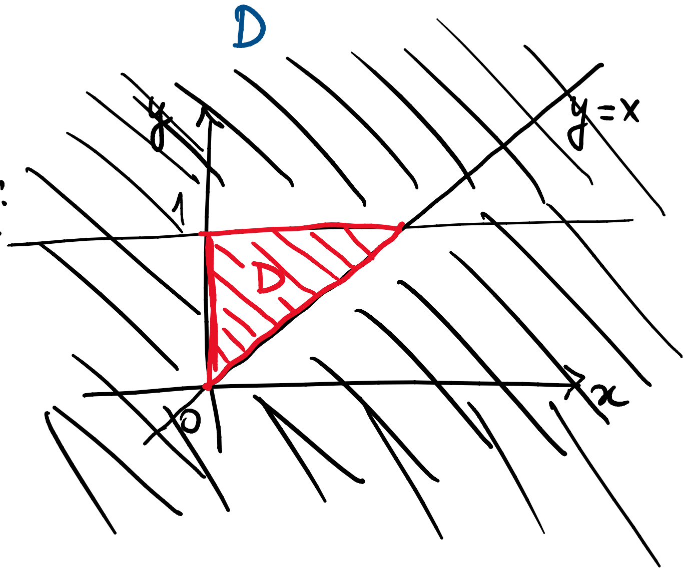
# Ex 4.5.3

\# 1 $\displaystyle\int_D x\sqrt{y^2-x^2}\,dx\,dy$ $\quad D=\{(x,y): 0\le y\le 1,\ 0\le x\le y\}$

Sol:




$f(x,y)=x\sqrt{y^2-x^2}\in \mathcal{C}(D)$, $D$ closed, $f\geq 0$ on $D$

To check $f\in L^1(D)$ we apply FT checking if

$$\int_{\mathbb{R}}\int_{D_x}|f|\,dy\,dx \overset{f\geq 0}{=} \int_{\mathbb{R}}\int_{D_x} f\,dy\,dx < +\infty$$

or

$$\int_{\mathbb{R}}\int_{D^y}|f|\,dx\,dy \overset{f\geq 0}{=} \int_{\mathbb{R}}\int_{D^y} f\,dx\,dy < +\infty$$

$$\int_{\mathbb{R}}\int_{D^y} f\,dx\,dy = \int_0^1\left(\int_0^y x\sqrt{y^2-x^2}\,dx\right)dy \overset{(*)}{=}$$

$$D=\{0\le y\le 1,\ 0\le x\le y\}$$

$$\int_0^y x\sqrt{y^2-x^2}\,dx = -\frac{1}{2}\int_0^y (y^2-x^2)^{1/2}(-2x)\,dx$$

$$\partial_x(y^2-x^2) = -2x \qquad f^\alpha \quad f'$$

$$= -\frac{1}{2}\frac{\cancel{2}}{3}\left[\frac{(y^2-x^2)^{3/2}}{\cancel{3/2}}\right]_{x=0}^{x=y}$$

$$= -\frac{1}{3}\left[0 - (y^2)^{3/2}\right] = \frac{1}{3}|y|^3$$

$$= \frac{1}{3}y^3 \qquad y \geq 0$$

$$\overset{(*)}{=} \int_0^1 \frac{1}{3}y^3\,dy = \frac{1}{3}\left[\frac{y^4}{4}\right]_{y=0}^{y=1}$$

$$= \frac{1}{3}\left(\frac{1}{4}-0\right) = \frac{1}{12}. \implies f \in L^1 \text{ and}$$

