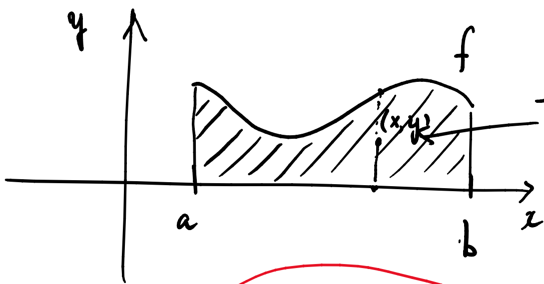
# Multiple Integration

Goal: To def and develop method of calculus for

$$\int_D f(x_1, \dots, x_d)\, dx_1 \dots dx_d = \int_D f(\vec{x})\, d\vec{x}$$

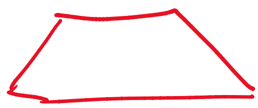
$$f = f(x_1, \dots, x_d) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$$

If $d = 1$, $f = f(x) : [a,b] \longrightarrow [0, +\infty[$

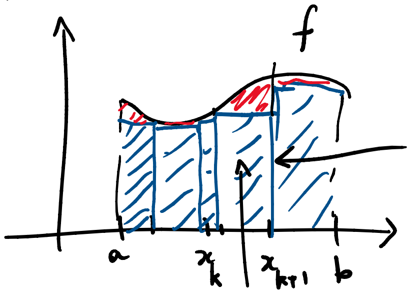



$\mathrm{Trap}(f) = \{(x,y) \in \mathbb{R}^2 : a \le x \le b,\ 0 \le y \le f(x)\}$

$$\int_a^b f(x)\, dx := \mathrm{Area}(\mathrm{Trap}(f))$$



where $\mathrm{Area}(\mathrm{Trap}(f))$ is built by doing a certain approx procedure




$$h_k = \min_{x \in [x_k, x_{k+1}]} f(x)$$

$$\underline{S}(\pi) = \sum_k (x_{k+1} - x_k) \cdot h_k \qquad \pi = \{ a = x_0 < x_1 < \dots < x_n = b \}$$

$$\sup \{ \underline{S}(\pi) : \pi \text{ subdiv of } [a,b] \} =: \underline{A}(\mathrm{Trap}(f))$$

best approx from below

$$\inf \{ \overline{S}(\pi) : \quad " \quad " \quad " \quad \} =: \overline{A}(\mathrm{Trap}(f))$$

$J =: A(\mathrm{Trap}(f))$

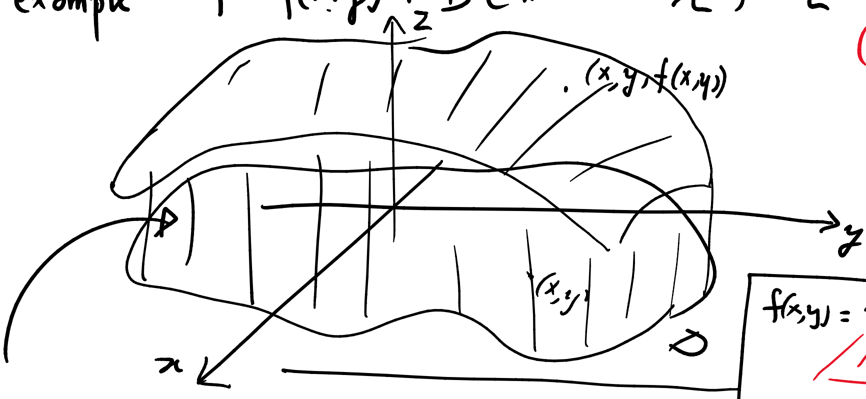
If $\underline{A} = \bar{A}$ we say that $\exists \int_a^b f(x)dx \equiv \underline{A} \equiv \bar{A}$.

