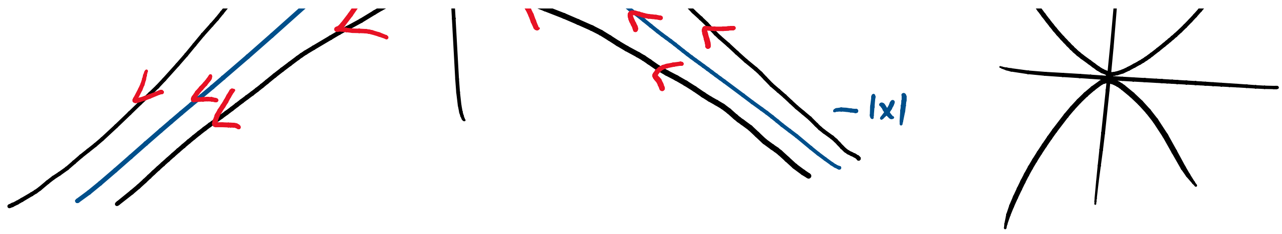
$C = 0$

$y = \pm\sqrt{x^2} = \pm|x|$








Orientation: $x \nearrow \iff 0 \leq x' = y\,(x^2+y^2) \iff y \geq 0$

$\phantom{Orientation: x \nearrow \iff 0 \leq x' = y} \rotatebox{90}{$\leq$}\ 0$

No periodic sol's. About global sol we cannot say too much at this point

iv) We solve CP $x(0)=0,\ y(0)=1$

We know

$$E(x(t),y(t)) \equiv E(x(0),y(0)) = E(0,1)$$
$$= 1^2 - 0^2 = 1$$

$$\Rightarrow \quad y(t)^2 - x(t)^2 \equiv 1 \implies y(t)^2 \equiv 1 + x(t)^2$$

$$\Rightarrow \quad y = \pm\sqrt{1+x^2} \qquad \left(\text{at } t=0 \quad 1 = \pm\sqrt{1+0^2} \right.$$

$$y^2 = 1 + x^2 \qquad\qquad 1 = \pm 1$$

$$\Downarrow$$

$$\left. \oplus \right)$$

$$\Rightarrow \quad y = \sqrt{1+x^2}$$

$$\Rightarrow \quad x' = y\,(x^2+y^2) = \sqrt{1+x^2}\,(x^2+1+x^2)$$

$$x' = \sqrt{1+x^2}\,(2x^2+1)$$

$$\iff \frac{dx}{\sqrt{1+x^2}(2x^2+1)} = dt$$

$$\iff \boxed{\int \frac{dx}{\sqrt{1+x^2}\,(2x^2+1)} = t + C} \quad (*)$$

$$a^2 + (\quad)^2$$

$$Ch^2 - Sh^2 = 1$$

$$Ch^2 = 1 + Sh^2$$

$$\boxed{x = Shu} \quad u = Sh^{-1}x$$

$$dx = Chu\,du$$

$$\| \qquad \int \frac{\cancel{Ch}u\,du}{\sqrt{1+(Shu)^2}\,(2(Shu)^2+1)}$$

$$\sqrt{1+(Shu)^2} = \sqrt{(Chu)^2} = |Chu| = \cancel{Ch}u, \quad Chu > 0$$

$$= \int \frac{1}{2(Shu)^2+1}\,du$$

$$\frac{1}{2(Shu)^2+1} = \frac{1}{2\left(\frac{e^u - e^{-u}}{2}\right)^2 + 1} = \frac{4}{2\,\frac{e^{2u} + \frac{1}{e^{2u}} - 2}{\cancel{4}} + 4}$$