$$\int_D f = 1/12. \quad \blacksquare$$

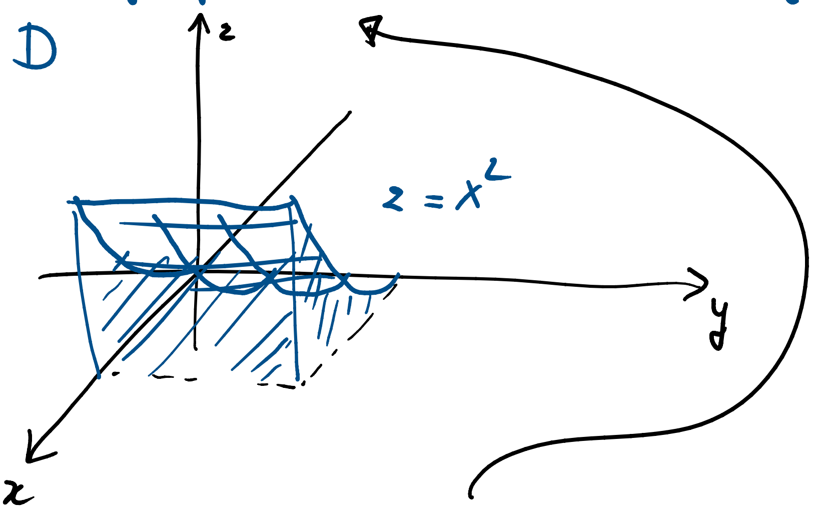
Rmk: If $f \in \underbrace{\mathcal{C}(D) \quad D \text{ compact}}_{\text{by W.} \Rightarrow |f| \leq C \text{ const.}} \Rightarrow f \in L^1(D)$

$$\begin{array}{l}\displaystyle\int_D |f| \\ \quad\wedge\!\!| \\ \displaystyle\int C = C\int 1 = C\,\lambda_d(D) < +\infty\end{array}$$

$$\int_D C = C \int_D 1 = C\, \lambda_d(D) < +\infty$$

bded ∎

#3 $\displaystyle\int_D z y^2 \sqrt{x^2 + zy}\; dx\,dy\,dz \qquad D = \left\{ 0 \leq z \leq x^2,\ 0 \leq x \leq 1,\ 0 \leq y \leq 1 \right\}$




In this case $f(x,y,z) \in \mathcal{C}(D)$, $D$ compact $\Rightarrow$ $f \in L^1(D)$. According to red. form.

$$\Rightarrow \int_D f = \int_{\substack{0 \leq x \leq 1 \\ 0 \leq y \leq 1}} \left( \int_0^{x^2} f(x,y,z)\, dz \right) dx\,dy$$

$$= \int_{\substack{0 \leq x \leq 1 \\ 0 \leq y \leq 1}} y^2 \left( \int_0^{x^2} z \sqrt{x^2 + zy}\; dz \right) dx\,dy \overset{(*)}{=}$$

$$\int_0^{x^2} z \underbrace{\sqrt{x^2 + zy}}_{\substack{(x^2+zy)^{1/2} \\ \| \\ \partial_z \left[ \frac{2}{3y} (x^2+zy)^{3/2} \right]}}\; dz = \left[ z \cdot \frac{2}{3y} (x^2 + zy)^{3/2} \right]_{z=0}^{z=x^2} - \int_0^{x^2} \frac{2}{3y} (x^2 + zy)^{3/2}\, dz$$

$$\partial_z \left[\frac{2}{3y}(x^2+zy)^{3/2}\right] \qquad \int_0 \frac{}{3y}(x^2+zy) \, dz$$

$$= \frac{2}{3}\frac{x^2}{y}(x^2)^{3/2}(1+y)^{3/2} - \frac{2}{3y}\int_0^{x^2}(x^2+zy)^{3/2}\,dz$$

$$\int_0^{x^2}(x^2+zy)^{3/2}\,dz = \partial_z\left[\frac{2}{5y}(x^2+zy)^{5/2}\right]$$

$$\overset{x\geq 0}{=} \frac{2}{3}\frac{x^5}{y}(1+y)^{3/2} - \frac{4}{15y}\left[(x^2+zy)^{5/2}\right]_{z=0}^{z=x^2}$$

$$= \frac{2}{3}\frac{x^5}{y}(1+y)^{3/2} - \frac{4}{15y}\left[x^5(1+y)^{5/2} - x^5\right]$$

