Ex 4.5.6

#8 Volume of $S = \{(x,y,z) \in \mathbb{R}^3 : z \geq x^2+y^2, \ z \leq 18 - x^2 - y^2 \}$

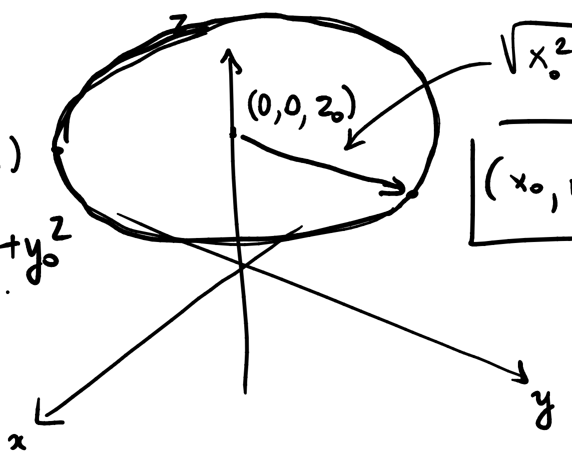
<u>Sol:</u> $S = \{ x^2+y^2 \leq z \leq 18 - (x^2+y^2) \}$

<u>Rmk:</u> If $S = \{(x,y,z) : g(x^2+y^2, z) \geq 0\} \Rightarrow$ S is rotations around z - axis invariant.

$x^2+y^2+z\sin(x^2+y^2) = 0$

$S \ni (x,y,z_0)$

$\underbrace{x^2+y^2} = x_0^2+y_0^2$



$\boxed{(x_0, y_0, z_0) \in S}$

$\Downarrow$

$(x_0, y_0, z_0) \in S$

$\Updownarrow$

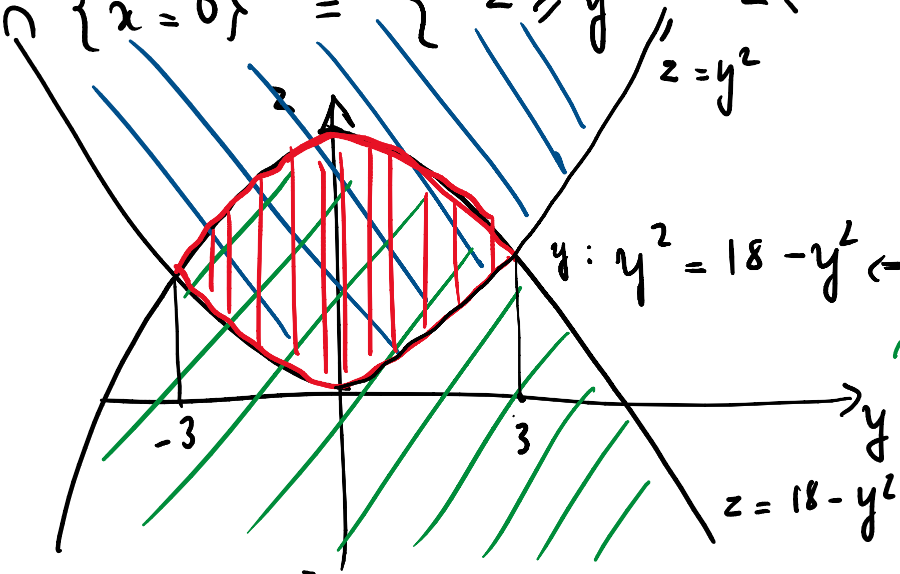
Indeed $g(x^2+y^2, z_0) = g(x_0^2+y_0^2, z_0) \geq 0$

So to plot S we may plot its section on xz (or yz) plane then we rotate
$(y=0)$ $(x=0)$ around the z-axis. ☒

