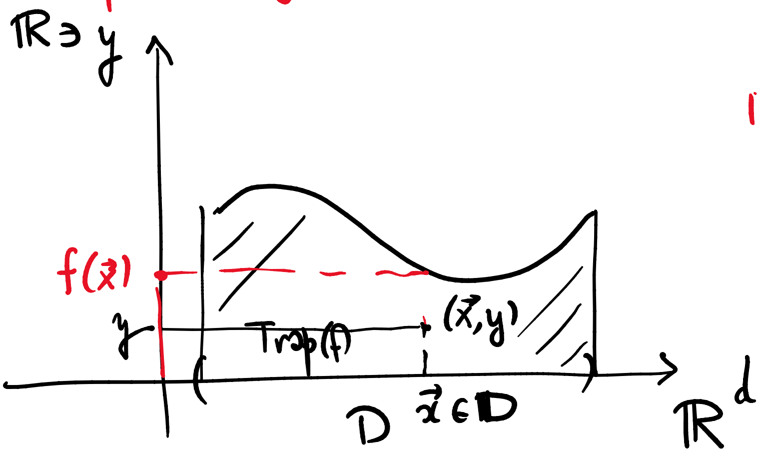
martedì 3 marzo 2020 11:06

Let $f : D \subset \mathbb{R}^d \to [0, +\infty[$.

**Thm:** If $f \in \mathcal{C}(D)$, $D$ is open or closed, then

$$\mathrm{Trap}(f) = \{ (\vec{x}, y) \in \mathbb{R}^{d+1} : \vec{x} \in D,\ 0 \leq y \leq f(\vec{x}) \}$$

is measurable in $\mathbb{R}^{d+1}$




and we pose

$$\int_D f \equiv \int_D f(\vec{x})\,dx \equiv \int_D f(x_1, \dots, x_d)\, dx_1 \dots dx_d$$

$$:= \lambda_{d+1}(\mathrm{Trap}(f))$$

**Rmk:** $f \in \mathcal{C}(D)$ $D$ closed described as

$$D = \{ \vec{x} \in \mathbb{R}^d : g_1(\vec{x}) \geq 0,\ g_2(\vec{x}) \geq 0, \dots, g_m(\vec{x}) \geq 0 \}$$

where $g_1 \dots g_m \in \mathcal{C}(\mathbb{R}^d)$

Then

$$\mathrm{Trap}(f) = \{ (\vec{x}, y) \in \mathbb{R}^{d+1} : \overset{\vec{x} \in D}{\overbrace{g_1(\vec{x}) \geq 0,\ g_2(\vec{x}) \geq 0, \dots, g_m(\vec{x}) \geq 0}},\ \ \in \mathcal{C}$$

$\mathrm{Trap}(f) = \{ (x, y) \in \mathbb{R} \quad : \quad g_1(x, y, 0), \; g_2(x, y, 0), \ldots$

$$0 \leqslant y \leqslant f(\vec{x}) \}$$

$\Rightarrow$ Trap(f) is defd through large inequalities involving cont functs $\Rightarrow$ Trap(f) is closed