$$= \frac{4}{2e^{2u} + \frac{2}{e^{2u}} \cancel{-4} \cancel{+4}}$$

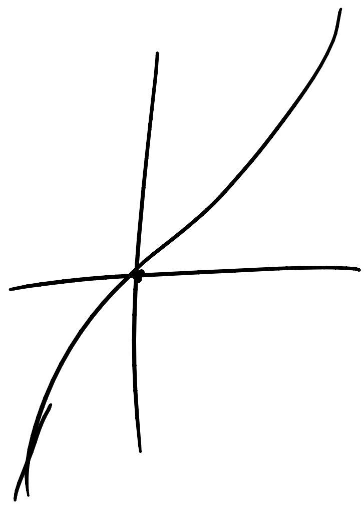
$$\frac{2\cancel{4}e^{2u}}{}$$

$$(\operatorname{arctg} e^{2u})' = \frac{1}{\cdot} \cdot 2e^{2u}$$

$$= \frac{2\cancel{4}e^{2u}}{\cancel{2}(e^{4u}+1)} = \frac{1}{1+(e^{2u})^2}\cdot 2e^{2u}$$

$$= \int \frac{2e^{2u}}{1+(e^{2u})^2}\,du = \int \frac{1}{1+(e^{2u})^2}(e^{2u})'\,du$$

$$= \operatorname{arctg}(e^{2u}) = \operatorname{arctg} e^{2\,\mathrm{Sh}^{-1}x}.$$



Therefore $(*)$ becomes

$$\boxed{\operatorname{arctg} e^{2\,\mathrm{Sh}^{-1}x} = t + C}$$

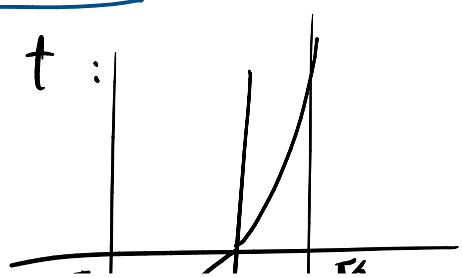
$t=0$ $\quad C = \operatorname{arctg} e^{2\,\mathrm{Sh}^{-1}\overset{0}{\overset{\shortparallel}{x(0)}}=0} = \operatorname{arctg} \underset{\underset{1}{\shortparallel}}{e^0} = \pi/4$

$$e^{2\,\mathrm{Sh}^{-1}x} = \operatorname{tg}(t+\pi/4)$$

$$\cancel{2}\,\mathrm{Sh}^{-1}x = \frac{1}{2}\log(\operatorname{tg}(t+\pi/4))$$

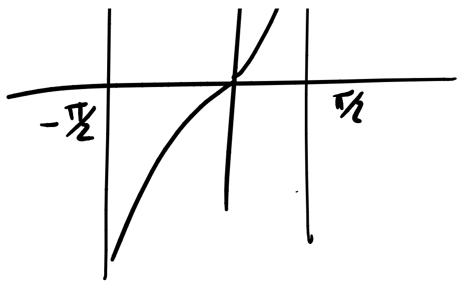
$$\boxed{x(t) = \mathrm{Sh}\left(\frac{1}{2}\log \operatorname{tg}(t+\pi/4)\right)}$$

Rmk: This sol. is defined for $t$:

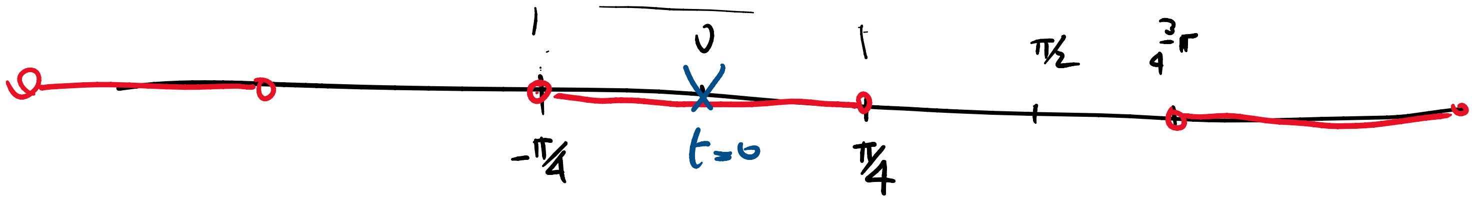


$\Big\{ -\frac{\pi}{2} + k\pi < \cancel{t + \pi/4 < \frac{\pi}{2}} + k\pi$

$$\begin{cases} \cancel{-\frac{\pi}{2} + k\pi < t + \frac{\pi}{4} < \frac{\pi}{2} + k\pi} \\ k\pi + 0 < t + \frac{\pi}{4} < \frac{\pi}{2} + k\pi \end{cases}$$



