mercoledì 4 marzo 2020 08:18

Ex 4.5.1

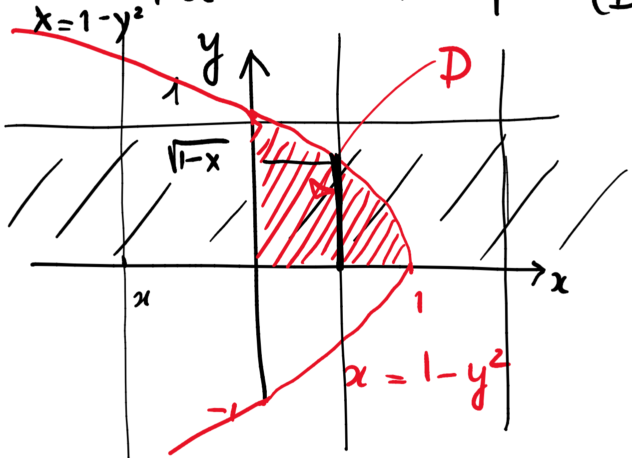
Compute

#1 $$\int_{\boxed{0\le y\le 1,\ 0\le x\le 1-y^2}} x e^{y}\, dx\, dy .$$

<u>Sol</u>: Here $f(x,y) = x e^{y} \in \mathcal{C}(\mathbb{R}^2)$

$$D = \{ (x,y)\in\mathbb{R}^2 : 0\le y\le 1,\ 0\le x\le 1-y^2 \}$$

D is closed. and $f\in\mathcal{C}(D)$.



$0\le y\le 1$

$$D_x = \begin{cases} \emptyset & x\notin[0,1] \\ [0,\sqrt{1-x}\,] & x\in[0,1] \end{cases}$$

$$x = 1-y^2 \Leftrightarrow y^2 = 1-x \Longleftrightarrow y = \pm\sqrt{1-x}$$

We have to check first if $f\in L^1(D) \Leftrightarrow \int_D |f| < +\infty$

On D, $f \geq 0 \Rightarrow |f| = f$. Thus to check $f\in L^1(D)$

we may apply F-T thm checking if

$$+\infty > \int_{\mathbb{R}} \left(\int_{D_x} |f|\, dy \right) dx = \int_{\mathbb{R}} \int_{D_x} f\, dy\, dx$$

(so once this is achieved we have automatically also

The value of $\int_D f \overset{RF}{=} \int_{\mathbb{R}} \left(\int_{D_x} f \, dy \right) dx.$)

$$\Rightarrow \int_{\mathbb{R}} \int_{D_x} f \, dy \, dx = \int_0^1 \int_0^{\sqrt{1-x}} x e^y \, dy \, dx$$

$$= \int_0^1 x \underbrace{\int_0^{\sqrt{1-x}} e^y \, dy}_{= \left[e^y \right]_{y=0}^{y=\sqrt{1-x}}} dx$$

$$= \int_0^1 x \left(e^{\sqrt{1-x}} - e^0 \right) dx$$

$$= \int_0^1 x \left(e^{\sqrt{1-x}} - 1 \right) dx.$$

$$= \int_0^1 x e^{\sqrt{1-x}} \, dx - \underbrace{\int_0^1 x \, dx}_{= \left[\frac{x^2}{2} \right]_0^1 = 1/2}$$

$$\int x e^{\sqrt{1-x}} \, dx \underset{\substack{u = \sqrt{1-x} \\ x = 1-u^2 \\ dx = -2u\,du}}{=} \int (1-u^2) e^u (-2u) \, du$$

$$= -2 \int (u - u^3) \overbrace{e^u}^{(e^u)'} du$$

$$= -2 \left[(u-u^3) e^u - \int (1 - 3u^2) \overbrace{e^u}^{(e^u)'} du \right]$$

$(e^u)'$

$$= -2\left[(u-u^3)e^u - \int (1-3u^2)e^u\,du\right]$$

$$= -2\left[(u-u^3)e^u - \left[(1-3u^2)e^u + \int 6u\,\underset{}{\overset{(e^u)'}{\overset{\shortparallel}{e^u}}}\,du\right]\right]$$

$$= -2\left[e^u(-u^3+3u^2+u-1) - 6\left[ue^u - \int \underset{e^u}{\underset{\shortparallel}{e^u\,du}}\right]\right]$$

