venerdì 6 marzo 2020 11:15

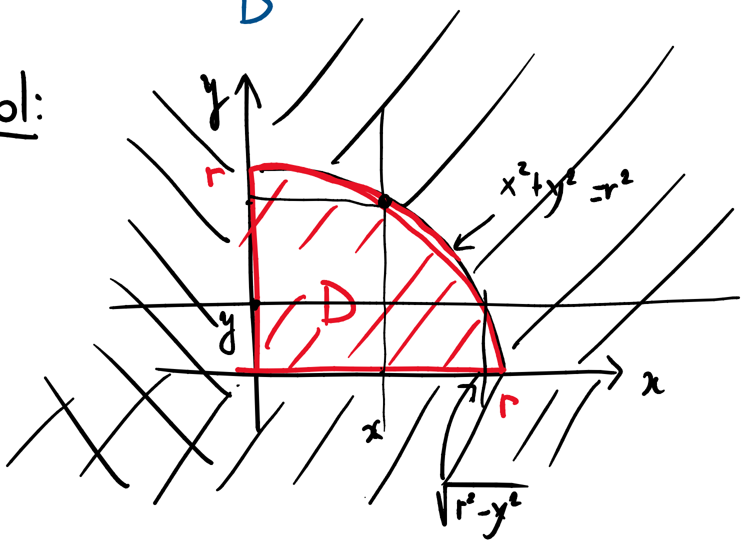
Ex 4.5.5.

$D = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x^2+y^2 \leq r^2\}$

Draw D, determine its barycenter and compute

$$\int_D \frac{x+y}{x^2+y^2} \, dx\,dy.$$

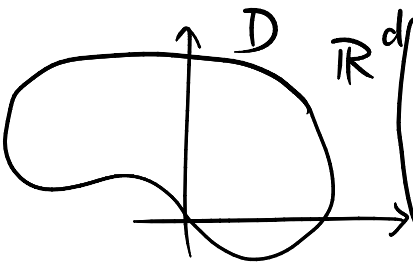
Sol:



$y^2 = r^2 - x^2$

$y = \sqrt{r^2-x^2}$

Def: We call barycenter of a domain $D \subset \mathbb{R}^d$ a point $\vec{x}_G = (x_G^1, x_G^2, \ldots, x_G^d)$



$$\boxed{\frac{1}{\lambda_d(D)} \int_D x_j \, dx_1 \ldots dx_d = x_G^j}$$

Rmk: $d=1$ $\qquad \frac{1}{b-a}\int_a^b f(x)\,dx$

To det the baryc. of D, we have to det

$$(x_G, y_G) \quad \text{where} \quad x_G = \frac{1}{\lambda_2(D)} \int_D x \, dx\, dy$$

$$y_G = \frac{1}{\lambda_2(D)} \int_D y \, dx\, dy$$

In our case $\lambda_2(D) = \frac{\pi r^2}{4}$.

$$\int_D x \, dx\, dy \overset{RF}{=} \int_0^r \int_0^{\sqrt{r^2-x^2}} x \, dy \, dx$$

$\parallel$

$f(x,y) \in \mathcal{C}(D)$ D compact $\Rightarrow f \in L^1(D)$

$$= \int_0^r x \underbrace{\int_0^{\sqrt{r^2-x^2}} dy}_{\sqrt{r^2-x^2}} \, dx$$

$$= \int_0^r x\sqrt{r^2-x^2} \, dx = -\frac{1}{3}\left[(r^2-x^2)^{3/2}\right]_{x=0}^{x=r}$$

$(r^2-x^2)^{1/2} \cdot x$

$$\partial_x (r^2-x^2)^{3/2} = \frac{3}{\cancel{2}}(r^2-x^2)^{1/2}(-\cancel{2}x)$$

$$= -\frac{1}{3}\left[0 - (r^2)^{3/2}\right] = \frac{r^3}{3}.$$

$$\Rightarrow x_G = \frac{1}{\frac{\pi r^2}{4}} \cdot \frac{r^3}{3} = \frac{4}{3}\frac{r}{\pi}.$$

4

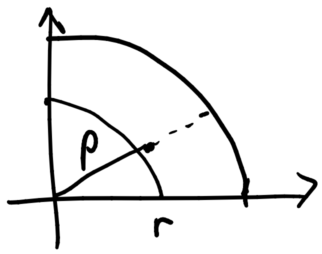
$$y_G = \frac{1}{\frac{\pi r^2}{4}} \int_D y \, dx\,dy$$

$$\overset{RF}{=} \boxed{\int_0^r \int_0^{\sqrt{r^2-y^2}} y \; dx \; dy} = \frac{4}{3}\frac{r}{\pi}.$$

Let's now compute

$$\int_D \frac{x+y}{x^2+y^2}\, dx\,dy = \int \frac{\rho\cos\theta + \rho\sin\theta}{\rho^2}\cdot \rho \; d\rho\, d\theta$$

$$\Phi^{-1} \begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \quad \begin{matrix} 0 \le \rho \le r \\ 0 \le \theta \le \pi/2 \end{matrix}$$



$$= \int_{\substack{0 \le \rho \le r \\ 0 \le \theta \le \pi/2}} \underbrace{(\cos\theta + \sin\theta)}\; d\rho\, d\theta$$

$$\overset{RF}{=} \int_0^{\pi/2} \left(\int_0^r \overset{1}{(\cos\theta + \sin\theta)} \; d\rho \right) d\theta$$

$$= \int_0^{\pi/2} (\cos\theta + \sin\theta)\, r \, d\theta = r\left(\int_0^{\pi/2} \cos\theta \, d\theta + \int_0^{\pi/2} \sin\theta \, d\theta \right)$$

$$\| \qquad [\sin\theta]_0^{\pi/2} + [-\cos\theta]_0^{\pi/2}$$

$$= r\left([1-0] + [-0-(-1)] \right) = 2r. \quad \blacksquare$$

4.5.7.