$\Downarrow$

Trap(f) is meas. $\blacksquare$

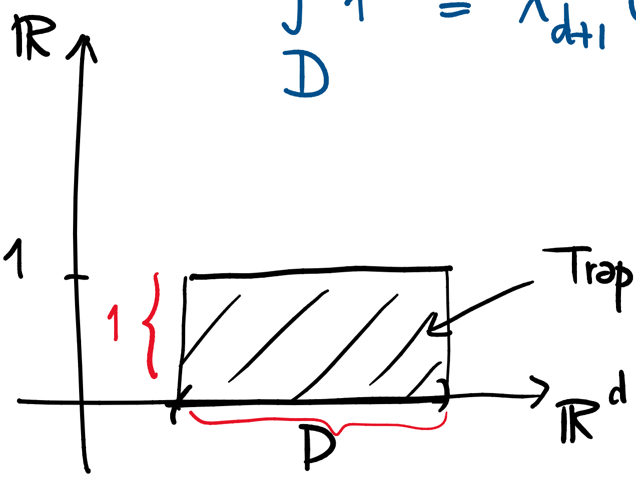
Rmk: If $D$ is open or closed, taking

$$f \equiv 1 \quad \text{on } D$$

$$\mathbb{1}_D(x) = \begin{cases} 1 & x \in D \\ 0 & x \notin D \end{cases} \quad \text{(unit funct of } D\text{)}$$

then

$$\int_D 1 = \lambda_{d+1}(\mathrm{Trap}(\mathbb{1}_D)) = \lambda_{d+1}(D \times [0,1])$$




$$\mathrm{Trap}(\mathbb{1}_D) = \{ (\vec{x}, y) \in \mathbb{R}^{d+1} : \vec{x} \in D, \; 0 \leqslant y \leqslant 1 \}$$

$$= D \times [0,1]$$

$$\overset{\text{fact}}{=} \lambda_d(D) \cdot \underbrace{\lambda_1([0,1])}_{=1}$$

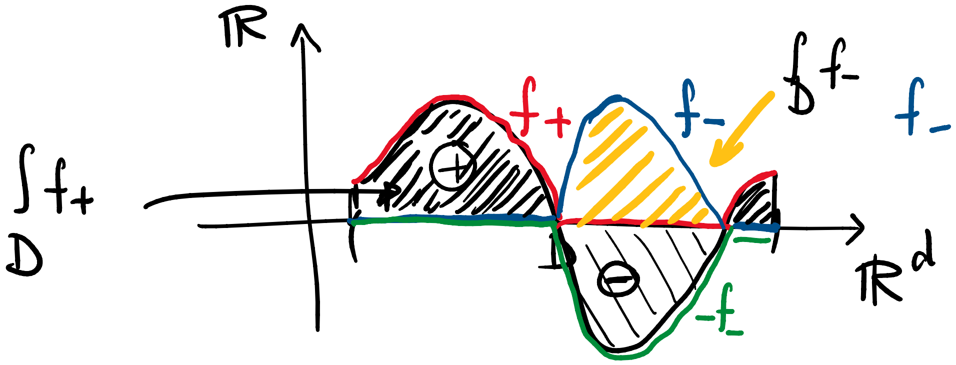
$$= \lambda_d(D)$$

$$\Rightarrow \qquad \lambda_d(D) = \int_D 1 \, d\vec{x}.$$

Let's pass now to the integral of

$$f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$$

$f \in \mathcal{C}(D)$, $D \subset \mathbb{R}^d$ closed or open.




$$f_+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{if } f(x) < 0 \end{cases} \qquad \geq 0 \quad \text{always}$$

$$f_-(x) := \begin{cases} -f(x) & \text{if } f(x) \leq 0 \\ 0 & \text{if } f(x) > 0 \end{cases} \qquad \geq 0 \quad \text{"}$$

We have:

- $f_+, f_- : D \longrightarrow [0, +\infty[$
- $f_+ + f_- = |f|$, $\quad f_+ - f_- = f$
- $f_+, f_- \in \mathcal{C}(D)$ if $f \in \mathcal{C}(D)$

For $f_+, f_-$ it is well def'd

$$\underbrace{\int_D f_+}_{\in [0,+\infty]} , \quad \underbrace{\int_D f_-}_{\in [0,+\infty]}$$

Idea:

$$\int_D f := \int_D f_+ - \int_D f_-$$

To define this we need

$$\int_D f_+ , \int_D f_- < +\infty \quad (\text{not } = +\infty)$$

$$\int_D |f| = \int_D f_+ + \int_D f_- < +\infty$$

<u>Def</u>: Let $f \in \mathcal{C}(D)$, $D$ open or closed. We say that <u>$f$ is integrable on $D$</u> (notation $f \in L^1(D)$) if

$$\int_D |f| < +\infty.$$

In this case we pose $\int_D f := \int_D f_+ - \int_D f_-$.