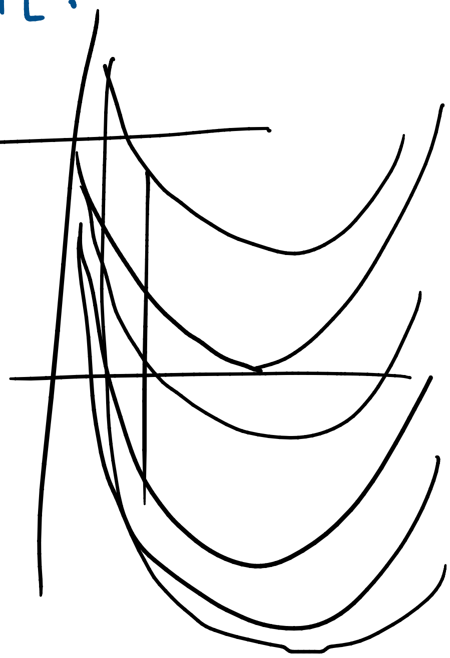

$$0 + k\pi < t + \frac{\pi}{4} < \frac{\pi}{2} + k\pi \iff -\frac{\pi}{4} + k\pi < t < +\frac{\pi}{4} + k\pi$$



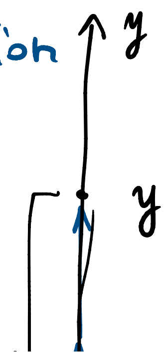

Because $t = 0 \in \left]\alpha, \beta\right[$ where sol is def'd

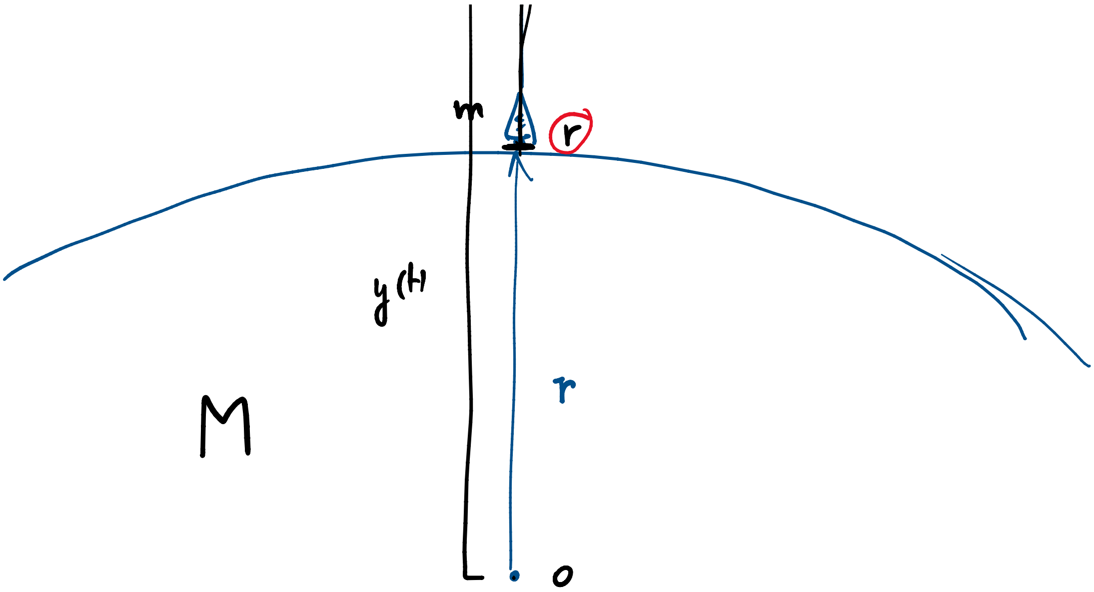
$\Rightarrow \quad \left]\alpha, \beta\right[ = \left]-\frac{\pi}{4}, \frac{\pi}{4}\right[.$