$$= \int_{\substack{0\leq x\leq 1\\ 0\leq y\leq 1}} y^{\cancel{2}}\frac{1}{\cancel{y}}\left[\frac{2}{3}x^5(1+y)^{3/2} - \frac{4}{15}x^5\left((1+y)^{5/2}-1\right)\right]dx\,dy$$

$$\overset{RF}{=} \int_0^1 y\left(\int_0^1 \frac{2}{3}x^5(1+y)^{3/2} - \frac{4}{15}x^5\left((1+y)^{5/2}-1\right)dx\right)dy$$

$$\frac{2}{3}(1+y)^{3/2}\int_0^1 x^5 \qquad - \frac{4}{15}\left((1+y)^{5/2}-1\right)\int_0^1 x^5\,dx$$

$$\int_0^1 x^5 = \left[\frac{x^6}{6}\right]_0^1 = \frac{1}{6} \qquad \int_0^1 x^5\,dx = \frac{1}{6}$$

$$= \int_0^1 y\left[\frac{1}{9}(1+y)^{3/2} - \frac{1}{45}\left((1+y)^{5/2}-1\right)\right]dy$$

$$= \frac{1}{9}\Big[ \qquad\qquad\qquad\qquad \Big] = \text{you finish.}$$

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4.5.2

\# 6 $\displaystyle\int_{[1,+\infty[^3} \underbrace{y^3 z^8 e^{-xy^2z^3}}_{f(x,y,z)} \, dx\, dy\, dz$

$f \in \mathscr{C}([1,+\infty[^3)$ , $f \geq 0$ on $[1,+\infty[^3$

closed but unbded

$1 \leq x < +\infty$
$1 \leq y < +\infty$
$1 \leq z < +\infty$

To check if $f \in L^1([1,+\infty[^3)$ because $f \geq 0$ we check one of the iterated int is finite.

$$\int_{1 \leq y,z < +\infty} \left( \int_1^{+\infty} y^3 z^8 e^{-xy^2z^3} dx \right) dy\, dz$$

$$= \int_{1 \leq y,z < +\infty} \left( y^3 z^8 \int_1^{+\infty} e^{-xy^2z^3} dx \right) dy\, dz$$

$e^{\alpha x}$

$$e$$

$$\|$$

$$\partial_x\left(\frac{e^{\alpha x}}{\alpha}\right)$$

$$= \int_{1\le y,z<+\infty} y^3 z^8 \left[\frac{e^{-xy^2z^3}}{-y^2z^3}\right]_{x=1}^{x=+\infty} dy\,dz$$

$$+\frac{1}{y^2z^3}\, e^{-y^2z^3}$$

$$= \int_{1\le y,z<+\infty} y^{\cancel{3}} z^{\cancel{8}\,5} \frac{1}{\cancel{y^2}\,\cancel{z^3}}\, e^{-y^2z^3}\, dy\,dz$$

$$= \int_{1\le y,z<+\infty} y z^5 e^{-y^2z^3}\, dy\,dz$$

$$= \int_{1\le z<+\infty}\left(\int_{1\le y<+\infty} y z^5 e^{-y^2z^3}\, dy\right) dz$$

$$= -\frac{1}{2}\int_1^{+\infty} z^{\cancel{5}\,2} \int_1^{+\infty} -2z^3 y\, e^{-y^2z^3}\, dy \quad dz$$

$$\|$$

$$\partial_y\left(e^{-y^2z^3}\right) = e^{-y^2z^3}\left(-2yz^3\right)$$

$$= -\frac{1}{2}\int^{+\infty} z^2 \left[e^{-y^2z^3}\right]^{y=+\infty} dz$$

$$= -\frac{1}{2}\int_1 z^2 \left[ e^{-y^2 z^3} \right]_{y=1}^{y=+\infty} dz$$

$$0 - e^{-z^3}$$

$$= \frac{1}{-3\cdot 2}\int_1^{+\infty} (-3) z^2 e^{-z^3}\, dz \qquad = -\frac{1}{6}\left[ e^{-z^3} \right]_1^{+\infty}$$