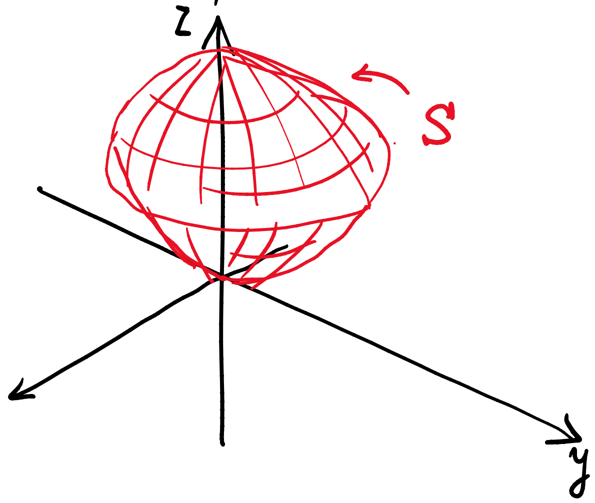
In our case $S = \{ z \geq x^2+y^2, \ z \leq 18 - (x^2+y^2)\}$ is not invariant around z - axis.

Let's see the sect of S on plane $x=0$

$$S \cap \{x=0\} = \{ z \geq y^2,\ z \leq 18 - y^2 \}$$



$y : y^2 = 18 - y^2 \Longleftrightarrow 2y^2 = 18 \quad y^2 = 9 \quad y = \pm 3$



$$\mathrm{Vol}(S) = \int_S 1 \, dx\, dy\, dz$$

$$= \int_{x^2+y^2 \leq z \leq 18 - (x^2+y^2)} 1 \quad dx\, dy\, dz$$

$$= \int_{\substack{\rho^2 \leq z \leq 18 - \rho^2 \\ 0 \leq \theta \leq 2\pi}} 1 \cdot \rho \quad d\rho\, d\theta\, dz$$

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z \end{cases}$$

$\rho \geq 0, \quad \theta \in [0, 2\pi], \quad z \in \mathbb{R}$

$$\overset{RF}{=} \int_{\rho^2 \leq z \leq 18-\rho^2} \int_0^{2\pi} \rho \, d\theta \quad d\rho \, dz$$

$$= 2\pi \int_{\boxed{\rho^2 \leq z \leq 18-\rho^2}} \rho \, d\rho \, dz$$

$\rho^2 \leq 18 - \rho^2$

$\Leftrightarrow \; 2\rho^2 \leq 18 \overset{\rho > 0}{\Leftrightarrow} \rho \leq 3$

$$\cancel{= 2\pi \int_0^{+\infty}} \; ?$$

$$= 2\pi \int_0^3 \left(\int_{\rho^2}^{18-\rho^2} \rho \, dz \right) d\rho$$

$$= 2\pi \int_0^3 \rho \left(18 - \rho^2 - \rho^2 \right) d\rho$$

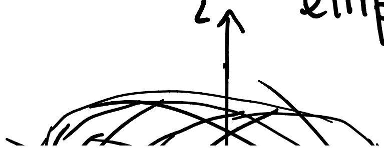
$$= 2\pi \int_0^3 \left(18\rho - 2\rho^3 \right) d\rho$$

$$= 2\pi \left(18 \cdot \frac{\rho^2}{2} \Big|_0^3 - 2 \, \frac{\rho^4}{4} \Big|_0^3 \right)$$

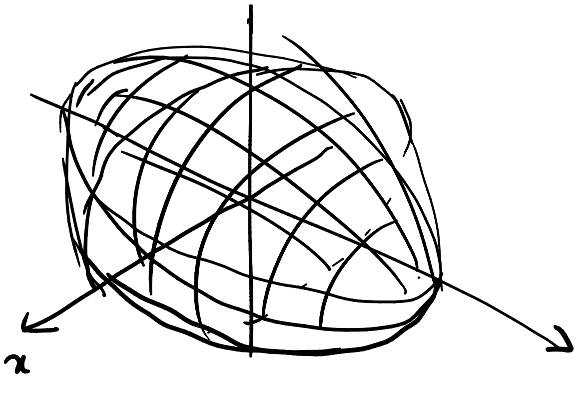
$$= 2\pi \left(18 \, \frac{9}{2} - 2 \, \frac{81}{4} \right) \quad \blacksquare$$