#2 $$\int_{x^2+y^2\leq 4} \sqrt{4-x^2-y^2}\,dx\,dy$$

Here $f(x,y) = \sqrt{4-x^2-y^2} = \sqrt{4-(x^2+y^2)}$

is well defd and cont on $\underbrace{\{x^2+y^2\leq 4\}}_{D}$, compact

$\Rightarrow f\in L^1(D)$. Changing vars $\rho^2\leq 4 \Rightarrow 0\leq\rho\leq 2$

$\theta\in[0,2\pi]$

$$\int_D f\,dx\,dy \underset{\substack{x=\rho\cos\theta\\ y=\rho\sin\theta}}{=} \int_{\substack{0\leq\rho\leq 2\\ 0\leq\theta\leq 2\pi}} \sqrt{4-\rho^2}\cdot\rho\,d\rho\,d\theta$$

$$\overset{\text{RF}}{=} \int_0^2 \rho\sqrt{4-\rho^2}\underbrace{\int_0^{2\pi} d\theta}_{2\pi}\,d\rho = \frac{2\pi}{-3}\int_0^2 (-3)\rho(4-\rho^2)^{1/2}\,d\rho$$

$$\partial_\rho\left[(4-\rho^2)^{3/2}\right] = \frac{3}{2}(4-\rho^2)^{1/2}(-2\rho)$$

$$= -3(4-\rho^2)^{1/2}\rho$$

$$= -\frac{2}{3}\pi\underbrace{\left[(4-\rho^2)^{3/2}\right]_{\rho=0}^{\rho=2}}_{0-4^{3/2}=-8} = \frac{16}{3}\pi. \quad \boxtimes$$

#3 $$\int_{x^2+2y^2\leq 1} \frac{1}{1+x^2+2y^2}\,dx\,dy =$$

$x = \rho\cos\vartheta$

$y = \rho\sin\vartheta$

$$= \int \frac{1}{1+\rho^2+\rho^2(\sin\vartheta)^2}\,\rho\,d\rho\,d\vartheta \quad ?$$

$$\rho^2+\rho^2(\sin\vartheta)^2 \leq 1$$

$$x^2+2y^2 = x^2+(\sqrt{2}y)^2 \qquad \boxed{\begin{array}{l} x = \rho\cos\vartheta \\ \sqrt{2}y = \rho\sin\vartheta \end{array}}$$

$$\Updownarrow$$

$$\phi^{-1} \qquad \begin{cases} x = \rho\cos\vartheta \\ y = \frac{\rho}{\sqrt{2}}\sin\vartheta \end{cases}$$

Let's compute $|\det((\phi^{-1})'(\rho,\vartheta))|$

More in general

$$\phi^{-1} \quad \begin{cases} x = \alpha\rho\cos\vartheta \\ y = \beta\rho\sin\vartheta \end{cases} \qquad \det((\phi^{-1})'(\rho,\vartheta)) =$$

$$= \alpha\beta\rho$$

$$\Rightarrow \int_{x^2+2y^2\leq 1} \frac{1}{1+x^2+2y^2}\,dx\,dy =$$

$$= \int_{0\leq\rho^2\leq 1} \frac{1}{1+\rho^2}\,\frac{1}{\sqrt{2}}\,\rho\,d\rho\,d\vartheta$$

$0\leq\rho\leq 1 \iff 0\leq\rho^2\leq 1$

$0 \le \rho \le 1 \iff 0 \le \rho^2 \le 1 \qquad 1+\rho^2 \qquad \sqrt{2}$

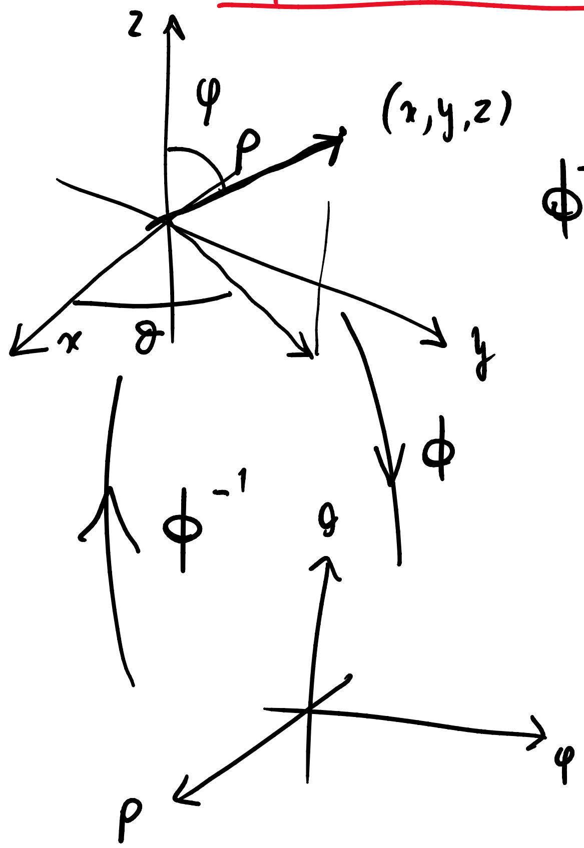
$0 \le \theta \le 2\pi$

$$\text{RF} = \frac{1}{\sqrt{2}} \int_0^1 \int_0^{2\pi} \frac{\rho}{1+\rho^2} \, d\theta \, d\rho$$

$$= \frac{\cancel{2}\pi}{\sqrt{2}} \int_0^1 \frac{2\rho}{1+\rho^2} \, d\rho = \frac{\pi}{\sqrt{2}} \log(1+\rho^2) \Big|_{\rho=0}^{\rho=1}$$

$$= \frac{\pi}{\sqrt{2}} \log 2. \quad \blacksquare$$

Spherical Coords



$$\Phi^{-1} \begin{cases} x = \boxed{\rho \sin\varphi \cos\theta} \\ y = \boxed{\rho \sin\varphi \sin\theta} \\ z = \rho\cos\varphi \end{cases} \qquad \begin{array}{l} S = \sin\varphi \\ C = \cos\varphi \\ s = \sin\theta \\ c = \cos\theta \end{array}$$