$$y = (x-1)^2 + c$$



Ex. Let's consider a mass $m$ in movement subj to gravitation






According to Newton second law

$$\cancel{m}\, y''(t) = F = -\frac{kM\cancel{m}}{y(t)^2}$$

$$\boxed{y'' = -\frac{kM}{y^2} \equiv -\frac{K}{y^2} \qquad K>0}$$

This is cons. system

$$y'' = -\frac{K}{y^2} = \partial_y\left(\frac{K}{y}\right)$$

and the mechanical energy is

$$E(y,v) = \frac{v^2}{2} - \frac{K}{y}$$

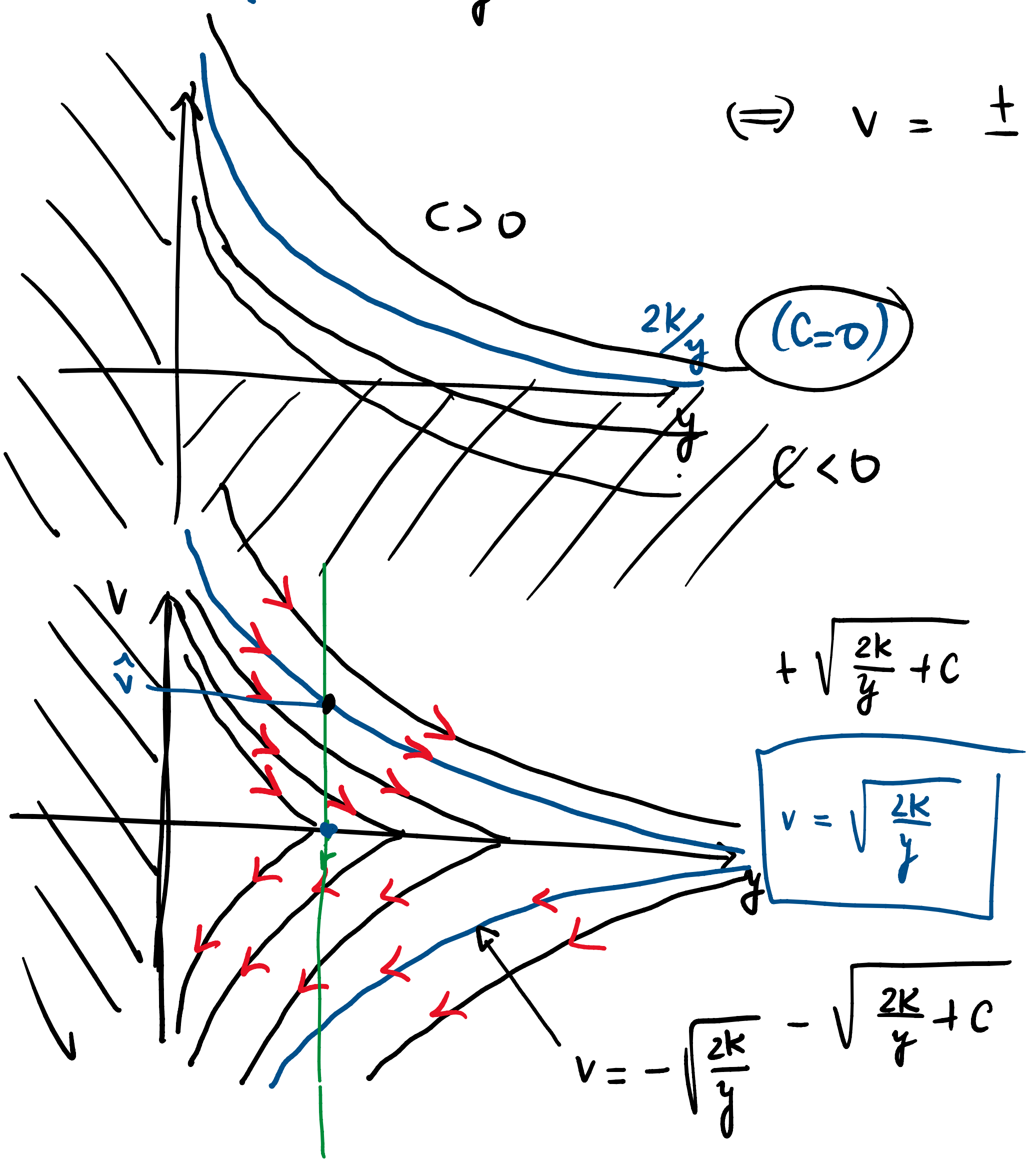
Let's plot the phase portr for this system