#3. $\mathrm{Vol} \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \right\} \qquad a, b, c > 0$

ellipsoid.



$$= \int 1 \, dx \, dy \, dz$$



$$\int_{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1} 1 \; dx\,dy\,dz$$

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2\le 1$$

We adapt spher. coords:

$$\left\{\begin{aligned} \frac{x}{a} &= \rho\sin\varphi\cos\theta \\ \frac{y}{b} &= \rho\sin\varphi\sin\theta \\ \frac{z}{c} &= \rho\cos\varphi \end{aligned}\right. \qquad \Bigg| \qquad \left\{\begin{aligned} u &= \frac{x}{a} \\ v &= \frac{y}{b} \\ w &= \frac{z}{c} \end{aligned}\right. \leftarrow \Phi$$

$$\left\{\begin{aligned} &\boxed{x = au} \\ &y = bv \\ &z = cw \end{aligned}\right. \qquad \Phi^{-1}$$

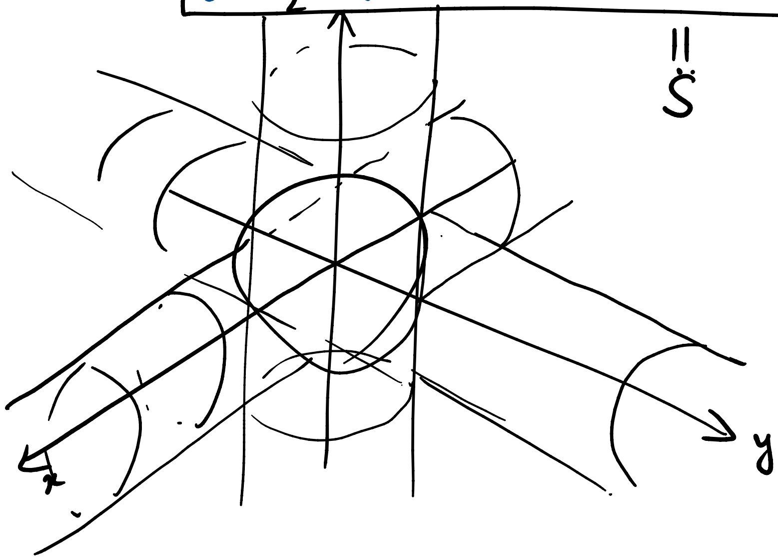
$$\left|\det(\Phi^{-1})'\right| = \left|\det \begin{array}{c} \\ \\ \end{array}\begin{array}{ccc} \partial_u & \partial_v & \partial_w \end{array} \hspace{-7em}\begin{array}{c}\\ \left[\begin{array}{ccc} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{array}\right]\end{array}\right| = |abc| \overset{>0}{=} abc.$$

$$\Rightarrow \int_{\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2\le 1} dx\,dy\,dz =$$

$$= \int_{u^2+v^2+w^2\le 1} abc \; du\,dv\,dw = abc\,\frac{4}{3}\pi. \qquad \boxtimes$$

#4. Vol $\left\{ x^2+y^2\le 1, \boxed{y^2+z^2\le 1, \quad x^2+z^2\le 1} \right\}$