$\rho \ge 0, \quad \varphi \in [0,\pi], \quad \theta \in [0, 2\pi]$

$$\left| \det (\Phi^{-1})'(\rho,\theta,\varphi) \right|$$

$$\det \begin{array}{c} \begin{array}{ccc} \partial_\rho & \partial_\varphi & \partial_\theta \end{array} \\ \begin{bmatrix} Sc & \rho Cc & -\rho Ss \\ Ss & \rho Cs & \rho Sc \\ C & -\rho S & 0 \end{bmatrix} \end{array}$$

$$= C \det \begin{bmatrix} \rho Cc & -\rho Ss \\ \rho Cs & \rho Sc \end{bmatrix} - (-\rho S) \det \begin{bmatrix} Sc & -\rho Ss \\ Ss & \rho Sc \end{bmatrix}$$

$+0 \cdot \ldots \quad =$

$= C\rho^2 \big(\underbrace{CS c^2 + SCs^2}_{CS(c^2+s^2) \,=\, CS \cdot 1} \big) + \rho^2 S \big(\underbrace{S^2c^2 + S^2 s^2}_{S^2(c^2+s^2) \,=\, S^2 \cdot 1} \big)$

$= \rho^2 S \left[C^2 + S^2 \right] = \rho^2 S$

$= \rho^2 \sin \varphi$

$\Longrightarrow \quad \left| \det (\phi^{-1})'(\rho, \varphi, \theta) \right| = \rho^2 \sin \varphi.$

Ex. 4.3.6 Compute the volume of a sphere of radius r.

Sol: $D = \{ (x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leq r^2 \}$

We want $\lambda_3(D) = \int_D 1 \, dx\,dy\,dz$

$= \int_{\rho \leq r^2 \Leftrightarrow x^2+y^2+z^2 \leq r^2} 1 \quad dx\,dy\,dz$

$= \int 1 \cdot \rho^2 \sin\varphi \; d\rho \, d\varphi \, d\theta$

$x = \rho \sin\varphi \cos\theta \qquad \boxed{0 \leq \rho \leq r}$

$y = \rho \sin\varphi \sin\theta \qquad \boxed{0 \leq \theta \leq 2\pi}$

$y = \rho \sin\varphi \sin\theta$
$z = \rho \cos\varphi$

$$\boxed{\begin{array}{l} 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \pi \end{array}}$$

$$\overset{\mathrm{RF}}{=} \int_0^{\pi} \int_0^{2\pi} \left(\int_0^r \rho^2 \sin\varphi \, d\rho \right) d\theta \, d\varphi$$

$$= \int_0^{\pi} \sin\varphi \int_0^{2\pi} \left(\int_0^r \rho^2 \, d\rho \right) d\theta \, d\varphi$$

$$\int_0^r \rho^2 \, d\rho = \left[\frac{\rho^3}{3} \right]_0^r = \frac{r^3}{3}$$

$$= \frac{r^3}{3} \int_0^{\pi} \sin\varphi \int_0^{2\pi} d\theta \, d\varphi = \frac{2\pi r^3}{3} \int_0^{\pi} \sin\varphi \, d\varphi$$

$$\int_0^{2\pi} d\theta = 2\pi$$

$$\int_0^{\pi} \sin\varphi \, d\varphi = \left[-\cos\varphi \right]_0^{\pi} = 1 - (-1) = +2$$