<u>Properties:</u>

Properties:

1. (linearity), $f, g \in L^1(D) \Rightarrow \alpha f + \beta g \in L^1(D)$ $\forall \alpha, \beta \in \mathbb{R}$

and $$\int_D (\alpha f + \beta g) = \alpha \int_D f + \beta \int_D g$$

2. (isotonicity). if $f, g \in L^1$ $f \leq g$ on $D$ $\left( f(\vec{x}) \leq g(\vec{x}) \; \forall \vec{x} \in D \right)$

$$\Rightarrow \int_D f \leq \int_D g$$

3. (triang. inequality). $f \in L^1(D)$

$$\left| \int_D f \right| \leq \int_D |f|$$

4. (decomposition) $f \in L^1(D)$, $D = E \cup F$, $E \cap F = \emptyset$ (or, less, $\lambda_d(E \cap F) = 0$)

$$\int_D f = \int_E f + \int_F f.$$

How do we compute multiple integrals?

There're two main techniques of Calculus:

- Reduction Formula

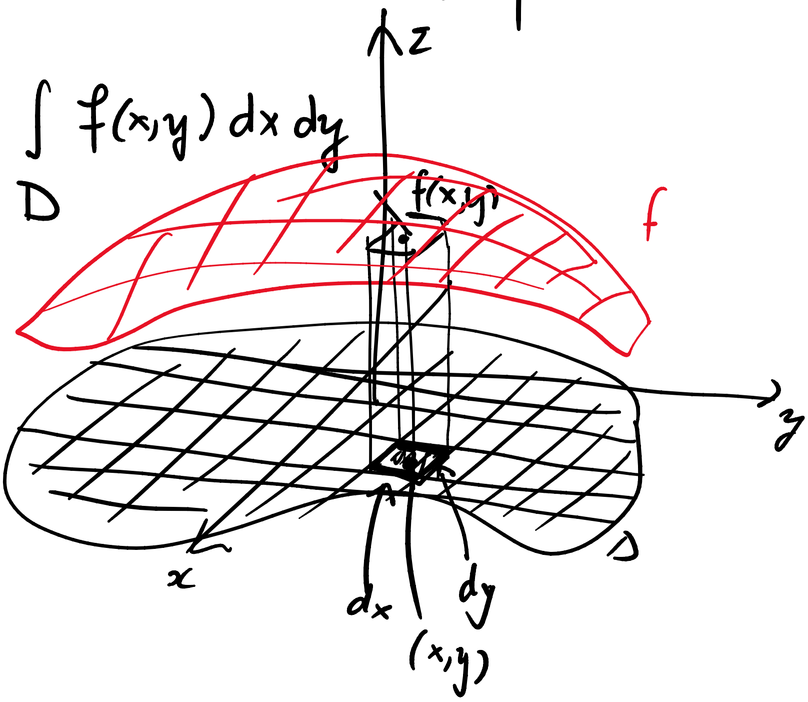
• Change of Var Formula

## Reduction Formula

Let's consider a function $f = f(x,y) : D \subset \mathbb{R}^2 \to \mathbb{R}$, $f \in L^1(D)$. We want to compute