#4. Vol $\underbrace{\{x^2+y^2 \le 1, \ y^2+z^2 \le 1, \ \ldots\}}_{\substack{\| \\ S}}$



$$\int_{x,y} \quad \int_{z}$$

$$\mathrm{Vol}(S) = \int_{\boxed{x^2+y^2\le 1}} 1 \quad dx\,dy\,dz$$

$$\left[\begin{array}{l} z^2 \le 1-x^2 \\ z^2 \le 1-y^2 \end{array}\right. \iff \boxed{|z| \le \sqrt{1-x^2}}$$

$$\Updownarrow$$

$$-\sqrt{1-x^2} \le z \le +\sqrt{1-x^2}$$

$$\Updownarrow$$

$$-\sqrt{1-y^2} \le z \le +\sqrt{1-y^2}$$

$$S_{(x,y)}$$

$$\begin{array}{l} |z| \le \sqrt{1-x^2} \\ |z| \le \sqrt{1-y^2} \end{array} \implies \boxed{|z| \le \min\left(\sqrt{1-x^2}, \sqrt{1-y^2}\right)}$$

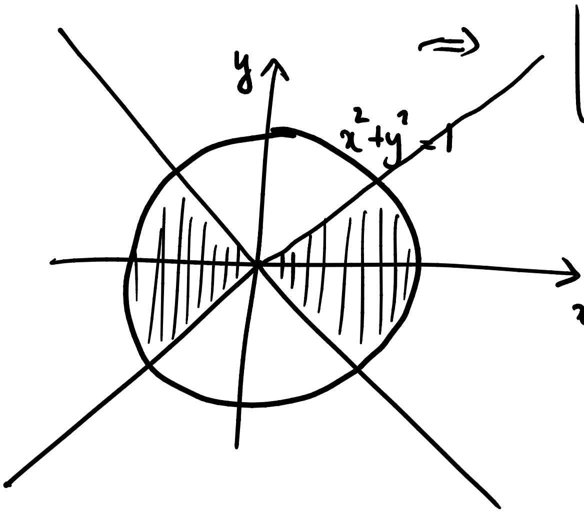
$$= \begin{cases} \sqrt{1-x^2} & \text{if } \sqrt{1-x^2} \le \sqrt{1-y^2} \\ \sqrt{1-y^2} & \text{if } \sqrt{1-x^2} \ge \sqrt{1-y^2} \end{cases}$$

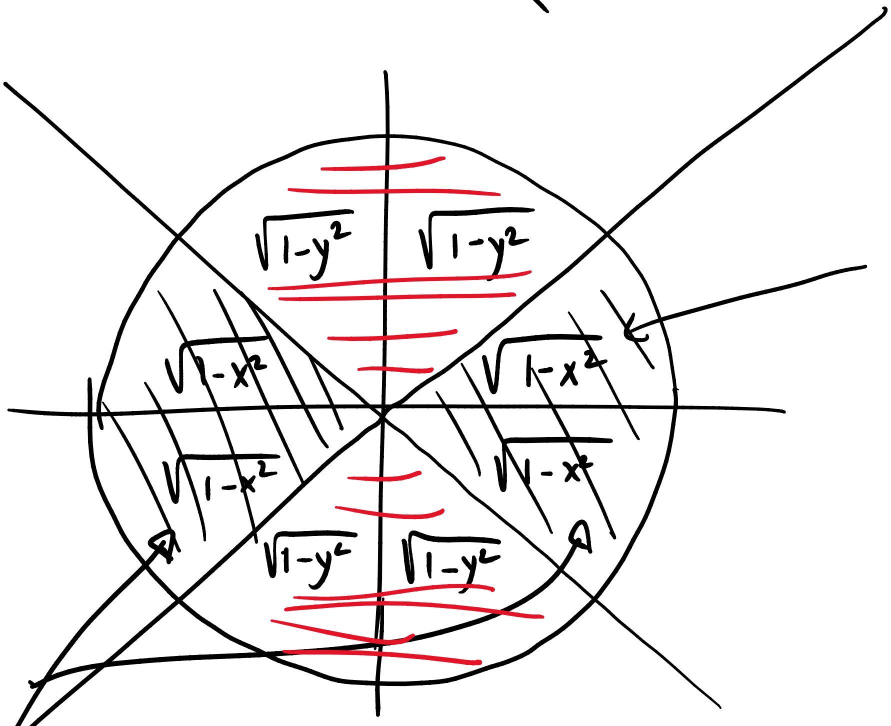
$$\sqrt{1-x^2} \le \sqrt{1-y^2} \iff \cancel{1}-x^2 \le \cancel{1}-y^2 \iff y^2 \le x^2$$