$$= \frac{4}{3} \pi r^3 . \quad \blacksquare$$

Ex 4.5.6

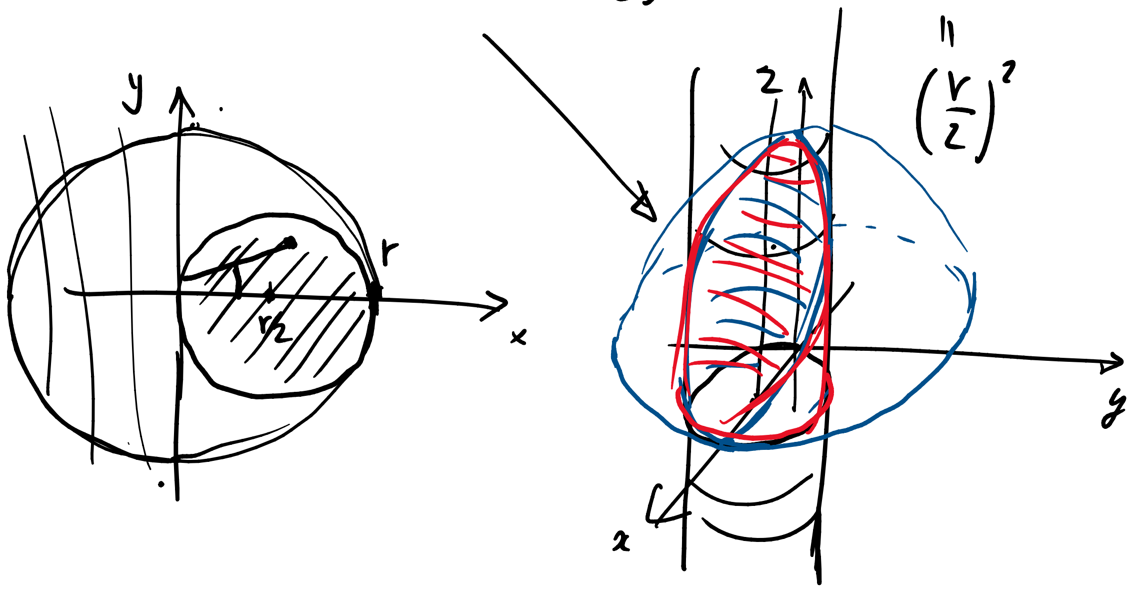
#2. Compute the volume of

$$D := \left\{ (x,y,z) \in \mathbb{R}^3 : \; x^2 + y^2 + z^2 \le r^2 , \; \left(x - \frac{r}{2}\right)^2 + y^2 \le \frac{r^2}{4} \right\}$$

$x^2 - rx + \frac{r^2}{4} + y^2$

Sol: To draw D, let's first ~~see~~ give a look to

$$\left\{ (x,y,z) : \; \left(x - \frac{r}{2}\right)^2 + y^2 \le \frac{r^2}{4} \right\}$$



$$\lambda_3(D) = \int_D 1 \, dx\,dy\,dz =$$

$$\begin{cases} x = \rho \sin\varphi \cos\theta \\ y = \rho \sin\varphi \sin\theta \\ z = \rho \cos\varphi \end{cases}$$

$$x^2+y^2+z^2 \le r^2 \iff \rho^2 \le r^2$$

$$\left(x - \frac{r}{2}\right)^2 + y^2 \le \frac{r^2}{4} \iff$$

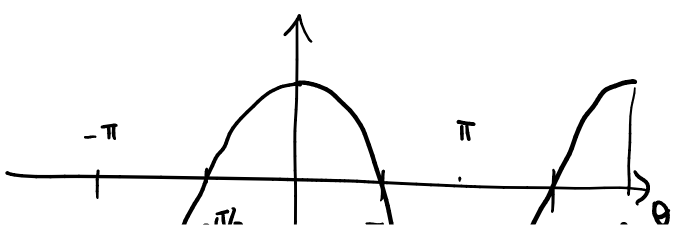
$$x^2 + \cancel{\frac{r^2}{4}} - rx + y^2 \le \cancel{\frac{r^2}{4}}\ 0$$

$$\iff x^2 + y^2 - rx \le 0 \iff \rho^2 (\sin\varphi)^2 - r\rho\sin\varphi\cos\theta \le 0$$

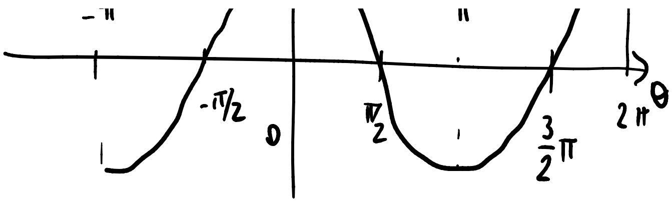
$$\overset{\rho,\ \sin\varphi \ge 0}{\iff} \rho\sin\varphi - r\cos\theta \le 0$$

$$\iff \underbrace{\rho\sin\varphi}_{\oplus} \le r\cos\theta \Rightarrow \cos\theta \ge 0$$

$$\Updownarrow$$



$0 \le \theta \le \pi/2 \ \vee\ 3\pi/2 \le \theta \le 2\pi$



$\Downarrow$

$0 \leq \theta \leq \pi/2 \ \vee \ \frac{3}{2}\pi \leq \theta \leq 2\pi$

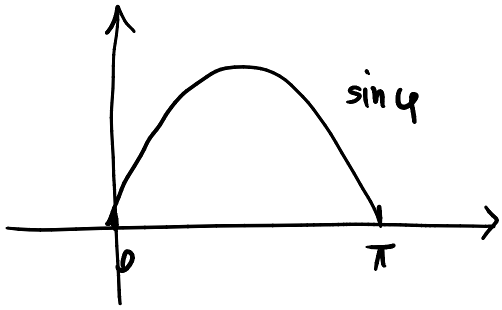
or, better

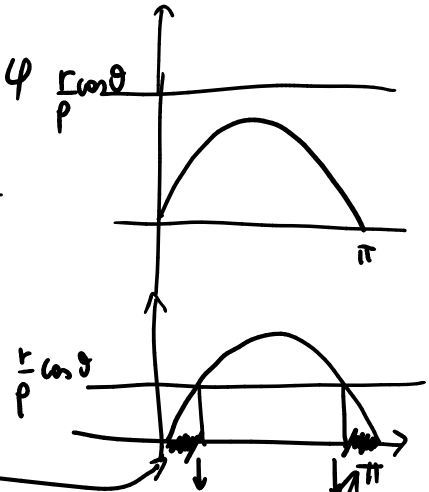
$$\boxed{-\pi/2 \leq \theta \leq \pi/2}$$