$$= -2\,e^u\left(-u^3+3u^2+u-1-6u+6\right)$$

$$= 2e^u\left(u^3-3u^2+5u-5\right).$$

$$= 2e^{\sqrt{1-x}}\left((1-x)^{3/2} - 3(1-x) + 5(1-x)^{1/2} - 5\right)$$

$$\Rightarrow \int_0^1 x e^{\sqrt{1-x}}\,dx = \Big[\qquad // \qquad \Big]_0^1$$

$$= 2(-5) - 2e(-2)$$

$$= -10 + 4e$$

$$\Rightarrow \int_{\mathbb{R}} \int_{D_x} f\,dy\,dx = -10 + 4e - \tfrac{1}{2}$$

$$\Rightarrow f \in L^1 \quad \text{and} \quad \int_D f = \qquad \blacksquare$$

Rmk: In this case it would have been more convenient to use the original def of D because it gives automatically the y-sect:

$$D = \{(x,y) : 0 \leq y \leq 1,\ 0 \leq x \leq 1-y^2\}$$

$$\Rightarrow \qquad D^y = \begin{cases} \emptyset & y \notin [0,1] \\ [0, 1-y^2] & y \in [0,1] \end{cases}$$

$$\int_{\mathbb{R}} \int_{D^y} f \, dx \, dy = \int_0^1 \left(\int_0^{1-y^2} x e^y \, dx \right) dy$$

$$= \int_0^1 e^y \int_0^{1-y^2} x \, dx \, dy$$

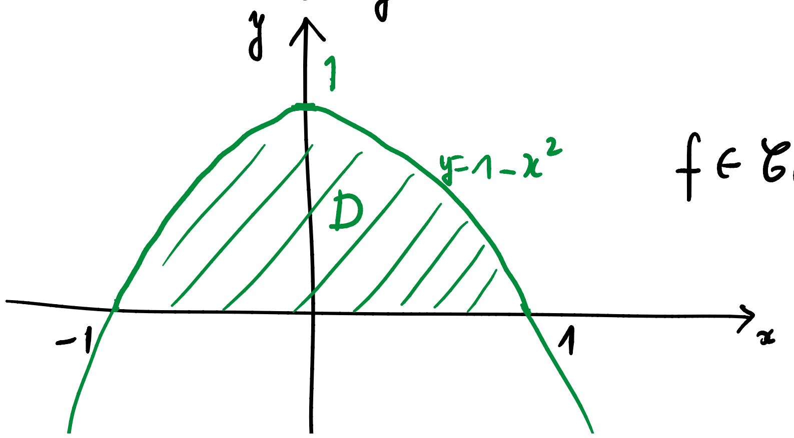
$$\int_0^{1-y^2} x \, dx = \frac{x^2}{2} \Big|_0^{1-y^2}$$

$$= \int_0^1 e^y \left(\frac{(1-y^2)^2}{2} - 0 \right) dy$$

$$= \frac{1}{2} \int_0^1 e^y (1-y^2)^2 \, dy \overset{\text{you}}{=} \text{finish} \ldots \qquad \blacksquare$$

#2. $$\int_{\boxed{0 \leq y \leq 1-x^2}} \frac{x}{2+y} \, dx \, dy$$

Here $f = f(x,y) = \dfrac{x}{2+y} \in \mathcal{C}(\mathbb{R}^2 \setminus \{y=-2\})$



$f \in \mathcal{C}(D)$

To check if $f \in L^1(D)$ we have to check if $\int_D |f| < +\infty$. It is convenient to use FT in form

$$\int_{\mathbb{R}} \left(\int_{D_x} |f| \, dy \right) dx = \int_{-1}^{1} \int_{0}^{1-x^2} \frac{|x|}{2+y} \, dy \, dx$$

$$0 \leq y \leq 1 - x^2 \implies \boxed{1 - x^2 \geq 0} \iff x^2 \leq 1 \iff |x| \leq 1$$

$$= \int_{-1}^{1} |x| \int_0^{1-x^2} \frac{1}{2+y} \, dy \, dx$$

$$\| $$

$$\left[\log|2+y| \right]_{y=0}^{y=1-x^2}$$

$$= \int_{-1}^{1} \underbrace{|x| \left(\log(3-x^2) - \log 2 \right)}_{\text{even funct of } x.} dx$$

$$= 2 \int_0^1 x \left(\log(3-x^2) - \log 2 \right) dx$$

$$= -2 \log 2 \int_0^1 x \, dx + 2 \int_0^1 x \log(3-x^2) \, dx$$

$$x = \left(\frac{x^2}{2} \right)'$$

$$\| \qquad\qquad \|$$

$$x^2 \, | \, 1$$

$$\begin{array}{ccc} \| & & \| \\ \frac{x^2}{2}\Big|_0^1 & & \frac{x^2}{2}\log(3-x^2)\Big|_0^1 - \int_0^1 \frac{x^2}{\cancel{2}}\,\frac{-\cancel{2}x}{3-x^2}\,dx \\ 1/2 & & \end{array}$$

$$= -\log 2 + \frac{1}{2}\log 2 + \int_0^1 \frac{x^3}{3-x^2}\,dx.$$

you finish.