$$\iff |y| \le |x|$$

$\Longleftrightarrow \; |y| \le |x|$

$\Longrightarrow \; \boxed{-|x| \le y \le |x|}$





$\min(\sqrt{1-x^2}, \sqrt{1-y^2})$

$$RF = \int_{x^2+y^2\le 1} \int_{|z| \le \min(\quad)} dz \; dx\,dy$$

$$= \int_{\substack{x^2+y^2\le 1 \\ |y|\le|x|}} \int_{|z|\le\sqrt{1-x^2}} dz \; dx\,dy$$

$$\|$$

$$+ \int \int dz \; dx\,dy$$

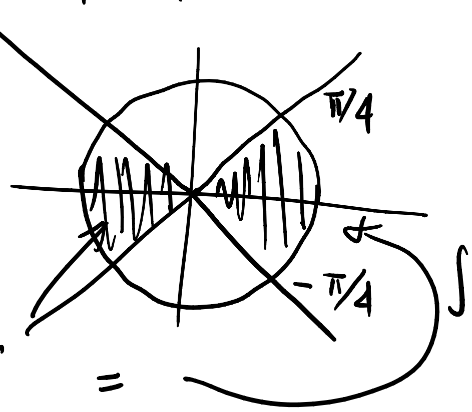
$$+ \int_{\substack{x^2+y^2 \leq 1 \\ |y| > |x|}} \overset{\shortparallel}{\int_{|z| \leq \sqrt{1-y^2}}} dz \quad dx\,dy$$

$$= 2 \int_{\substack{x^2+y^2 \leq 1 \\ |y| \leq |x|}} \Big(\underbrace{\int_{-\sqrt{1-x^2}}^{+\sqrt{1-x^2}} dz}_{\substack{\shortparallel \\ 2\sqrt{1-x^2}}} \Big) dx\,dy$$

$$= 4 \int_{\substack{x^2+y^2 \leq 1 \\ |y| \leq |x|}} \sqrt{1-x^2} \; dx\,dy$$

$$= 2 \cdot 4 \int_{\substack{\rho^2 \leq 1 \\ -\frac{\pi}{4} \leq \theta \leq \pi/4}} \sqrt{1-\rho^2(\cos\theta)^2} \cdot \rho \; d\rho\, d\theta$$

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$



$$\int \; = \; \int$$

$$\overset{RF}{=} 8 \int_{-\pi/4}^{\pi/4} \frac{1}{-3(\cos\theta)^2} \int_0^1 \overset{-3(\cos\theta)^2}{\vphantom{x}} \left(1-\rho^2(\cos\theta)^2\right)^{1/2} \rho \; d\rho \quad d\theta$$

$$(1-c\rho^2)^{1/2}\rho = \partial_\rho \left(1-c\rho^2\right)^{3/2}$$

$$\shortparallel$$

$$\frac{3}{2}(1-c\rho^2)^{1/2} \cdot (-c\cancel{2}\rho)$$

$$\frac{3}{\cancel{2}}(1-c\rho^2)^{1/2}\cdot(-c\cancel{2}\rho)$$

$$= -\frac{8}{3}\int_{-\pi/4}^{\pi/4} \frac{1}{(\cos\theta)^2}\left[1-\rho^2(\cos\theta)^2\right]^{3/2}\Big|_0^1 \, d\theta$$

$$\left(1-(\cos\theta)^2\right)^{3/2} - 1$$

$$\left((\sin\theta)^2\right)^{3/2} = |\sin\theta|^3$$

$$= -\frac{8}{3}\int_{-\pi/4}^{\pi/4} \frac{1}{(\cos\theta)^2}\left(|\sin\theta|^3 - 1\right) d\theta$$