$$\int_D f(x,y)\,dx\,dy$$

Intuitively




$$\int_D f(x,y)\,dx\,dy = \sum_{(x,y)\in D} f(x,y)\,dx\,dy$$

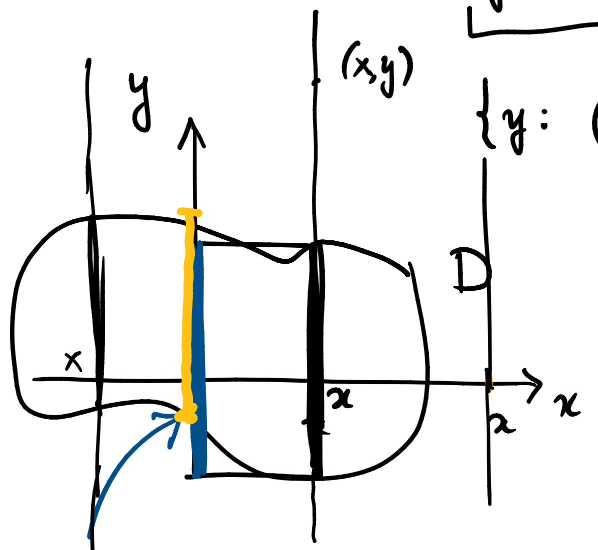
$$= \sum_x \Big( \sum_{y\,:\,(x,y)\in D} f(x,y)\,dx\,dy \Big)$$

$$= \sum_x \Big( \underbrace{\sum_{y\,:\,(x,y)\in D} f(x,y)\,dy}_{} \Big)\,dx$$

$$\parallel$$

$$\int_{y\,:\,(x,y)\in D} f(x,y)\,dy = \int_{D_x} f(x,y)\,dy$$

$\underbrace{y : (x,y) \in D}$ $\qquad D_x$




$\{y : (x,y) \in D\}$

$\{y : (x,y) \in D\} = D_x \qquad x$ fixed

$$\Rightarrow \int_D f(x,y)\,dx\,dy = \sum_x \left( \int_{D_x} f(x,y)\,dy \right) dx$$

$$\sum_x \boxed{\quad x \quad}\ dx$$

$$= \int_{\mathbb{R}} \left( \int_{D_x} f(x,y)\,dy \right) dx$$

$$= \int_{x:\, D_x \neq \phi} \left( \int_{D_x} f(x,y)\,dy \right) dx$$

<u>Thm:</u> If $\boxed{f \in L^1(D)}$ then

$$\int_D f(x,y)\,dx\,dy = \int_{\mathbb{R}} \left( \int_{D_x} f(x,y)\,dy \right) dx$$

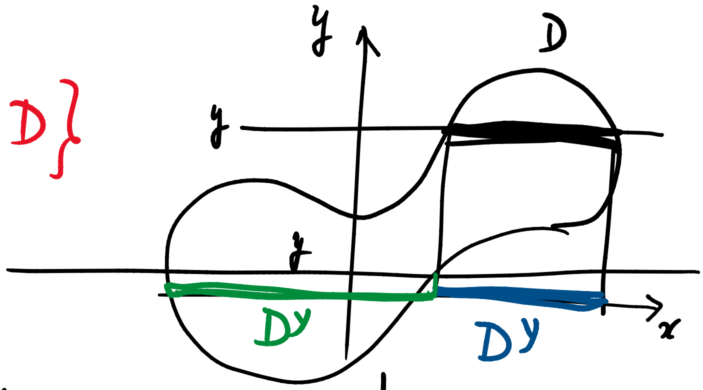
(reduction - formula)

where $D_x = \{ y \in \mathbb{R} : (x,y) \in D \}$ is called $x$-section of $D$.

Moreover

$$\int_D f(x,y)\,dx\,dy = \int_{\mathbb{R}} \left( \int_{D^y} f(x,y)\,dx \right) dy$$

$$D^y = \{ x \in \mathbb{R} : (x,y) \in D \}$$




To apply the Red Formula we need to know

$$f \in L^1(D)$$

$$\Updownarrow$$

$$\int_D |f(x,y)|\,dx\,dy < +\infty .$$

If it were possible to apply the RF

$$+\infty > \int_D |f(x,y)|\,dx\,dy = \int_{\mathbb{R}} \left( \int_{D_x} |f(x,y)|\,dy \right) dx$$

$$= \int_{\mathbb{R}} \left( \int_{D^y} |f(x,y)|\,dx \right) dy$$

finite

A viceversa holds

Thm: (Fubini - Tonelli)

$f \in \mathcal{C}(D)$, $D$ closed / open s.t. one of the

following integrals be finite

following integrals be finite

$$\int_{\mathbb{R}} \left( \int_{D_x} |f(x,y)|\, dy \right) dx , \qquad \int_{\mathbb{R}} \left( \int_{D^y} |f(x,y)|\, dx \right) dy$$

Then $f \in L^1(D)$ and reduction formula applies to $f$.