$\vdots$

$\Rightarrow f \in L^1(D)$. Now because f is not const sign we should repeat the RF for f

$$\Rightarrow \int_D f \overset{RF}{=} \int_{\mathbb{R}} \left(\int_{D_x} f\,dy \right) dx = \int_{-1}^{1} \left(\int_0^{1-x^2} \frac{x}{2+y}\,dy \right) dx$$

$$= \int_{-1}^{1} x \int_0^{1-x^2} \frac{1}{2+y}\,dy\,dx$$

$$\log|2+y|\Big|_0^{1-x^2}$$

$$= \int_{-1}^{1} \underbrace{x\left(\log(3-x^2) - \log 2\right)}_{\text{odd}} dx = 0. \qquad \blacksquare$$

Do # 3

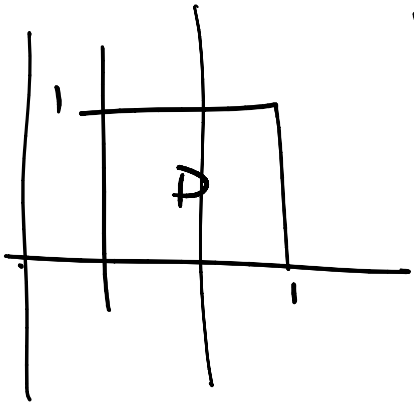
Ex 4.5.2.

Compute

#1 $$\int_{[0,1]^2} e^{\max\{x^2,y^2\}}\, dx\, dy .$$

Here $f(x,y) = e^{\max\{x^2,y^2\}} \in \mathcal{C}(\mathbb{R}^2)$. To compute and check at once $f \in L^1$

$\vee_0$



we apply FT

$$\int_{\mathbb{R}} \int_{D_x} |f|\, dy\, dx = \int_{\mathbb{R}} \int_{D_x} f\, dy\, dx$$

$$= \int_0^1 \int_0^1 e^{\max\{x^2,y^2\}}\, dy\, dx \overset{(*)}{=}$$

$$\max\{x^2,y^2\} = \begin{cases} x^2 & \text{if } x^2 \geq y^2 \overset{x,y\in[0,1]}{\iff} \boxed{x \geq y} \\ y^2 & \text{if } x^2 < y^2 \iff x < y \end{cases}$$

$$\int_0^1 e^{\max\{x^2,y^2\}}\, dy = \int_0^x e^{x^2}\, dy + \int_x^1 e^{y^2}\, dy$$

$$= e^{x^2} \underbrace{\int_0^x dy}_{= x} + \int_x^1 e^{y^2}\, dy$$

$$= x e^{x^2} + \int_x^1 e^{y^2}\, dy$$

$$= xe^{x^2} + \int_x e^{y^2}\, dy$$

$$\Rightarrow \overset{(*)}{=} \int_0^1 \left(xe^{x^2} + \int_x^1 e^{y^2}\, dy \right) dx$$

$$= \int_0^1 \underset{\partial_x\left(\frac{e^{x^2}}{2}\right)}{\underset{\parallel}{x e^{x^2}}}\, dx + \int_0^1 \underset{(x)'}{\underset{\parallel}{1}} \cdot \left(\int_x^1 e^{y^2}\, dy \right) d\underline{x}$$

$$= \left[\frac{e^{x^2}}{2} \right]_0^1 + \left[x \overset{0}{\overset{\parallel}{\int_x^1 e^{y^2}\, dy}} \right]_{x=0}^{x=1} - \int_0^1 x\left(-e^{x^2}\right) dx$$

$$\partial_x \left(\int_x^1 e^{y^2}\, dy \right) = -\partial_x \left(+ \underbrace{\int_1^x e^{y^2}\, dy}_{\text{integral function}} \right) \overset{\text{F.T.I.C.}}{=} -e^{x^2}$$

$$= \frac{1}{2}(e-1) + 0 + \int_0^1 \underset{\left[\frac{e^{x^2}}{2}\right]_0^1 = \frac{1}{2}(e-1)}{\underset{\parallel}{x e^{x^2}\, dx}}$$

$$= e - 1. \quad \blacksquare$$

#3 $\displaystyle\int_{[0,+\infty[\times[1,+\infty[} e^{-xy^4}\,dx\,dy$

Sol: Here $f(x,y) = e^{-xy^4} \in \mathcal{C}(\mathbb{R}^2)$,

$D = [0,+\infty[\times [1,+\infty[$

Clearly D is closed, $f \geq 0$ so to check if $f \in L^1(D)$ we need to check if (F.T.)