$\cos\theta \geq 0 \iff$

$\iff \sin\varphi \leq \frac{r}{\rho}\cos\theta$

Now:





if $\frac{r}{\rho}\cos\theta \geq 1 \implies 0 \leq \varphi \leq \pi$

if $\frac{r}{\rho}\cos\theta < 1 \implies$

$0 \leq \varphi \leq \arcsin\left(\frac{r}{\rho}\cos\theta\right) \ \vee$

$\pi - \arcsin\left(\frac{r}{\rho}\cos\theta\right) \leq \varphi \leq \pi$

$\frac{r}{\rho}\cos\theta \geq 1 \iff \rho \leq r\cos\theta$

$\frac{r}{\rho}\cos\theta < 1 \iff \rho > r\cos\theta$

$$\implies \lambda_3(D) = \int_D 1 \, dx\,dy\,dz = \int_{\substack{-\frac{\pi}{2} \leq \theta \leq \pi/2 \\ 0 \leq \rho \leq r \\ \sin\varphi \leq \frac{r}{\rho}\cos\theta}} \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

$$\int^{\pi/2} \left(\int^{r\cos\theta} \int^{\pi} \rho^2 \ldots \, d\rho \ldots + \int^{r} \rho^2 \left(\int^{\arcsin(\ )} \rho^2 \sin\varphi \, d\varphi + \int^{\pi} \rho^2 \sin\varphi \, d\varphi \right) d\rho \right) d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left(\int_0^{r\cos\theta} \int_0^{\pi} \rho^2 \sin\varphi \, d\varphi \, d\rho + \int_{r\cos\theta}^{r} \rho^2 \left(\int_0^{\arcsin(\ )} \rho^2 \sin\varphi \, d\varphi + \int_{\pi-\arcsin}^{\pi} \rho^2 \sin\varphi \, d\varphi \right) d\rho \right) d\theta$$

$(*) \parallel$

$\downarrow$

below

$$\int_0^{r\cos\theta} \int_0^{\pi} \rho^2 \sin\varphi \, d\varphi \, d\rho =$$

$$= \int_0^{r\cos\theta} \rho^2 \underbrace{\int_0^{\pi} \sin\varphi \, d\varphi}_{\substack{\parallel \\ -\cos\varphi\big|_0^{\pi}}} \, d\rho = 2\int_0^{r\cos\theta} \rho^2 \, d\rho$$

$$= \frac{2}{3} \rho^3 \Big|_0^{r\cos\theta}$$

$$= \frac{2}{3} r^3 (\cos\theta)^3.$$

$$\int_0^{\arcsin(\ )} \sin\varphi \, d\varphi = -[\cos\varphi]_0^{\arcsin(\ )} = -\left[\sqrt{1-\sin(\arcsin(\ ))^2} - 1\right]$$

$$= 1 - \sqrt{1 - \left(\frac{r}{\rho}\cos\theta\right)^2}$$

$$\int_{\pi-\arcsin(\ )}^{\pi} \sin\varphi \, d\varphi = -[\cos\varphi]_{\pi-\arcsin(\ )}^{\pi} = -\left[-1 - \underbrace{\cos(\pi - \arcsin(\ ))}_{\substack{\parallel \\ (-\cos(\arcsin(\ )))}}\right]$$

$$= 1 - \sqrt{1 - \left(\frac{r}{\rho}\cos\theta\right)^2}$$

$$\int_{r\cos\theta}^{r} 2\rho^2 \left[1 - \sqrt{1 - \frac{r^2}{\rho^2}(\cos\theta)^2}\right] d\rho = 2\left(\left[\frac{\rho^3}{3}\right]_{r\cos\theta}^{r} - \int_{r\cos\theta}^{r} \underbrace{\rho\sqrt{\rho^2 - r^2(\cos\theta)^2}}_{\substack{\parallel \\ \rho(\rho^2-(r\cos\theta)^2)^{1/2} = \frac{1}{3}\partial_\rho (\rho^2-(r\cos\theta)^2)^{3/2}}} d\rho\right) =$$

$$= \frac{2}{3}\left(r^3 - r^3(\cos\theta)^3 - \left[\rho^2 - r^2(\cos\theta)^2\right]^{3/2}\Big|_{r\cos\theta}^{r}\right)$$

$$= \frac{2}{3}\left(r^3 - r^3(\cos\theta)^3 - \left(\left(\underset{\substack{\| \\ r^3|\sin\theta|^3}}{\left(r^2(1-(\cos\theta)^2)\right)^{3/2}} - 0\right)\right)\right) \qquad \theta)^2$$

$$= \frac{2}{3}r^3\left(1 - (\cos\theta)^3 - |\sin\theta|^3\right)$$

Returning to $\overset{(*)}{=}$ ↑ above

$$\lambda_3(D) = \int_{-\pi/2}^{\pi/2} \cancel{\frac{2}{3}r^3(\cos\theta)^3} + \frac{2}{3}r^3\left(1 - \cancel{(\cos\theta)^3} - |\sin\theta|^3\right) d\theta$$

$$= \frac{2}{3}r^3\left[\pi - \int_{-\pi/2}^{\pi/2} |\sin\theta|^3 \, d\theta\right]$$

$$= \frac{2}{3}r^3\left[\pi - 2\int_0^{\pi/2} (\sin\theta)^3 \, d\theta\right].$$

Now

$$\int_0^{\pi/2} (\sin\theta)^3 \, d\theta = \int_0^{\pi/2} (\sin\theta)^2 (-\cos\theta)' \, d\theta \qquad 1-(\sin\theta)^2$$

$$= \underbrace{-(\sin\theta)^2\cos\theta\Big|_0^{\pi/2}}_{=0} + \int_0^{\pi/2} 2\sin\theta \overset{\substack{1-(\sin\theta)^2 \\ \|}}{(\cos\theta)^2} \, d\theta$$

$$= 2\int_0^{\pi/2} \sin\theta \, d\theta - 2\int_0^{\pi/2} (\sin\theta)^3 \, d\theta$$