<u>Rmk</u>: In particular, if $f \in \mathcal{C}(D)$, $D$ closed/open

$f \geq 0 \Rightarrow |f| = f$

$$\int_{\mathbb{R}} \left( \int_{D_x} |f(x,y)|\, dy \right) dx = \int_{\mathbb{R}} \left( \int_{D_x} f(x,y)\, dy \right) dx$$

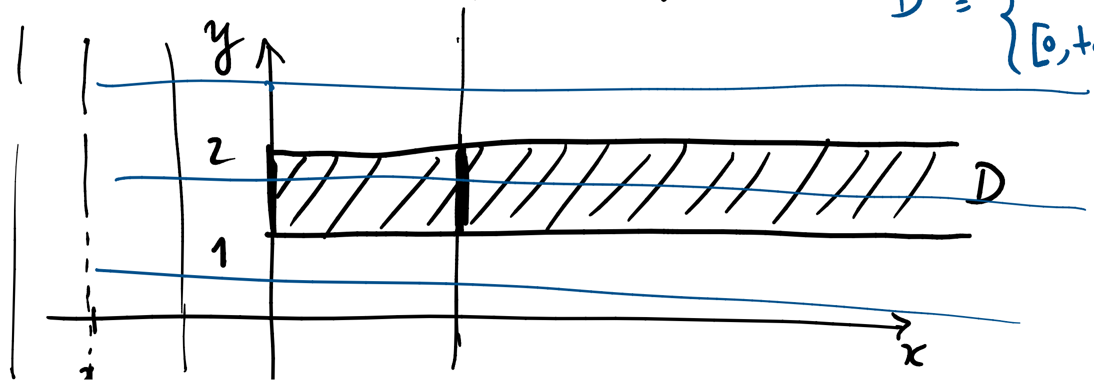
once this is finite, by red form this $= \int_D f(x,y)\, dx\, dy$

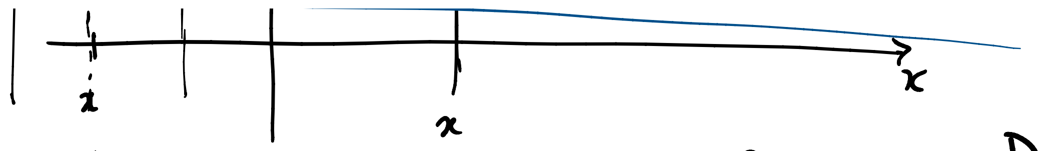
<u>Example 4.2.4</u>

Discuss if $f(x,y) = x^3 e^{-yx^2} \in L^1(\underbrace{[0,+\infty[ \times [1,2]}_{D})$ and compute $\int_D f$.

<u>Sol</u>: $D$ is closed, $f \in \mathcal{C}(D)$

$$D^y = \begin{cases} \emptyset & y \notin [1,2] \\ [0,+\infty[ & y \in [1,2] \end{cases}$$






Because on $D$ $x \geq 0 \Rightarrow f \geq 0$ on $D$.