$$\int_{\mathbb{R}} \Big(\int_{D_x} \underset{\overset{\shortparallel}{f}}{|f|}\, dy \Big)\, dx < +\infty \quad \text{or} \quad \int_{\mathbb{R}} \Big(\int_{D^y} \underset{\overset{\shortparallel}{f}}{|f|}\, dx \Big)\, dy < +\infty$$

$$\int_{\mathbb{R}} \int_{D_x} f \, dy . dx = \int_0^{+\infty} \underbrace{\Big(\int_1^{+\infty} e^{-xy^4} dy \Big)}_{\text{impossible}} dx$$

impossible task

$$\int_{\mathbb{R}} \int_{D^y} f \, dx\, dy = \int_1^{+\infty} \Big(\int_0^{+\infty} e^{-xy^4} dx \Big) dy$$

$$\shortparallel$$

$$\partial_x \Big(\frac{e^{-xy^4}}{-y^4} \Big)$$

$$\int^{+\infty} \Big[\frac{1}{\ } - xy^4 \Big]^{x=+\infty}$$

$$= \int_1^{+\infty} \left[-\frac{1}{y^4} e^{-xy^4} \right]_{x=0}^{x=+\infty}$$

$$= \int_1^{+\infty} -\frac{1}{y^4} \left[e^{-xy^4} \right]_{x=0}^{x=+\infty} dy$$

$$0 - 1$$

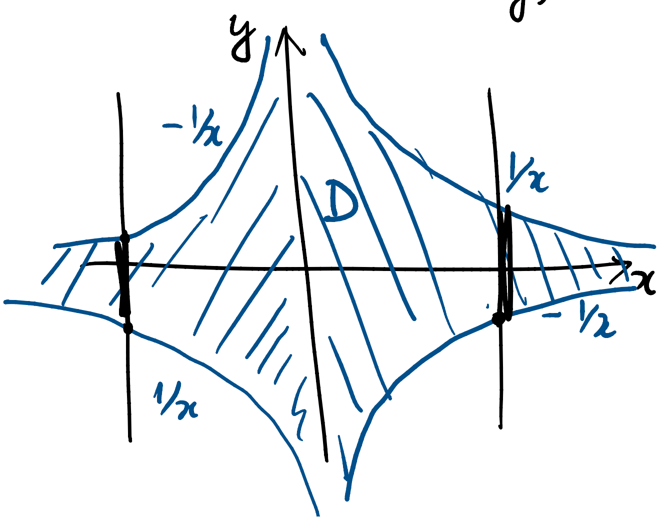
$$= \int_1^{+\infty} -\frac{1}{y^4} (-1)\, dy = \int_1^{+\infty} \frac{1}{y^4}\, dy = \left[+ \frac{y^{-3}}{-3} \right]_1^{+\infty}$$

$$= +\frac{1}{3}. \quad \blacksquare$$

Ex. 4.5.3

#2 $$\int_D \frac{x^2 e^{-x^2}}{1+(xy)^2}\, dx\, dy \qquad D = \{(x,y) \in \mathbb{R}^2 : |xy| \leq 1\}$$

Sol: $$f(x,y) := \frac{x^2 e^{-x^2}}{1+(xy)^2} \in \mathcal{C}(\mathbb{R}^2)$$



$$-1 \leq xy \leq 1$$

$$\Updownarrow \quad x > 0$$

$$-\frac{1}{x} \leq y \leq 1/x$$

$$x < 0 \qquad 1/x \leq y \leq -1/x$$

$$x<0 \quad \frac{1}{x} < y < -\frac{1}{x}$$

To check $f \in L^1(D)$ we may apply FT thm

and check $\displaystyle\int_{\mathbb{R}} \int_{D_x} \overset{f}{\overset{=}{|f|}}\, dy\, dx < +\infty$ or $\displaystyle\int_{\mathbb{R}} \int_{D^y} \overset{f}{\overset{=}{|f|}}\, dx\, dy < +\infty$

$$\int_{\mathbb{R}} \int_{D_x} f \, dy\, dx = \int_{-\infty}^{0} \left(\int_{1/x}^{-1/x} \frac{x^2 e^{-x^2}}{1+(xy)^2}\, dy \right) dx + \int_{0}^{+\infty} \left(\int_{-1/x}^{1/x} \frac{x^2 e^{-x^2}}{1+(xy)^2}\, dy \right) dx$$