$$= 2 - 2 \int_0^{\pi/2} (\sin\vartheta)^3 \, d\vartheta$$

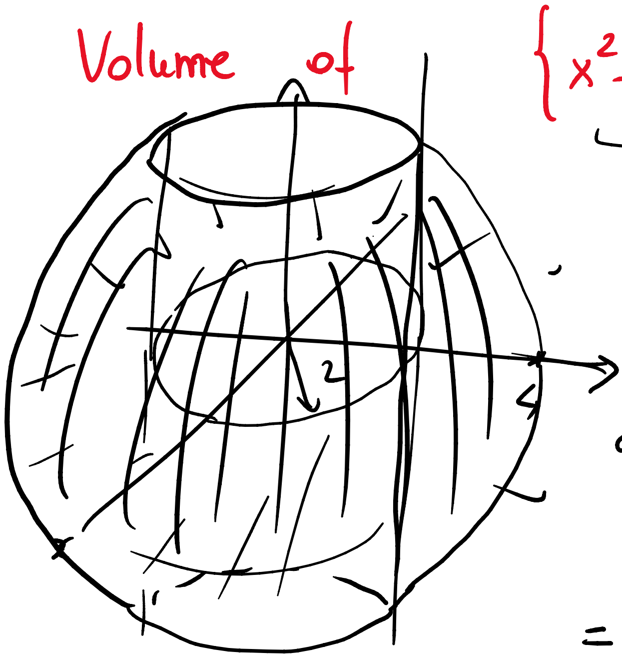
$$\Rightarrow \int_0^{\pi/2} (\sin\vartheta)^3 \, d\theta = \frac{2}{3}.$$

In conclusion

$$\Rightarrow \lambda_3(D) = \frac{2}{3} r^3 \left[\pi - \frac{4}{3}\right] \quad \blacksquare$$

This exercise is really tough! For an alternative and easier solution, use cylindrical coordinates. Solution is given at the end of these slides. (but first you try to do it!)

#6 Volume of $\{\underbrace{x^2+y^2+z^2 \leq 16}_{\text{sphere centered at } (0,0,0)}, \underbrace{x^2+y^2 \geq 4}\}$



$$\lambda_3(D) = \int_D 1 \, dx\,dy\,dz$$

$$= \int_{\substack{\rho^2 \leq 16 \\ \rho^2(\sin\varphi)^2 \geq 4 \\ 0 \leq \theta \leq 2\pi}} 1 \cdot \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

$x = \rho \sin\varphi \cos\theta$
$y = \rho \sin\varphi \sin\theta$
$z = \rho \cos\varphi$

$$= \int_{\substack{\boxed{0 \leq \rho \leq 4} \\ \rho \sin\varphi \geq 2 \\ 0 \leq \theta \leq 2\pi}} \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

$\boxed{\sin\varphi \geq \frac{2}{\rho}}$

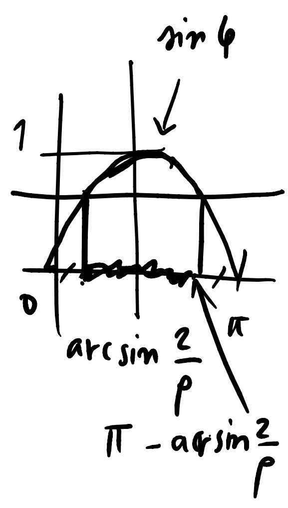
$\frac{2}{\rho} > 1 \quad \rho < 2$

$0 \leq \rho \leq 4$

$\frac{2}{\rho} < 1 \iff 2 < \rho$

$$= \int_{2 \leq \rho \leq 4,\ \sin\varphi \geq \frac{2}{\rho},\ 0 \leq \theta \leq 2\pi} \rho^2 \sin\varphi \quad d\rho\, d\varphi\, d\theta$$

$2/\rho$



RF
$$= \int^{2\pi} \left(\int^{4} \int^{\pi - \arcsin\frac{2}{\rho}} \rho^2 \sin\varphi \quad d\varphi \, d\rho \right) d\theta$$

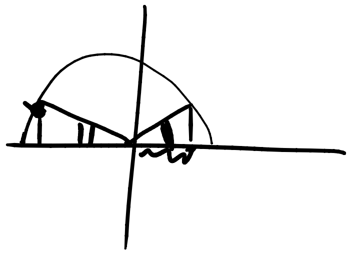
$$RF = \int_0^{2\pi} \underbrace{\left(\int_2^4 \int_{\arcsin \frac{2}{\rho}}^{\pi - \arcsin \frac{2}{\rho}} \rho^2 \sin\varphi \quad d\varphi \; d\rho \right)} d\theta \qquad \pi - \arcsin \frac{2}{\rho}$$

$$= -2\pi \int_2^4 \rho^2 \; (+\cos\varphi)\Big|_{\arcsin \frac{2}{\rho} \;\in [-\pi/2, \pi/2]}^{\pi - \arcsin \frac{2}{\rho} \;\in\; ''} \qquad \overset{(*)}{=}$$

$$\to \cos \arcsin \frac{2}{\rho} = \sqrt{1 - \left(\frac{2}{\rho}\right)^2} = \sqrt{\frac{\rho^2 - 4}{\rho^2}} = \frac{1}{\rho}\sqrt{\rho^2 - 4}$$

$$\cos = \sqrt{1 - \sin^2}$$

$$\cos\left(\pi - \arcsin \frac{2}{\rho}\right)$$



$$= -\cos\left(\arcsin \frac{2}{\rho}\right)$$

$$\to = -\frac{1}{\rho}\sqrt{\rho^2 - 4}$$

$$\Rightarrow \left[\cos\theta\right]_{\theta = \arcsin 2/\rho}^{\theta = \pi - \arcsin \frac{2}{\rho}} = -\frac{2}{\rho}\sqrt{\rho^2 - 4}.$$

$$\Rightarrow \overset{(*)}{=} -2\pi \int_2^4 \rho^2 \cdot \left(-\frac{2}{\rho}\sqrt{\rho^2 - 4} \right) d\rho$$