To check integrability we apply F-T thm checking one of the following iterated integrals

$$\int_{\mathbb{R}}\left(\int_{D_x} |f(x,y)|\,dy\right)dx \underset{f\geq 0}{=} \int_{\mathbb{R}}\left(\int_{D_x} f(x,y)\,dy\right)dx$$

$$= \int_{\mathbb{R}}\left(\int_{D_x} x^3 e^{-yx^2}\,dy\right)dx$$

$$D_x = \begin{cases} \emptyset & x < 0 \\ [1,2] & x \geq 0 \end{cases}$$

$$= \underbrace{\int_{-\infty}^{0}\Big(\underbrace{\int_{\emptyset} x^3 e^{-yx^2}\,dy}_{0}\Big)dx}_{=0} + \int_{0}^{+\infty}\left(\int_{1}^{2} x^3 e^{-yx^2}\,dy\right)dx$$

$$= \int_{0}^{+\infty}\left(x^3 \int_{1}^{2} e^{-yx^2}\,dy\right)dx \overset{(*)}{=}$$

$$\int_{1}^{2} e^{-yx^2}\,dy = \left[\frac{e^{-yx^2}}{-x^2}\right]_{y=1}^{y=2} \qquad e^{\alpha y} = \partial_y\left(\frac{e^{\alpha y}}{\alpha}\right)$$

$y=1$

$\alpha = -x^2$

$$= -\frac{1}{x^2}\left[e^{-2x^2} - e^{-x^2}\right]$$

$$\overset{(*)}{=} \int_0^{+\infty} x^3 \left(-\frac{1}{x^2}\left[e^{-2x^2} - e^{-x^2}\right]\right) dx$$

$$= + \int_0^{+\infty} -x\, e^{-2x^2} \mp (-x) e^{-x^2} \quad dx$$

$$\partial_x\left(\frac{e^{-2x^2}}{4}\right) - \partial_x \frac{e^{-x^2}}{2}$$

$$= \left[\frac{e^{-2x^2}}{4}\right]_0^{+\infty} - \left[\frac{e^{-x^2}}{2}\right]_0^{+\infty} = -\frac{1}{4} + \frac{1}{2} = \frac{1}{4}$$

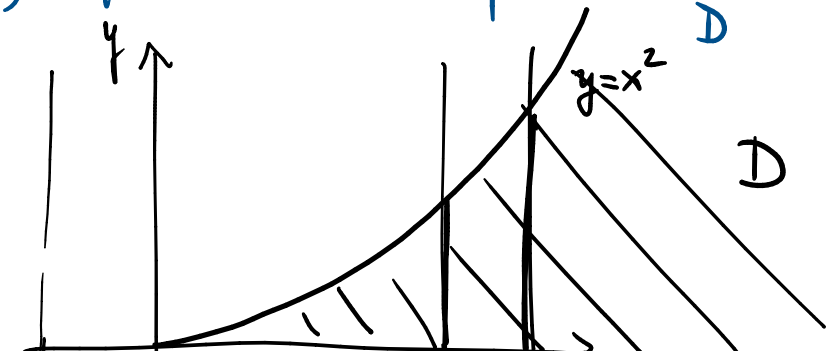
$\Rightarrow f \in L^1$ and because $f \geq 0$ $\int_D f = \frac{1}{4}$. ■

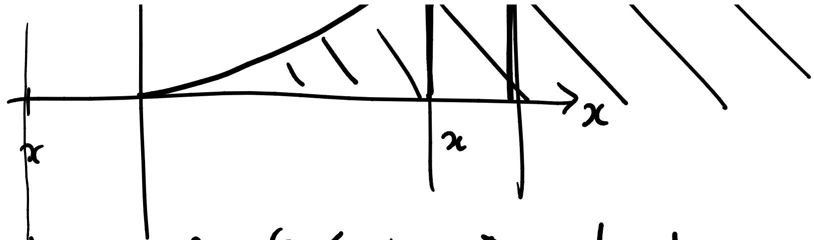
**Ex. 4.2.5** $f(x,y) = e^{-x} \in L^1(D)$

$$D = \{(x,y) \in \mathbb{R}^2 : \underline{x \geq 0}, \underline{0 \leq y \leq x^2}\}$$

and, if it $\exists$, compute $\int_D f$.