$$= \int_{-\infty}^{0} \left(x^{\cancel{2}} e^{-x^2} \int_{1/x}^{-1/x} \frac{x}{1+(xy)^2}\, dy \right) dx$$

$$+ \int_{0}^{+\infty} \left(x^{\cancel{2}} e^{-x^2} \int_{-1/x}^{1/x} \frac{x}{1+(xy)^2}\, dy \right) dx \overset{(*)}{=}$$

$$\int_{1/x}^{-1/x} \frac{x}{1+(xy)^2}\, dy = \Big[\operatorname{arctg}(xy) \Big]_{y=1/x}^{y=-1/x}$$

$$\partial_y \operatorname{arctg}(xy) = \frac{1}{1+(xy)^2} \cdot x$$

$$= \operatorname{arctg}(-1) - \operatorname{arctg} 1 = -\frac{\pi}{4} - \frac{\pi}{4} = -\frac{\pi}{2}$$

$$\int_{-1/x}^{1/x} \frac{x}{1+(xy)^2}\,dx = \operatorname{arctg} xy \Big|_{y=-1/x}^{y=1/x} = +\pi/2$$

$$\overset{(*)}{=} \int_{-\infty}^{0} x e^{-x^2} \left(-\pi/2\right) dx + \int_0^{+\infty} x e^{-x^2} \frac{\pi}{2}\, dx$$

$$= \frac{\pi}{2}\left[+\frac{1}{2}\int_{-\infty}^{0} \underbrace{-2x\, e^{-x^2}}_{\partial_x e^{-x^2}} dx - \frac{1}{2}\int_0^{+\infty} -2x e^{-x^2}\, dx \right]$$

$$\frac{1}{2}\left[e^{-x^2}\right]_{-\infty}^{0} - \frac{1}{2}\left[e^{-x^2}\right]_0^{+\infty}$$

$$= \frac{\pi}{2}\left[\frac{1}{2}\cdot 1 - \frac{1}{2}(-1)\right] = \pi/2 .$$

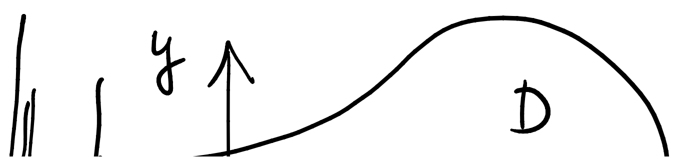
$$\Rightarrow f \in L^1 \quad \text{and} \quad \int_D f = \pi/2 . \quad \boxtimes$$

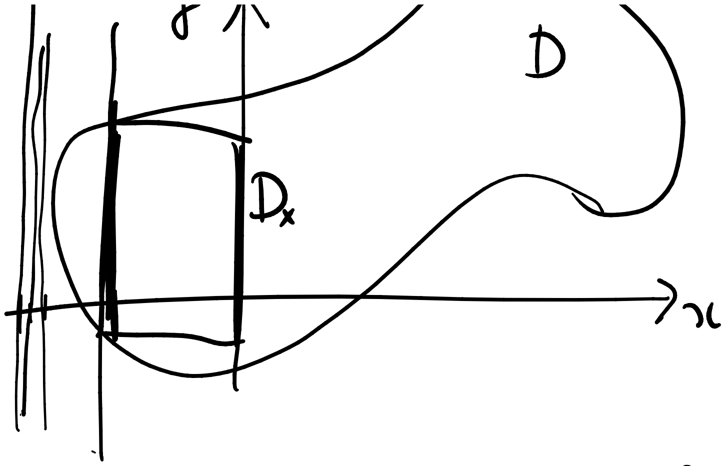
R.F. for $f = f(x,y,z)$.

Remind that

$$f = f(x,y) \in L^1(D) \Rightarrow \int_D f = \int_{\mathbb{R}} \left(\int_{D_x} f(x,y)\,dy \right) dx .$$