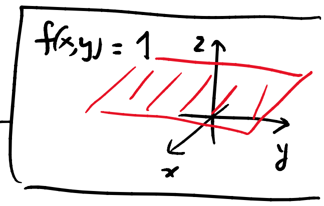
Imagine we try to repeat this procedure: when

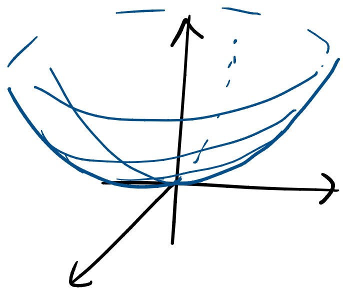
$$f = f(x_1, \dots, x_d)$$

For example $f = f(x,y) : D \subset \mathbb{R}^2 \longrightarrow [0, +\infty[$

$(P, f(P))$



$$\mathrm{Trap}(f) = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in D,\ 0 \leq z \leq f(x,y)\}$$

To def

$$\int_D f(x,y)\,dx\,dy$$

I should def $\mathrm{Vol}(\mathrm{Trap}(f))$.

If $f = f(x,y,z) : D \subset \mathbb{R}^3 \longrightarrow [0, +\infty[$

$$\mathrm{Trap}(f) = \{(x,y,z,w) \in \mathbb{R}^4 : (x,y,z) \in D,\ 0 \leq w \leq f(x,y,z)\}$$

to def

$$\int_D f(x,y,z)\,dx\,dy\,dz$$

we should def

4-dim $\mathrm{Vol}(\mathrm{Trap}(f))$

$D \subset \mathbb{R}^d \longrightarrow [0, +\infty[$

4 - dim vol ( ... p(4 0))

and more generally, if $f = f(\underbrace{x_1, \dots, x_d}_{\vec{x}}) : D \subset \mathbb{R}^d \to [0, +\infty[$

$$\mathrm{Trap}(f) = \{ (\underset{\mathbb{R}^d}{\underset{\cap}{\vec{x}}}, y) \in \mathbb{R}^{d+1} : \vec{x} \in D, \ 0 \leq y \leq f(\vec{x}) \}$$

to def

$$\int_D f(x_1, \dots, x_d)\, dx_1 \dots dx_d$$

we should dispose of d+1 dim measure of Trap(f)

where in dim = 2 $\Rightarrow$ area

dim = 3 $\Rightarrow$ volume

To define in gen the operation of int. we need a suitable concept of d - dim measure of a set.

Given a set $S \subset \mathbb{R}^d$ we will denote by

$$\lambda_d(S) \in [0, +\infty]$$

the d dim measure of S.

$$\lambda_d : \text{sets} \longrightarrow \text{numbers}$$

Thm: There exists a function

$$\lambda_d : \underset{\substack{\uparrow \\ \text{class of measurable sets}}}{\mathcal{M}_d} \subset \underset{\substack{\uparrow \\ \text{parts of } \mathbb{R}^d}}{\mathcal{P}(\mathbb{R}^d)} \longrightarrow [0, +\infty]$$

such that

class of measurable ...

such that

i) every closed and every open sets are measurable, and any set $S$ who differs by a measurable set by a meas $=0$ set, is meas.

Example:

$\{(x,y,z)\in\mathbb{R}^3 : g(x,y,z)\geq 0\}$ is closed $g\in C(\mathbb{R}^3)$

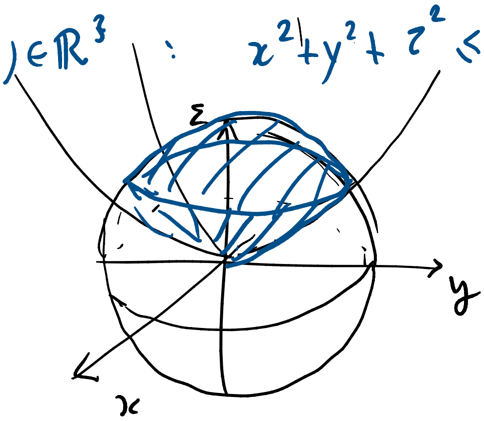
$x^2+y+z^2x\geq 0$ $\Downarrow$ it is meas.

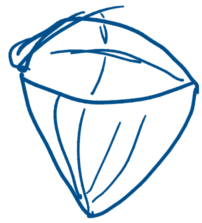
$g_1(x,y,z)\geq 1$ $\quad$ $g_2(x,y,z)\geq 0$

$\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2\leq 1,\ z\geq x^2+y^2\}$ is closed

$\Downarrow$

it is meas.



in gen a set

$$C:=\{\vec{x}\in\mathbb{R}^d : g_1(\vec{x})\geq 0,\ g_2(\vec{x})\geq 0 \ldots,\ g_m(\vec{x})\geq 0\}$$

$g_1, g_2 \ldots, g_m \in C \Rightarrow C$ is closed $\Rightarrow C$ is meas.