$$\partial_z \left(e^{-z^3}\right) = e^{-z^3}(-3z^2) \qquad 0 - e^{-1}$$

$$= \frac{e^{-1}}{6}. \qquad \blacksquare$$

## Change of variables.

Suppose we've to compute

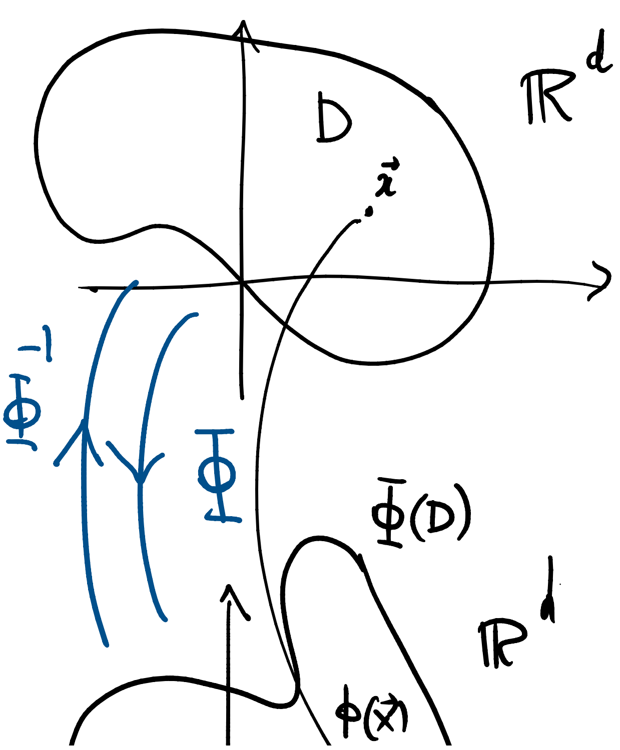
$$\int_D f(\vec{x})\, d\vec{x}.$$

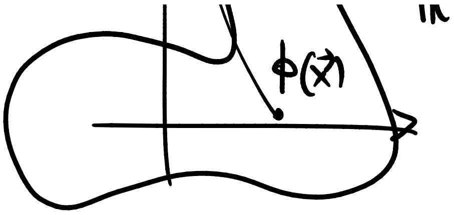
Suppose also that, for some reason, it seems to be better to introduce a new variable

$$\vec{y} = \Phi(\vec{x})$$

Normally, by change of var we mean a map $\Phi$ such that $\Phi$ is invertible

$$\vec{x} = \Phi^{-1}(\vec{y})$$






$x = \Phi(\vec{y})$

and $\phi$ is regular

$\exists\, \phi', (\phi^{-1})' \in \mathcal{C}$

Def: We say that a map $\Phi : D \subset \mathbb{R}^d \longrightarrow \Phi(D) \subset \mathbb{R}^d$ is a diffeomorfism if