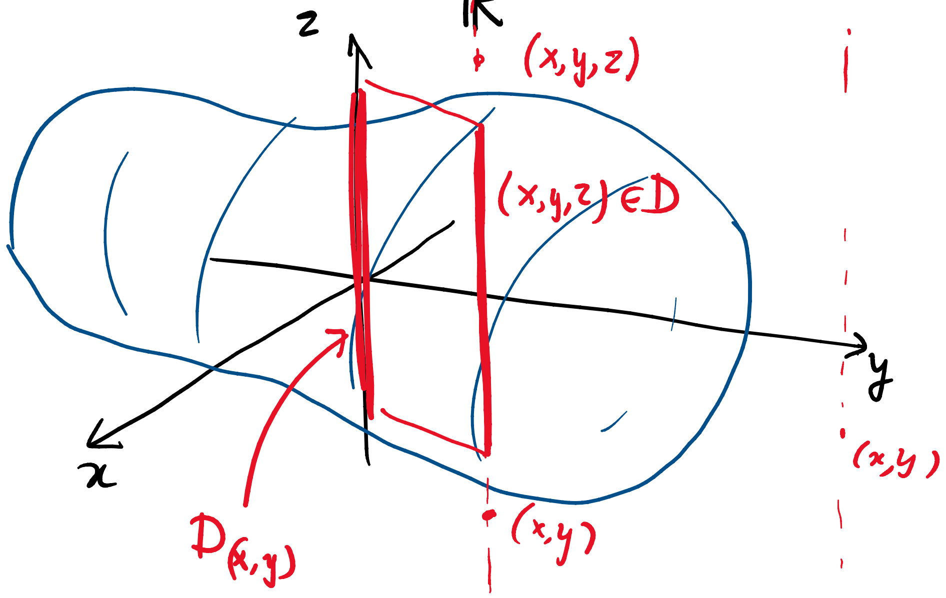
$$= \int_{\mathbb{R}} \left(\int_{D^y} f(x,y)\,dx \right) dy$$





Let's now consider $f = f(x,y,z)$

$$\int_D f(x,y,z)\,dx\,dy\,dz = \int_{\mathbb{R}^2} \left(\int_{\{z:\ (x,y,z)\in D\} = D_{(x,y)}} f(x,y,z)\,dz \right) dx\,dy$$



Other possibilities: if $f \in L^1(D)$

$$\int_D f(x,y,z)\,dx\,dy\,dz = \int_{\mathbb{R}^2} \left(\int_{D_{(x,y)}} f(x,y,z)\,dz \right) dx\,dy$$

$$= \int_{\mathbb{R}^2} \left(\int_{\{y:\ (x,y,z)\in D\} = D} f(x,y,z)\,dy \right) dx\,dz$$

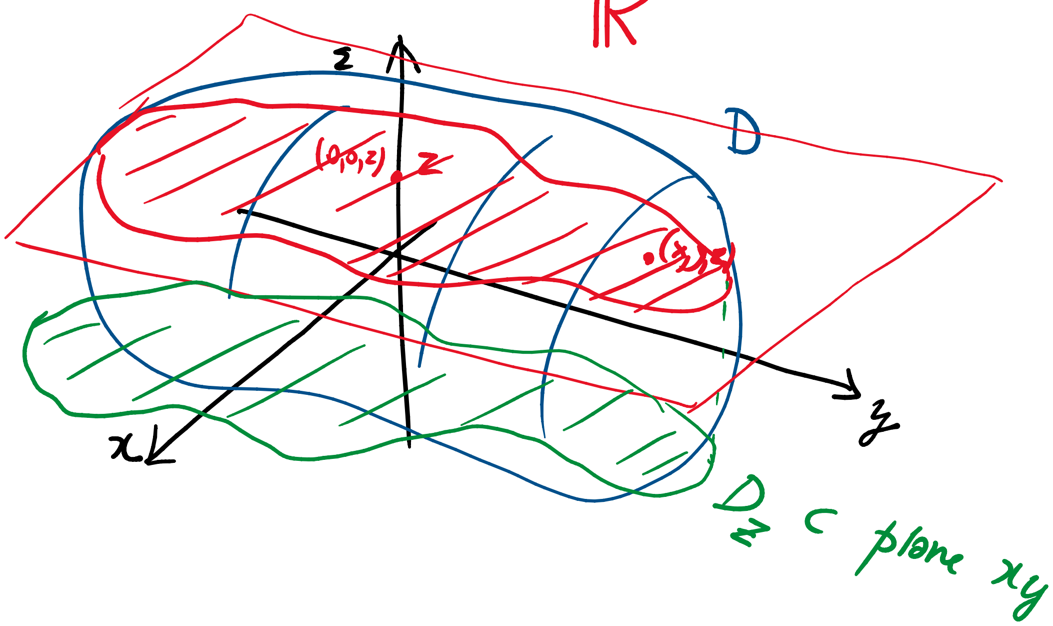
$$= \int_{\mathbb{R}^2} \left(\int_{\{y:\ (x,y,z)\in D\} = D_{(x,z)}} f(x,y,z)\, dy \right) dx\, dz$$

fixed

$$= \int_{\mathbb{R}^2} \left(\int_{D_{(y,z)}} f(x,y,z)\, dx \right) dy\, dz$$

These are not the unique possible ways we may apply R.F.