**Sol:**






Clearly $f \in \mathcal{C}(D)$, $D$ closed. and $f \geq 0$.

To check $f \in L^1(D)$ we apply F-T thm

$$\int_{\mathbb{R}} \left( \int_{D_x} |f| \, dy \right) dx = \int_{\mathbb{R}} \left( \int_{D_x} f \, dy \right) dx$$

$$D_x = \begin{cases} \emptyset & x < 0 \\ [0, x^2] & x \geq 0 \end{cases}$$

$$= \int_0^{+\infty} \left( \int_0^{x^2} e^{-x} \, dy \right) dx$$

$$= \int_0^{+\infty} \left( e^{-x} \underbrace{\int_0^{x^2} 1 \, dy}_{= x^2} \right) dx$$

$$= \int_0^{+\infty} x^2 \underbrace{e^{-x}}_{= \partial_x(-e^{-x})} dx = \left[ \underbrace{-x^2 e^{-x} \Big|_0^{+\infty}}_{= 0} - \int_0^{+\infty} 2x(-e^{-x}) \, dx \right]$$

$$= 2 \int_0^{+\infty} x \underbrace{e^{-x}}_{= \partial_x(-e^{-x})} dx$$

$$= 2 \left[ \underset{\|}{-x e^{-x} \Big|_0^{+\infty}} - \int_0^{+\infty} 1 \cdot (-e^{-x}) \, dx \right]$$

$$= 2\int_0^{+\infty} e^{-x}\,dx = 2\left[-e^{-x}\right]_0^{+\infty}$$

$$= 2\cdot 1 = 2 < +\infty \Rightarrow f \in L^1(D) \text{ and,}$$

because $f \geqslant 0$

$$\int_D f \overset{RF}{=} \int_{\mathbb{R}} \left( \int_{D_x} f\,dy \right) dx = 2. \qquad \blacksquare$$

## Example 4.2.7

**Warning:** It might happens that

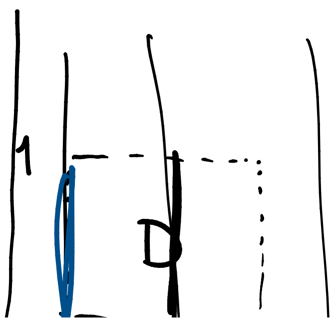
$$\int_{\mathbb{R}} \left( \int_{D_x} f\,dy \right) dx \in \mathbb{R}, \qquad \int_{\mathbb{R}} \left( \int_{D^y} f\,dx \right) dy \in \mathbb{R}$$

**but** $f \notin L^1$

Take this

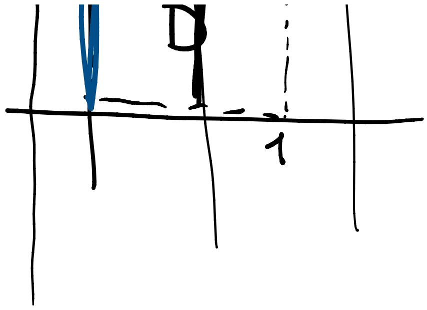
$$f(x,y) = \frac{x-y}{(x+y)^3} \qquad D = ]0,1[^2$$

and compute the two iterated integrals. What can be concluded on $f$?