$$= -\frac{8}{3}\cdot 2 \int_0^{\pi/4} \frac{1}{(\cos\theta)^2}\left((\sin\theta)^3 - 1\right) d\theta$$

$$= -\frac{16}{3}\left[\int_0^{\pi/4} \frac{(\sin\theta)^2}{(\cos\theta)^2}\sin\theta \, d\theta - \int_0^{\pi/4}\frac{1}{(\cos\theta)^2}\,d\theta\right]$$

$$(\sin\theta)^2 = 1-\cos^2 \qquad \frac{1}{(\cos\theta)^2} = \partial_\theta \operatorname{tg}\theta$$

$$= -\frac{16}{3}\left[\underbrace{\int_0^{\pi/4}\frac{\sin\theta}{(\cos\theta)^2}\,d\theta} + \int_0^{\pi/4} -\sin\theta\, d\theta\right]$$

$$-\sin\theta = \partial_\theta[\cos\theta]$$

$$\left[\cos\theta\right]_0^{\pi/4} - \left[\operatorname{tg}\theta\right]_0^{\pi/4}$$

$$\frac{\sqrt{2}}{2} - 1 \qquad - \qquad 1$$

$$\partial_\theta \frac{1}{\cos\theta} = +\frac{+\sin\theta}{(\cos\theta)^2}$$

$[\cos\theta]_0^{\pi/4} - [\operatorname{tg}\theta]_0 \Big]$

$\frac{\sqrt{2}}{2} - 1 \quad - \quad 1$

$$\partial_\theta \frac{1}{\cos\theta} = + \frac{+\sin\theta}{(\cos\theta)^2}$$

$$\left[\frac{1}{\cos\theta}\right]_0^{\pi/4} = \frac{2}{\sqrt{2}} - 1$$

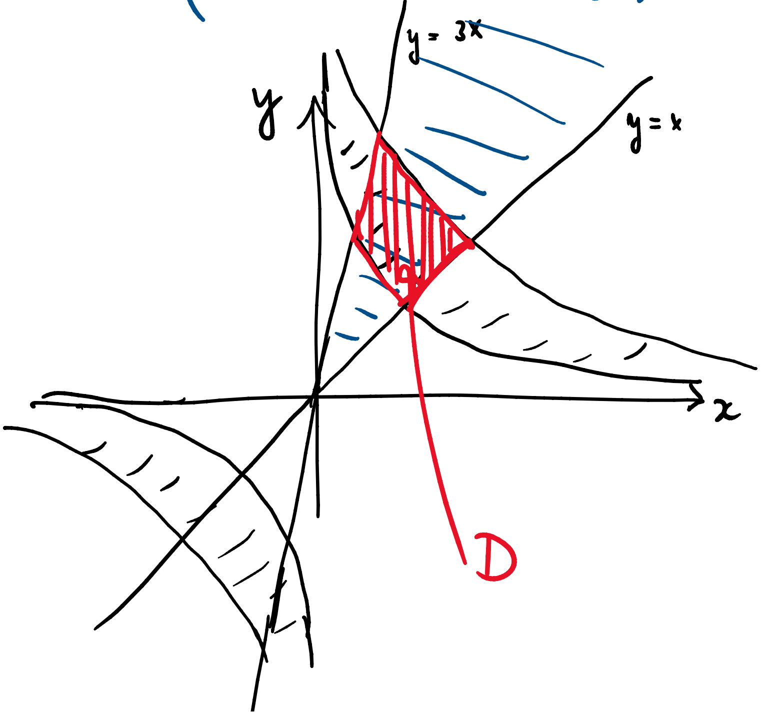
$$= -\frac{16}{3}\left[\frac{2}{\sqrt{2}} - 1 + \frac{\sqrt{2}}{2} - 1 - 1\right]$$

$$= \frac{16}{3}\left[3 - \left(\frac{2}{\sqrt{2}} + \frac{\sqrt{2}}{2}\right)\right] . \quad \blacksquare$$