$$\int_D f(x,y,z)\, dx\,dy\,dz = \int_{\mathbb{R}} \left(\int_{\{(x,y)\ :\ (x,y,z)\in D\} =: D_z} f(x,y,z)\, dx\,dy \right) dz$$



$$= \int_{\mathbb{R}} \int_{\{(x,z)\ :\ (x,y,z)\in D\} =: D_y} f(x,y,z)\, dx\,dz \quad dy$$

fixed

$$= \int_{\mathbb{R}} \int_{\{(y,z)\,:\,(x,y,z)\, \in D\} =: D_x} f(x,y,z)\, dy\, dz \qquad dx$$

fixed

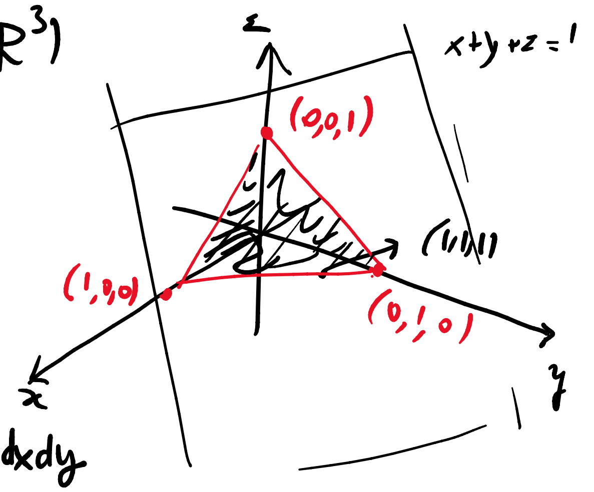
Ex 4.5.1

#5 $$\int_{z \geq 0,\ x \geq 0,\ y \geq 0,\ \boxed{x+y+z \leq 1}} xyz\, dx\, dy\, dz.$$

<u>Sol:</u> Here $f(x,y,z) = xyz \in \mathcal{C}(\mathbb{R}^3)$

$$D = \{ \boxed{z \geq 0},\ x \geq 0,\ y \geq 0,\ x+y+z \leq 1. \}$$

$$\boxed{z \leq 1-(x+y)}$$



Applying RF $\displaystyle\int_{\mathbb{R}^2} \int_{z:\,(x,y,z) \in D} f \quad dxdy$

$$\int_D f = \int_{\boxed{x+y \leq 1},\ \boxed{x \geq 0,\, y \geq 0}} \left(\int_0^{1-(x+y)} xyz\, dz \right) dx\, dy$$

$z \geq 0$

$x+y+z \leq 1$

$$\boxed{0 \leq z \leq 1-(x+y)}$$

$\Updownarrow$ if ≥ 0

$1-(x+y) \geq 0$

$x+y \leq 1$

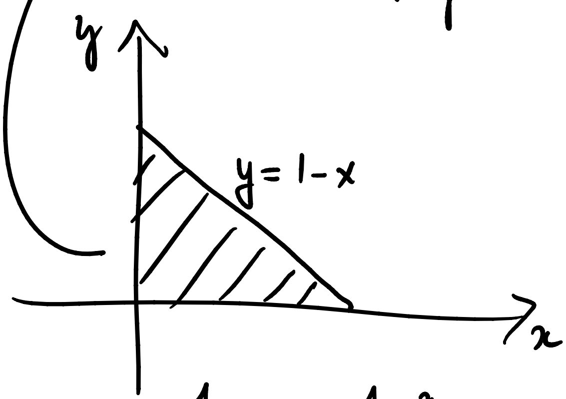
$$= \int_{\boxed{x+y \leq 1},\ \boxed{x \geq 0,\, y \geq 0}} \left(xy \int_0^{1-(x+y)} z\, dz \right) dx\, dy$$

$$\| $$

$$\frac{z^2}{2} \Big|_0^{1-(x+y)} = \frac{1}{2}\left(1-(x+y)\right)^2$$

$$= \frac{1}{2} \int_{\substack{x+y\leq 1 \\ x\geq 0,\, y\geq 0}} xy \cancel{\frac{1}{2}} \left(1 - (x+y)\right)^2 \, dx\,dy$$