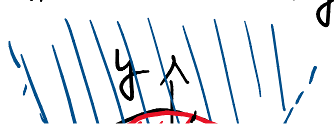
Similarly

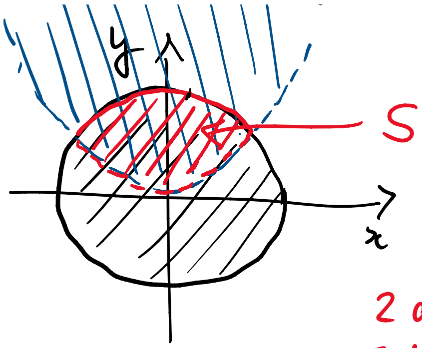
$$O=\{\vec{x}\in\mathbb{R}^d : g_1(\vec{x})>0,\ g_2(\vec{x})>0 \ldots\ g_m(\vec{x})>0\}$$

$g_1 \ldots g_m \in C \Rightarrow O$ is open $\Rightarrow O$ is meas.

but what if

$$S=\{(x,y)\in\mathbb{R}^2 : x^2+y^2\leq 1,\ \ y>x^2\}$$



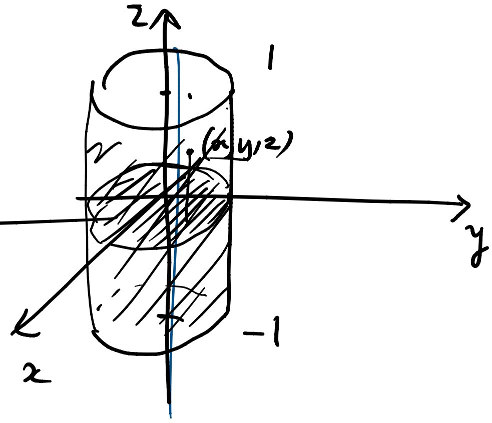




2 dim meas = 0 set

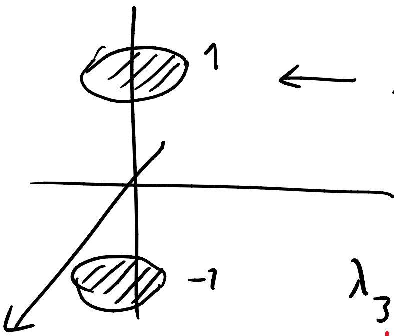
$S \setminus \tilde S =$ 

So for example




$$S = \{(x,y,z) \in \mathbb{R}^3 : -1 < z < 1, \; x^2+y^2 \le 1\}$$

$$\tilde S = \{(x,y,z) \in \mathbb{R}^3 : -1 \le z \le +1, \; x^2+y^2 \le 1\}$$ is closed $\Rightarrow$ it is meas.

$\tilde S \setminus S =$  $\leftarrow \lambda_3(\;) = 0$

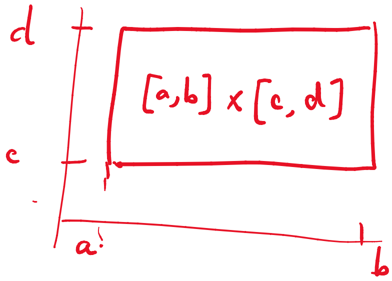


$\lambda_3(\cdot) = 0$

$\Downarrow$

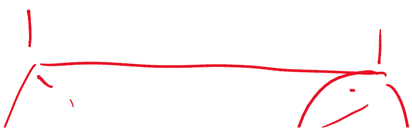
$S$ is meas.