i) $\phi$ is a bijection $(\exists\, \phi^{-1})$

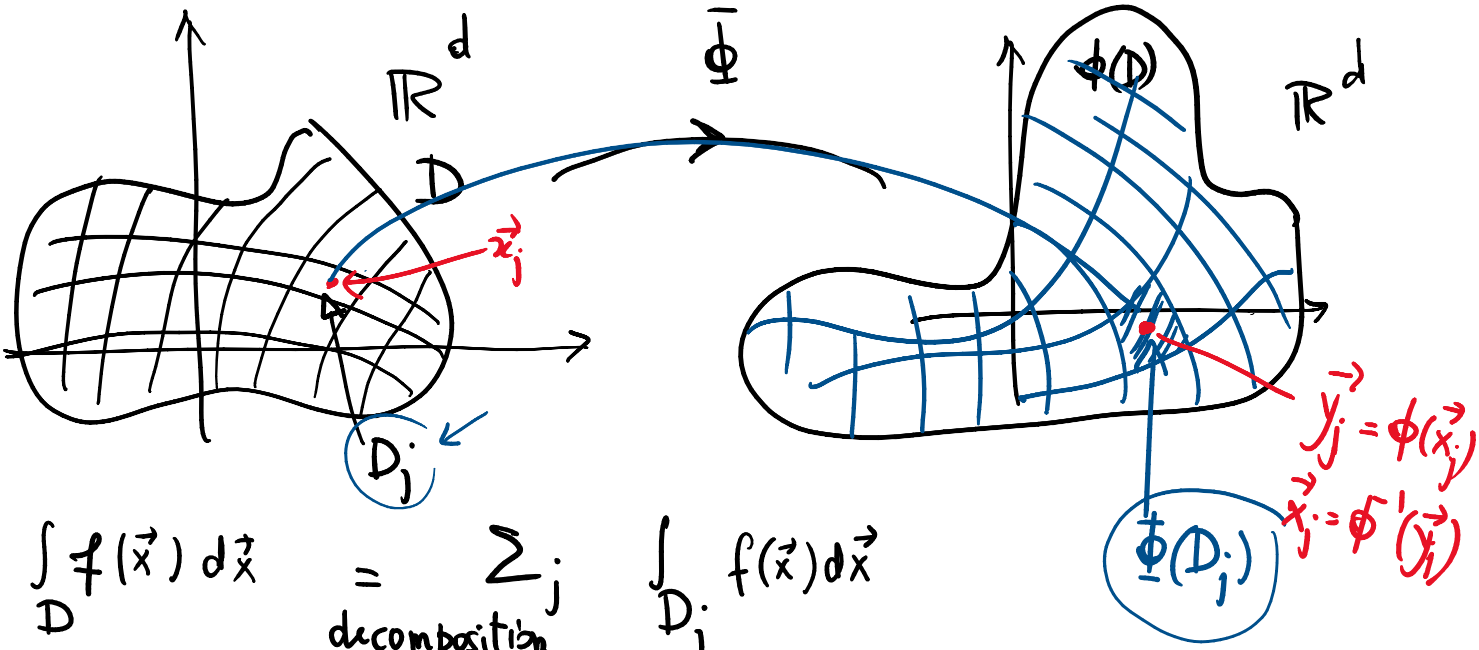
ii) $\phi, \phi^{-1}$ are diff with $\phi', (\phi^{-1})' \in \mathcal{C}$

(in other words the entries of jacobian matr are cont functs.)

$\phi = (\phi_1(x_1, \ldots, x_d), \phi_2(x_1 \ldots x_d), \ldots, \phi_d(x_1 \ldots x_d))$

$$\phi' = \begin{bmatrix} \nabla\phi_1 \\ \nabla\phi_2 \\ \vdots \\ \nabla\phi_d \end{bmatrix} = \begin{bmatrix} \partial_1\phi_1 & \partial_2\phi_1 & \cdots & \partial_d\phi_1 \\ \partial_1\phi_2 & \partial_2\phi_2 & \cdots & \partial_d\phi_2 \\ \vdots & & & \\ \partial_1\phi_d & \partial_2\phi_d & \cdots & \partial_d\phi_d \end{bmatrix}$$

Let $\phi : D \subset \mathbb{R}^d \longrightarrow \phi(D) \subset \mathbb{R}^d$ be a diffeomorphism




$$\int_D f(\vec{x})\, d\vec{x} \underset{\text{decomposition}}{=} \sum_j \int_{D_j} f(\vec{x})\, d\vec{x}$$

We divide $D = \bigcup D_j$ s.t. on $D_j$ $f(\vec{x}) \approx f(\vec{x}_j)$

$D_i \cap D_j = \emptyset$

$$\approx \sum_j \int_{D_j} \underbrace{f(\vec{x}_j)}_{\text{constant}}\, d\vec{x}$$

$$\overset{\text{linearity}}{=} \sum_j f(\vec{x}_j) \int_{D_j} 1\, d\vec{x} = \underline{\sum_j f(\vec{x}_j)\, \lambda_d(D_j)}$$