$$= \frac{4}{3}\pi \int_2^4 3\rho \sqrt{\rho^2 - 4} \; d\rho$$

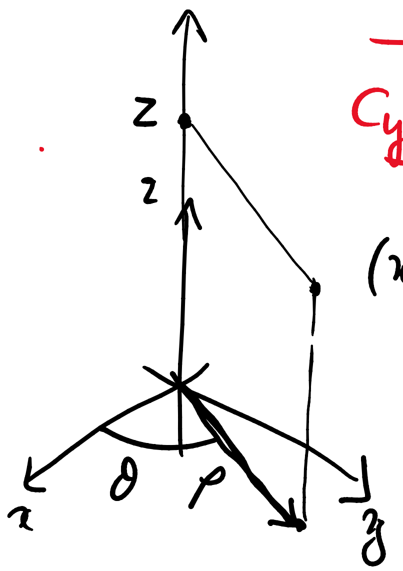
$$\|$$

$$\partial_\rho \left(\rho^2 - 4\right)^{3/2} = \frac{3}{\cancel{2}}\left(\rho^2 - 4\right)^{1/2} \cdot \cancel{2}\rho$$

$$\partial_\rho (\rho^2-4)^{3/2} = \frac{3}{\cancel{2}}(\rho^2-4)^{1/2}\cdot\cancel{2}\rho$$

$$= \frac{4}{3}\pi\left[(\rho^2-4)^{3/2}\right]_{\rho=2}^{\rho=4} = \frac{4}{3}\pi\left[12^{3/2}\right].$$

∎



Cylindrical Coords

$$(x,y,z) \quad \phi^{-1}: \begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z \end{cases} \qquad \boxed{\begin{array}{c} \rho > 0 \\ 0 \leq \theta \leq 2\pi \\ z \in \mathbb{R} \end{array}}$$

$$\det(\phi^{-1})' = \det \begin{array}{c} \begin{array}{ccc} \partial_\rho & \partial_\theta & \partial_z \end{array} \\ \begin{bmatrix} \cos\theta & -\rho\sin\theta & 0 \\ \sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$= \rho$$

$$\boxed{|\det(\phi^{-1})'| = \rho}$$

4.5.6

#1. $D = \left\{ (x,y,z) : 9\left(1-\sqrt{x^2+y^2}\right)^2 + 4z^2 \leq 1 \right\}$

$$\lambda_3(D) = \int_D 1 \, dx\,dy\,dz$$

$$=$$

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \qquad (x,y,z) \in D \Leftrightarrow$$

$$\begin{cases} y = \rho \sin\theta \\ z = z \end{cases} \qquad (x,y,z) \in D \Leftrightarrow$$

$$9\left(1 - \sqrt{\rho^2}\right)^2 + 4z^2 \leq 1$$

$$\Leftrightarrow 9(\rho - 1)^2 + 4z^2 \leq 1$$

$$= \int_{\substack{9(\rho-1)^2 + 4z^2 \leq 1 \\ 0 \leq \theta \leq 2\pi}} 1 \quad \rho \; d\rho \, d\theta \, dz$$

$$= \int_0^{2\pi} \underbrace{\left(\int_{-1/2}^{1/2} \int_{1-\sqrt{\frac{1-4z^2}{9}}}^{1+\sqrt{\frac{1-4z^2}{9}}} \rho \quad d\rho \right) dz}_{} \Big) \, d\theta \overset{(*)}{=}$$

$$9(\rho - 1)^2 + 4z^2 \leq 1$$

$$0 \leq \cancel{9}(\rho-1)^2 \leq \underbrace{1 - 4z^2} \implies \boxed{1 - 4z^2 \geq 0 \Leftrightarrow z^2 \leq 1/4 \Leftrightarrow |z| \leq 1/2}$$

$$(\rho - 1)^2 \leq \frac{1-4z^2}{9} \qquad 1 - \sqrt{} > 0 \text{ always.}$$

$$-\sqrt{\frac{1-4z^2}{9}} \leq \rho - 1 \leq \sqrt{\frac{1-4z^2}{9}} \qquad \Leftarrow \quad 4z^2 < -8 \text{ No}$$

$$\Updownarrow \qquad\qquad\qquad \Updownarrow$$

$$\underbrace{1 - \sqrt{\frac{1-4z^2}{9}}} \leq \rho \leq 1 + \sqrt{\frac{1-4z^2}{9}} \qquad 1 - 4z^2 > 9$$

$$\Updownarrow$$

$$1 - \sqrt{\frac{1-4z^2}{9}} \overset{?}{<} 0 \Leftrightarrow 1 < \sqrt{\frac{1-4z^2}{9}} \Leftrightarrow 1 < \frac{1-4z^2}{9}$$

$$(*) \quad \int^{1/2} \left(\int^{1 + \sqrt{\frac{1-4z^2}{9}}} \right.$$

$$\overset{(*)}{=} \cancel{2}\pi \int_{-1/2}^{1/2} \left(\int_{1-\sqrt{\frac{1-4z^2}{9}}}^{1+\sqrt{\frac{1-4z^2}{9}}} \rho \, d\rho \right) dz$$

$$\parallel \quad \frac{\rho^2}{\cancel{2}}\Big|$$

$$= \pi \int_{-1/2}^{1/2} \left(1+\sqrt{}\right)^2 - \left(1-\sqrt{}\right)^2 dz$$

$$\cancel{1} + \cancel{\frac{1-4z^2}{9}} + 2\sqrt{} - \left(\cancel{1} + \cancel{\frac{1-4z^2}{9}} - 2\sqrt{}\right)$$