$0 \leq y \leq 1-x \qquad 0 \leq x \leq 1$



$$= \frac{1}{2} \int_0^1 \int_0^{1-x} xy \left(1 - (x+y))^2 \, dy\right) dx$$

$$= \frac{1}{2} \int_0^1 x \int_0^{1-x} y \left(1 + (x+y)^2 - 2(x+y)\right) dy \, dx$$

$$1 + x^2 + y^2 + 2xy - 2x - 2y$$

$$= \frac{1}{2} \int_0^1 x \left(\int_0^{1-x} y(x-1)^2 + 2y^2(x-1) + y^3 \, dy \right) dx$$

$$= \frac{1}{2} \int_0^1 x \left[(x-1)^2 \frac{y^2}{2} \Big|_0^{1-x} + 2(x-1) \frac{y^3}{3} \Big|_0^{1-x} + \frac{y^4}{4} \Big|_0^{1-x} \right] dx$$

$$(x-1)^2 \frac{1}{2}(1-x)^2 + \frac{2}{3}(x-1)(1-x)^3 + \frac{1}{4}(1-x)^4$$

$$\frac{1}{2}(x-1)^4 - \frac{2}{3}(x-1)^4 + \frac{1}{4}(x-1)^4 \qquad \frac{3}{4} - \frac{2}{3} = \frac{9-8}{12} = \frac{1}{12}$$

$$= \frac{1}{24} \int_0^1 x(x-1)^4 \, dx = \frac{1}{24} \int_0^1 (x-1)^5 + (x-1)^4$$

$x-1+1$

$x-1+1$

$$= \frac{1}{24}\left[\frac{(x-1)^6}{6}\Big|_0^1 + \frac{(x-1)^5}{5}\Big|_0^1 \right]$$

$$= \frac{1}{24}\left[-\frac{1}{6} + \frac{1}{5} \right].$$ ∎

#6 $\displaystyle\int_{0\le x\le 1,\ 0\le y\le 2,\ 0\le z\le 6-x^2-y^2} x\log(1+y)\,dx\,dy\,dz$

Here $f(x,y,z) = x\log(1+y)$ on

$$D = \{ 0\le x\le 1,\ 0\le y\le 2,\ 0\le z\le 6-(x^2+y^2) \}$$

On D f is well defd and cont., $f\geq 0$.

To check int, being $|f| = f$ on D, we compute

$$\int_{\mathbb{R}^2} \int_{D_{(x,y)}} f(x,y,z)\,dz \quad dx\,dy$$

$$D_{(x,y)} = \{ z : (x,y,z)\in D \}$$

$$D_{(x,y)} \overset{?}{=} [0,\ 6-(x^2+y^2)] \quad \forall (x,y)?$$

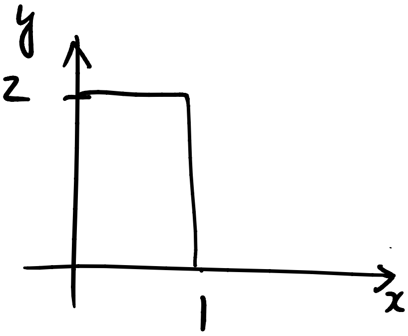
NO

$$D_{(x,y)} = \begin{cases} [0,\ 6-(x^2+y^2)] & \text{if } \begin{array}{l} 0 \le x \le 1 \\ 0 \le y \le 2 \\ \cancel{6-(x^2+y^2) \ge 0} \end{array} \\ \emptyset & \text{any other case} \end{cases}$$

$$\Longrightarrow \int_{\mathbb{R}^2} \left(\int_{D_{x,y}} f \, dz \right) dx\,dy = \int_{\substack{0 \le x \le 1 \\ 0 \le y \le 2}} \left(\int_0^{6-(x^2+y^2)} \underbrace{x \log(1+y)} \, dz \right) dx\,dy$$

$$6-(x^2+y^2) \ge 0 \iff x^2+y^2 \le 6$$

$$= \int_{\boxed{\substack{0 \le x \le 1 \\ 0 \le y \le 2}}} x \log(1+y) \left(6 - (x^2+y^2)\right) dx\,dy$$



$$= \int_0^2 \left(\int_0^1 x \log(1+y)(6-x^2-y^2)\, dx \right) dy$$

$$= \int_0^2 \log(1+y) \int_0^1 (6-y^2)x - x^3 \, dx \quad dy$$

2

$$= \int_0^2 \log(1+y)\left[(6-y^2)\ \frac{x^2}{2}\Big|_0^1 - \frac{x^4}{4}\Big|_0^1 \right] dy$$

$$= \int_0^2 \left(\frac{1}{2}(6-y^2) - \frac{1}{4}\right) \log(1+y)\, dy$$

= you finish.

Do 4.51 #4

4.5.2 #5, 6

4.5.3 #1, 3