What is the relation between

$$\lambda_d(\Phi(D_j)) \quad , \quad \lambda_d(D_j)$$

Recall that if $T$ is an inv. matrix

$$\lambda_d(TD_j + \vec{v}_j) = |\det T|\, \lambda_d(D_j)$$

$\phi(D_j)$

$\uparrow$ is non linear but $\phi(\vec{x}) = \overbrace{\phi(\vec{x}_j)}^{\vec{v}_j} + \overbrace{\phi'(x_j)}^{T_j}(\vec{x} - \vec{x}_j)$

$+$ error

$o(\|\vec{x} - \vec{x}_j\|)$

$\Downarrow$

$$\phi(D_j) = \vec{v}_j + T_j(D_j - \vec{v}_j)$$

$$\Rightarrow \lambda_d(\phi(D_j)) = \lambda_d(\vec{v}_j + T_j(D_j - \vec{v}_j)) = \lambda_d(T_j(D_j - \vec{v}_j))$$

$$= |\det T_j| \cdot \lambda_d(D_j - \vec{v}_j)$$

$$= |\det T_j| \cdot \lambda_j(D_j)$$

$$\Longrightarrow \boxed{\lambda_d(\phi(D_j)) = |\det \phi'(x_j)|\, \lambda_d(D_j)}$$

$$\Rightarrow \int_D f(\vec{x})\, d\vec{x} = \sum_j f(\vec{x}_j)\, \lambda_d(D_j)$$

$$= \sum_j f(\vec{x}_j) \frac{1}{|\det \phi'(x_j)|} \lambda_d(\phi(D_j)) \quad (*)$$

$\vec{x} \longrightarrow \vec{x}$

$(x_1 \dots x_d) \longrightarrow (x_1 \dots x_d)$

$\phi$ invertible

$\phi \circ \phi^{-1} = I \qquad \phi^{-1} \circ \phi = I \Rightarrow$

$jad(id) = \begin{bmatrix} \nabla x_1 \\ \nabla x_2 \\ \vdots \\ \nabla x_d \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots \end{bmatrix}$

$$\phi\circ\phi^{-1} = I \quad , \quad \phi^{-1}\circ\phi = I \Rightarrow$$

$$jad(id) = \begin{bmatrix} \nabla x_1 \\ \vdots \\ \nabla x_d \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots 0 \\ \vdots & & & \end{bmatrix} = I$$

$$\phi^{-1}(\phi(\vec{x})) = \vec{x} \Rightarrow$$

$$\Downarrow$$

$$\left[(\phi^{-1})'(\phi(x))\,\phi'(x)\right] = I$$

$$\boxed{\det(\phi^{-1})'(\phi(x)) \cdot \det\phi'(x) = \det I = 1}$$

$$\Rightarrow \quad \det(\phi^{-1})'(\phi(x)) = \frac{1}{\det\phi'(x)}$$

$$\Rightarrow \int_D f(\vec{x})\,d\vec{x} \overset{(*)}{=} \sum_j \boxed{f(\vec{x}_j)\left|\det(\phi^{-1})'(\phi(\vec{x}_j))\right|}\,\lambda_d(\phi(D_j))$$

$$\lambda_d(\phi(D_j)) = \int_{\phi(D_j)} 1\,d\vec{y}$$

$$= \sum_j \int_{\phi(D_j)} f(\phi^{-1}(\vec{y}_j))\left|\det(\phi^{-1})'(\vec{y}_j)\right|\,d\vec{y}$$

$$\approx f(\phi^{-1}(\vec{y}))\left|\det(\phi^{-1})'(\vec{y})\right|\,dy$$

$$\overset{\text{decomp}}{=} \int_{\phi(D)} f(\phi^{-1}(\vec{y}))\left|\det(\phi^{-1})'(\vec{y})\right|\,d\vec{y}$$

We obtained the following formula:

We obtained the following formula:

$$\boxed{\vec{y} = \phi(\vec{x}),\ \vec{x} = \phi^{-1}(\vec{y})}$$

$$\int_{\vec{x} \in D} f(\vec{x})\, d\vec{x} = \int_{\vec{y} \in \underline{\phi(D)}} \underbrace{f(\phi^{-1}(\vec{y}))}_{f \text{ in } \vec{y} \text{ coords}} \, |\det(\phi^{-1})'(\vec{y})|\, d\vec{y}$$