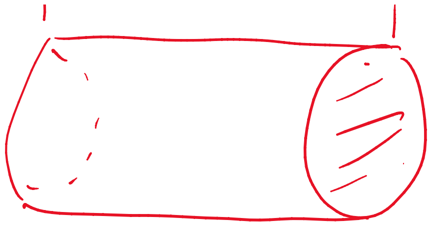
ii) (factorization)




$\lambda_2([a,b] \times [c,d]) = (b-a)\cdot(d-c)$

$\lambda_1([a,b])\,\lambda_1([c,d])$





$\lambda_1([a,b])\,\lambda_1([c,d])$

$A \times B \subset \mathbb{R}^{d+m}$ $\quad A \subset \mathbb{R}^d,\ B \subset \mathbb{R}^m$ $\quad A \in \mathcal{M}_d$, $B \in \mathcal{M}_m$.

$$\Rightarrow \quad \lambda_{d+m}(A \times B) = \lambda_d(A)\,\lambda_m(B)$$

In part

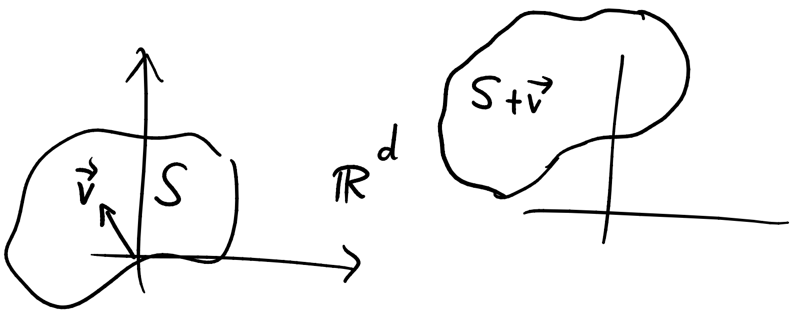
$$\lambda_d([a_1,b_1] \times [a_2,b_2] \times \dots \times [a_d,b_d])$$

$$= \lambda_1([a_1,b_1])\,\lambda_1([a_2,b_2]) \dots \lambda_1([a_d,b_d])$$

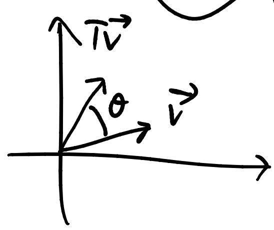
$$\equiv (b_1-a_1)(b_2-a_2)\dots(b_d-a_d)$$

iii) (invariances)

$$\lambda_d(S+\vec{v}) = \lambda_d(S)$$




$$T \doteq \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$




$$\lambda_2(TS) = \lambda_2(S) \qquad \forall\,\theta$$

In general a rotation is a lin. transformation defined by a matrix $T$ ($d \times d$ matrix) such that $T \cdot T^t = \mathbb{I}$ (we say $T$ orthogonal)

$$\lambda_d(TS) = \lambda_d(S) \qquad \forall S \in \mathcal{M}_d.$$

$T\vec{v} = c\vec{v}$ $\qquad \lambda_d(TS)$

$d=2$ $\quad c=2$ $\qquad \lambda_2(TS) = 2^2 \lambda_2(S)$

$c=3$ $\qquad 3^2$

$c>0$ $\qquad \lambda_2(TS) = |c|^2 \lambda_2(S)$

$$\lambda_d(TS) = |c|^d \lambda_d(S)$$

$T\vec{v} = (c_1 v_1, c_2 v_2, \dots, c_d v_d)$ $\qquad \lambda_d(TS) = |c_1||c_2|\dots|c_d|\,\lambda_d(S)$

$$= \begin{bmatrix} c_1 & 0 & \cdots & 0 \\ 0 & c_2 & 0 \cdots & 0 \\ & & \ddots & \\ & & & c_d \end{bmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_d \end{pmatrix} \qquad = |\det T|\,\lambda_d(S)$$

IN GEN: if $T$ is an invertible matrix

$$\lambda_d(TS + \vec{v}) = |\det T|\,\lambda_d(S) \qquad \forall S \in \mathcal{M}_d$$