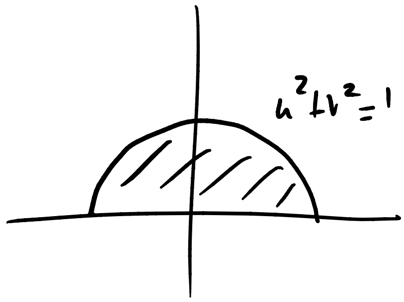
$$= 4\pi \int_{-1/2}^{1/2} \sqrt{\frac{1-4z^2}{9}} \, dz =$$

$2z = u$

$dz = \frac{du}{2}$

$$= \frac{\overset{2}{\cancel{4}}}{3}\pi \int_{-1}^{1} \sqrt{1-u^2} \, \frac{du}{\cancel{2}} = \frac{2}{3}\pi \cdot \frac{\pi}{2} = \frac{\pi^2}{3}. \blacksquare$$

$$\parallel \quad \frac{\pi}{2} \cdot 1^2$$



Do Ex 4.5.6 #3, #4 (**)
#5,
#7, #8

4.5.7 #6, #7.

Do Ex. 4.5.6 #2 by using cyl. coords (easier)

You may find this calculation here below (please, try first by yourself):

try first by yourself):

$$\lambda_3(D) = \int_D 1 \, dx\,dy\,dz$$

$$D = \left\{ x^2+y^2+z^2 \leq r^2, \ \left(x-\frac{r}{2}\right)^2+y^2 \leq \frac{r^2}{4} \right\}$$

$=$

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z \end{cases}$$

$x^2+y^2+z^2 \leq r^2 \iff \rho^2+z^2 \leq r^2$

$\left(x-\frac{r}{2}\right)^2+y^2 \leq \frac{r^2}{4}$

$\Updownarrow$

$x^2+y^2-rx \leq 0 \iff \rho^2 - r\rho\cos\theta \leq 0$

$\Updownarrow \ \rho \geq 0$

$\rho - r\cos\theta \leq 0$

$$= \int_{\substack{\rho^2+z^2 \leq r^2 \\ \boxed{\rho \leq r\cos\theta}}} \rho \, d\rho\,d\theta\,dz$$

because $\rho \geq 0 \Rightarrow \cos\theta \geq 0 \Rightarrow -\pi/2 \leq \theta \leq \pi/2$

$\iff z^2 \leq r^2-\rho^2$ ($\Rightarrow \rho \leq r$, this is automatic because $\rho \leq r\cos\theta \leq r$)

$\iff |z| \leq \sqrt{r^2-\rho^2}$

$$= \int_{-\pi/2}^{\pi/2} \left(\int_0^{r\cos\theta} \left(\int_{-\sqrt{r^2-\rho^2}}^{+\sqrt{r^2-\rho^2}} \rho \, dz \right) d\rho \right) d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left(\int_0^{r\cos\theta} \rho \int_{}^{+\sqrt{}} 1 \, dz \, d\rho \right) d\theta$$

$$= \int_{-\pi/2} \left(\int_0 \rho \underbrace{\int_{-\sqrt{\ }} 1 \, dz}_{= 2\sqrt{r^2-\rho^2}} \, d\rho \right) d\theta$$

$$= \int_{-\pi/2}^{+\pi/2} \underbrace{\int_0^{r\cos\theta} 2\rho\,(r^2-\rho^2)^{1/2} \, d\rho}_{= -\partial_\rho \frac{2}{3}(r^2-\rho^2)^{3/2}} \, d\theta$$

$$= \frac{2}{3} \int_{-\pi/2}^{\pi/2} - \left[(r^2-\rho^2)^{3/2} \right]_0^{r\cos\theta} d\theta$$

$$= \frac{2}{3} \int_{-\pi/2}^{\pi/2} - \left[\underbrace{\left(r^2 - r^2(\cos\theta)^2\right)^{3/2}}_{= r^3\left[(\sin\theta)^2\right]^{3/2} = r^3(\sin\theta)^3} - \underbrace{(r^2)^{3/2}}_{= r^3} \right] d\theta$$

$$= \frac{2}{3} \int_{-\pi/2}^{\pi/2} r^3 \left(1 - |\sin\theta|^3\right) d\theta$$

$$= \frac{2}{3} \, 2 r^3 \int_0^{\pi/2} 1 - (\sin\theta)^3 \, d\theta$$

$$= \frac{4}{3} r^3 \left[\frac{\pi}{2} - \int_0^{\pi/2} (\sin\theta)^3 \, d\theta \right]$$

$$\int_0^{\pi/2} (\sin\theta)^3 \, d\theta = \int_0^{\pi/2} (\sin\theta)^2 (-\cos\theta)' \, d\theta$$

$$= - \underbrace{\cos\theta(\sin\theta)^2 \Big|_0^{\pi/2}}_{=} + \int_0^{\pi/2} 2\sin\theta\,(\cos\theta)^2 \, d\theta$$

$$\begin{array}{c} \| \\ 0 \end{array}$$

$$= 2 \int_0^{\pi/2} \sin\theta \left(1 - (\sin\theta)^2\right) d\theta$$

$$= 2 \left[-\cos\theta \Big|_0^{\pi/2} - \int_0^{\pi/2} (\sin\theta)^3 \, d\theta \right]$$

$$= 2 - 2 \int_0^{\pi/2} (\sin\theta)^3 \, d\theta$$

$$\Rightarrow \quad 3 \int_0^{\pi/2} (\sin\theta)^3 \, d\theta = 2 \quad \Rightarrow \quad \int_0^{\pi/2} (\sin\theta)^3 \, d\theta = \frac{2}{3}$$

Conclusion: $\lambda_3(D) = \frac{4}{3} r^3 \left[\frac{\pi}{2} - \frac{2}{3} \right]$