We've to plot lines

$$\frac{v^2}{2} - \frac{K}{y} \equiv C \iff \frac{v^2}{\cancel{2}} = \frac{2K}{y} + C$$

$$\frac{v^2}{2} - \frac{K}{y} \equiv C \iff \frac{v^2}{\cancel{2}} = \frac{2K}{y} + C$$

$$\iff v = \pm\sqrt{\frac{2K}{y} + C}$$




Equiv syst

$$\begin{cases} \boxed{y' = v} \\ v' = -\frac{K}{y^2} \end{cases}$$

$$\boxed{\hat{v} = \sqrt{\frac{2K}{r}}}$$

Let's solve the CP

$$\begin{cases} y'' = -\frac{K}{y^2} \\ y(0) = r \\ y'(0) = \hat{v} \end{cases}$$

We know that E is conserved along our sol.:

$$E(y, v) = E(y(0), v(0)) = 0$$

sol.:

$$E(y,v) \equiv E(y(0), v(0)) = 0$$

$$\parallel$$

$$\frac{v^2}{2} - \frac{K}{y} = 0$$

$$\Rightarrow \quad v^2 = \frac{2K}{y} \quad \Rightarrow \quad v = \cancel{\pm} + \sqrt{\frac{2K}{y}}$$

$$\Rightarrow \quad y' = v = \sqrt{\frac{2K}{y}} \quad \Longleftrightarrow \quad \underbrace{y^{1/2} y'}_{\left(\frac{2}{3} y^{3/2}\right)' =} = \sqrt{2K}$$

$$= \sqrt{2K}$$

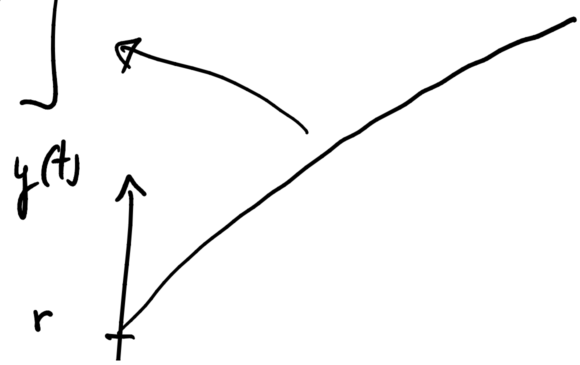
$$\Rightarrow \quad \frac{2}{3} y^{3/2} = \underbrace{\sqrt{2K}\, t} + C$$

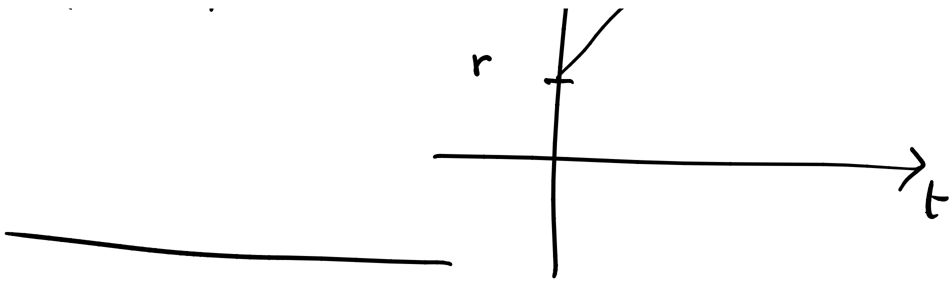
$$t = 0 \qquad C = \frac{2}{3} y(0)^{3/2} = \frac{2}{3} r^{3/2}$$

$$\Rightarrow \quad y^{3/2} = \frac{3}{2}\left(\sqrt{2K}\, t + \frac{2}{3} r^{3/2}\right) = \frac{3}{\sqrt{2}} \sqrt{K}\, t + r^{3/2}$$

$$\Rightarrow \quad y(t) = \left[ 3\sqrt{\frac{K}{2}}\, t + r^{3/2} \right]^{2/3}$$

$$(\alpha t + \beta)^{2/3}$$

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