Rmk: For functs $f(x)$ $x \in \mathbb{R}$

$x = \phi^{-1}(y)$

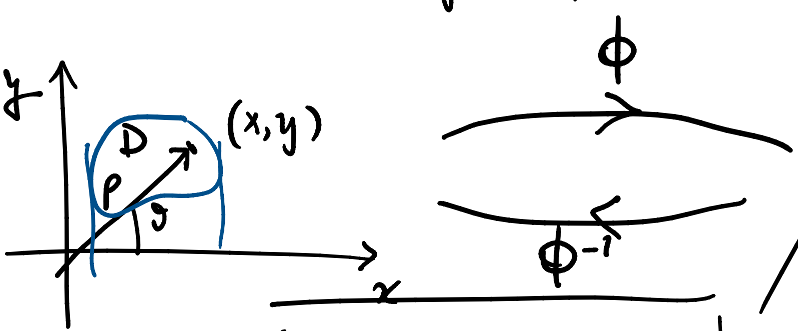
$y = \phi(x)$, $\phi \nearrow$

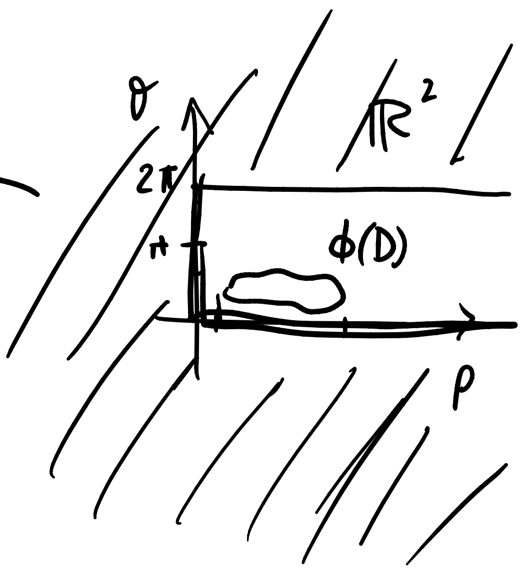
$$\int_a^b f(x)\,dx = \int_{\phi(a)}^{\phi(b)} f(\phi^{-1}(y)) \cdot (\phi^{-1})'(y)\, dy$$

$\phi \searrow$ $\underline{\ \ }$ $\int$ " "

## Special Change of Vars

I) Polar coords. The polar coord ch. of vars. is defined through position




$(\rho, \theta) = \Phi(x, y)$

$$\boxed{\phi^{-1} \begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}}$$

$\rho = \sqrt{x^2+y^2}$

$(x, y) = \phi^{-1}(\rho, \theta)$

old vars $\uparrow$ new vars $\uparrow$

$\theta = \int$ —

$\phi^{-1}(\rho, \theta)$

$\theta = \left\{ \begin{array}{l} - \\ - \\ - \\ - \end{array} \right.$

$$\text{So} \quad \int_D f(x,y)\,dx\,dy = \int_{\Phi(D)} f\big(\overbrace{\rho\cos\theta, \rho\sin\theta}^{\phi^{-1}(\rho,\theta)}\big) \cdot \left|\det\big((\phi^{-1})'(\rho,\theta)\big)\right| d\rho\, d\theta$$

old vars ↑ $\qquad$ ↑ new vars

Let's compute the jac. matrix of $\phi^{-1}$

$$(\phi^{-1})'(\rho,\theta) = \begin{array}{cc} \partial_\rho & \partial_\theta \\ \end{array}$$
$$\begin{bmatrix} \partial_\rho(\rho\cos\theta) & \partial_\theta(\rho\cos\theta) \\ \partial_\rho(\rho\sin\theta) & \partial_\theta(\rho\sin\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{bmatrix}$$

$$\Rightarrow \left|\det(\phi^{-1})'(\rho,\theta)\right| = \left|\det\left[\ \text{''}\ \right]\right| = \left|\rho(\cos\theta)^2 + \rho(\sin\theta)^2\right|$$

$$= |\rho| \overset{\rho \geq 0}{=} \rho$$

$$\boxed{\left|\det(\phi^{-1})'(\rho,\theta)\right| = \rho}$$

$$\Rightarrow \boxed{\int_D f(x,y)\,dx\,dy = \int_{\Phi(D)} f(\rho\cos\theta, \rho\sin\theta)\, \rho\, d\rho\, d\theta.}$$