$\forall T$ invertible mat

$\forall \vec{v} \in \mathbb{R}^d$

Rmk: If $T$ is orthogonal, ${}^tT\,T = \mathbb{I}$

$$\Downarrow$$

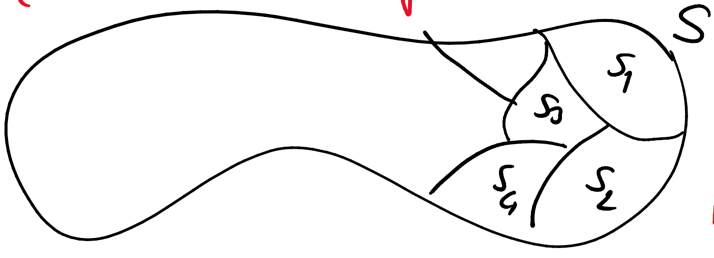
$$\det T \cdot \det {}^tT = \det T\,{}^tT = \det \mathbb{I} = 1$$

$$\det {}^tT = \det T \Rightarrow (\det T)^2 = 1 \Rightarrow |\det T| = 1$$

iv) (countable additivity) $\qquad S = \bigcup_{j=1}^{\infty} S_j$

S

iv) (countable additivity)

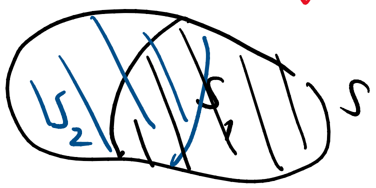



$$S = \bigcup_{j=1} S_j$$

$$\Downarrow$$

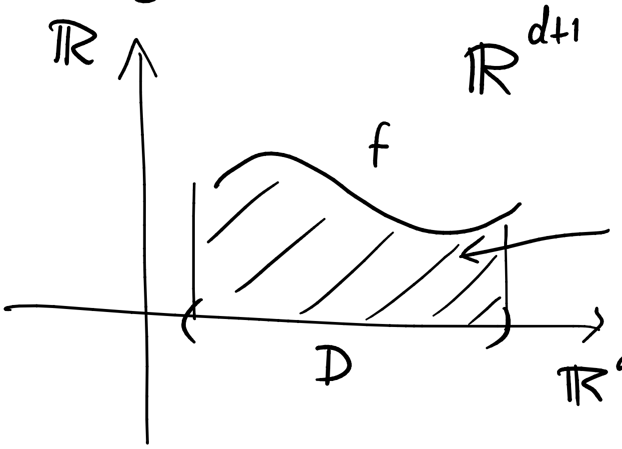
if $S_i \cap S_j = \phi \ \forall i \neq j$

then $\lambda_d(S) = \sum_{j=1}^{\infty} \lambda_d(S_j)$




Once the concept of measure is defined we can proceed to def **integrals**: we wish to set

$$(*) \quad \int_D f(\vec{x})\, d\vec{x} := \lambda_{d+1}(\underbrace{\mathrm{Trap}(f)}_{\subseteq \mathbb{R}^{d+1}})$$



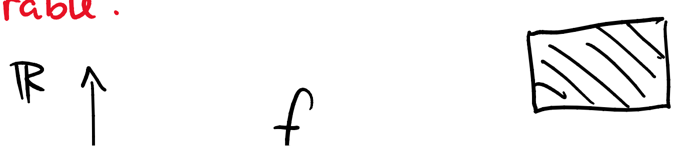

$\mathrm{Trap}(f) = \{(\vec{x}, y) \in \mathbb{R}^d \times \mathbb{R} \equiv \mathbb{R}^{d+1} \ \vec{x} \in D,\ 0 \leq y \leq f(\vec{x})\} \subset \mathbb{R}^{d+1}$

To give def $(*)$ we need to know if $\mathrm{Trap}(f)$ is meas.

Prop: Let $f \in \mathscr{C}(D)$ D closed or open.

$\Rightarrow$ $\mathrm{Trap}(f)$ is measurable.

Proof: D closed



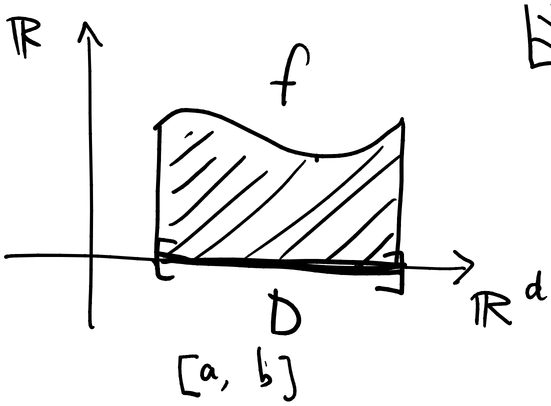
**Proof:** D closed

$\Downarrow$

Trap(f) is closed

$\Downarrow$

Trap(f) meas.




D open