$D$ is open $f \in \mathcal{C}(D)$

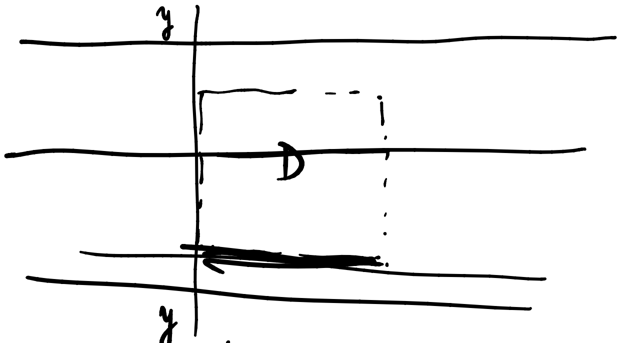
$$\int_{\mathbb{R}} \left( \int_{D} f(x,y)\,dy \right) dx$$



$$\int_{\mathbb{R}} \left( \int_{D_x} f(x,y)\, dy \right) dx$$

$$D_x = \begin{cases} \emptyset & x \notin [0,1] \\ [0,1] & x \in [0,1] \end{cases}$$

$$= \int_0^1 \left( \int_0^1 \frac{x-y}{(x+y)^3}\, dy \right) dx$$



$$\int_{\mathbb{R}} \left( \int_{D^y} f(x,y)\, dx \right) dy$$

$$= \int_0^1 \left( \int_0^1 \frac{x-y}{(x+y)^3}\, dx \right) dy$$

$$D^y = \begin{cases} \emptyset & y \notin [0,1] \\ [0,1] & y \in [0,1] \end{cases}$$

this is $-$

$$\int_0^1 \left( \int_0^1 \frac{\overset{y}{S} - \overset{x}{T}}{(S+T)^3}\, dS \right) dT$$

$$\int_0^1 \int_0^1 \frac{y-x}{(y+x)^3}\, dy\, dx$$

Let's compute the first int

$$\int_0^1 \left( \int_0^1 \frac{x-y}{(x+y)^3}\, dy \right) dx \qquad (x+y)^{-3} = \partial_y \left( \frac{(x+y)^{-2}}{-2} \right)$$

$$\int_0^1 \frac{x-y}{(x+y)^3}\, dy = x \int_0^1 \frac{\cancel{2x}}{(x+y)^3}\, dy - \int_0^1 \frac{y+x-x}{(x+y)^3}\, dy \qquad (x+y)^{-1}$$

$$= \cancel{2}x \left[ -\frac{1}{\cancel{2}} \frac{1}{(x+y)^{\cancel{2}}} \right]_{y=0}^{y=1} \left| \partial_y \left( -\frac{(x+y)^{-2}}{2} \right) \right| + \int_0^1 \frac{1}{(x+y)^2}\, dy \qquad = \partial_y \frac{1}{x+y}$$

$$\ldots \left[ \ldots \right]_{y=} + \int_0 \frac{1}{(x+y)^2}\,dy$$

$$+\, x \cancel{\int_0^1 \frac{1}{(x+y)^3}\,dy}$$

$$= -x\left[\frac{1}{(x+1)^2} - \frac{1}{x^2}\right] + \left[\frac{1}{x+y}\right]_{y=0}^{y=1}$$

$$= -\frac{x}{(x+1)^2} + \cancel{\frac{1}{x}} + \frac{1}{x+1} - \cancel{\frac{1}{x}}$$

$$\Rightarrow \int_0^1\left(\int_0^1 \frac{x-y}{(x+y)^3}\,dy\right)dx = \int_0^1 \frac{1}{x+1} - \frac{x+1-1}{(x+1)^2}\,dx$$

$$= \int_0^1 \cancel{\frac{1}{x+1}} - \cancel{\frac{1}{x+1}} + \frac{1}{(x+1)^2}\,dx$$

$$= \int_0^1 \frac{1}{(x+1)^2}\,dx = \left[-\frac{1}{x+1}\right]_{x=0}^{x=1} = -\frac{1}{2} + 1 = \frac{1}{2}$$

$$\Rightarrow \int_{\mathbb{R}}\left(\int_{D^y} f\,dx\right)dy = -\frac{1}{2}$$

Conclusion: $f \notin L^1$ (because if $f \in L^1$ RF says the two iterated integrals must coincide!) ☒

Do Ex. 4.5.1

Ex 4.5.3 #1, #2