Ex 4.3.4 Compute $\displaystyle\int_{\mathbb{R}^2} e^{-\sqrt{x^2+y^2}}\, dx\, dy.$

Sol. $f(x,y) = e^{-\sqrt{x^2+y^2}}$ ... on $\mathbb{R}^2$ ...

Sol: $f(x,y) = e^{-\sqrt{x^2+y^2}} \geq 0$ , $f \in \mathcal{C}(\mathbb{R}^2)$. Now

$$\int_{\mathbb{R}^2} e^{-\sqrt{x^2+y^2}}\, dx\, dy \underset{\substack{x=\rho\cos\vartheta \\ y=\rho\sin\vartheta}}{=} \int_{\substack{0\leq\rho<+\infty \\ 0\leq\theta\leq 2\pi}} \underbrace{e^{-\sqrt{\rho^2}}} \cdot \rho \, d\rho\, d\theta$$

$$= \int_{\substack{0\leq\rho<+\infty \\ 0\leq\theta\leq 2\pi}} e^{-\rho}\rho \, d\rho\, d\theta$$

$$\partial_\rho(-e^{-\rho})$$
$$\overset{RF}{=} \int_0^{2\pi} \underbrace{\left(\int_0^{+\infty} e^{-\rho}\rho\, d\rho\right)} d\theta \qquad \partial_\rho(-e^{-\rho})$$

$$= 2\pi\left[ \underbrace{-e^{-\rho}\rho\Big|_0^{+\infty}}_{=0} + \int_0^{+\infty} + e^{-\rho}\, d\rho\right]$$

$$= 2\pi\left[-e^{-\rho}\right]_0^{+\infty} = 2\pi[0-(-1)] = 2\pi \quad \blacksquare$$

Example 4.3.5 (Gaussian int)

$$\int_{-\infty}^{+\infty} e^{-x^2/2}\, dx = \sqrt{2\pi}.$$

Sol: Let's start with $\displaystyle\int_{\mathbb{R}^2} e^{-\frac{x^2+y^2}{2}} dx\,dy$

$+\infty \quad +\infty \quad x^2+y^2$

$\mathbb{K}$

$$\underset{RF}{=} \int_{-\infty}^{+\infty} \Big( \int_{-\infty}^{+\infty} \underbrace{e^{-\frac{x^2+y^2}{2}}}_{= e^{-x^2/2}\, e^{-y^2/2}} dx \Big) dy$$

$$= \int_{-\infty}^{+\infty} \Big( e^{-y^2/2} \underbrace{\int_{-\infty}^{+\infty} e^{-x^2/2} dx}_{I} \Big) dy = I^2.$$

So $I^2 = \int_{\mathbb{R}^2} e^{-\frac{x^2+y^2}{2}} dx\, dy \underset{\substack{x = \rho\cos\theta \\ y = \rho\sin\theta}}{=}$

$$= \int_{0 \leq \rho < +\infty,\, 0 \leq \theta \leq 2\pi} e^{-\rho^2/2} \rho \, d\rho \, d\theta$$

$$\underset{=}{RF} -\int_0^{2\pi} \overbrace{\Big( \int_0^{+\infty} \underbrace{e^{-\rho^2/2} (-\rho)}_{= \partial_\rho e^{-\rho^2/2}} d\rho \Big)} d\theta$$

$$= -2\pi \left[ e^{-\rho^2/2} \right]_0^{+\infty} = 2\pi \Rightarrow I = \sqrt{2\pi}$$

$\blacksquare$

Do 4.5.5

4.5.7 # 2, 3, 4, 1.