Ex. 4.5.8

1. $\int_D xy \, dxdy$ $\qquad D = \left\{(x,y) : \begin{array}{l} 1 \le xy \le 3 \\ x \le y \le 3x \end{array}\right\}$

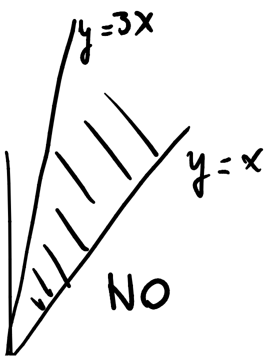
$\left(u = xy, \ v = \frac{y}{x}\right)$



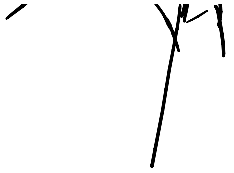
$1 \le xy \le 3 \iff \left\{\begin{array}{ll} \frac{1}{x} \le y \le \frac{3}{x} & x > 0 \\ \frac{3}{x} \le y \le \frac{1}{x} & x < 0 \end{array}\right.$

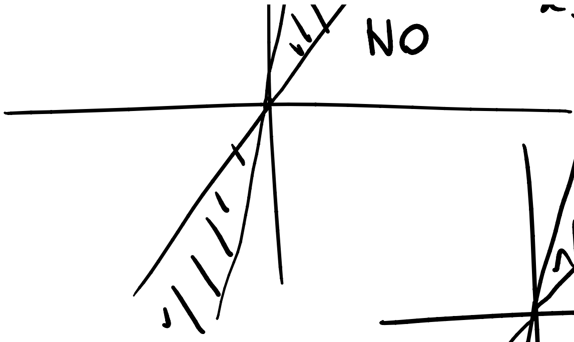
\\\ $1 \le xy \le 3$

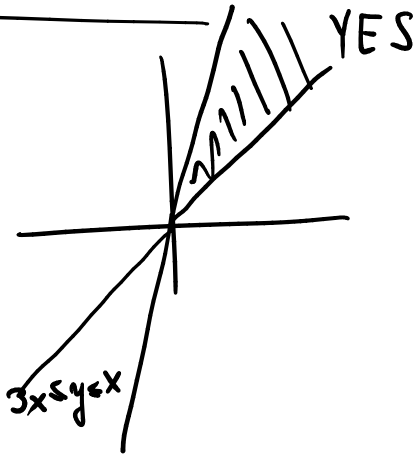
$x \le y \le 3x$



$x \le y \le 3x$







$$\int_D xy \, dx \, dy$$

$$\Phi : \begin{cases} u = xy \\ v = y/x \end{cases}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \Phi \begin{pmatrix} x \\ y \end{pmatrix}$$

$$(u,v) = \Phi(x,y)$$

$$(x,y) = \phi^{-1}(u,v)$$

$$\underline{\left| \det (\phi^{-1})'(u,v) \right|}$$

To compute this

1. either we invert Φ and then we compute the jac. of ϕ^{-1}

or

2. we use the following rmk:

$$\Phi\left(\phi^{-1}(u,v)\right) = \boxed{(u,v)}$$

$\Downarrow$ derive

$$\phi'\left(\phi^{-1}(u,v)\right)(\phi^{-1})'(u,v) = \begin{array}{cc} \partial_u & \partial_v \end{array} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbb{I}$$

$$\Rightarrow 1 = \det \mathbb{I} = \det \left[\phi'\left(\phi^{-1}(u,v)\right) \cdot (\phi^{-1})'(u,v) \right]$$

$$(f^{-1})' = \frac{1}{f'(f^{-1})} \qquad = \det \left[\phi'\left(\phi^{-1}(u,v)\right) \right] \det (\phi^{-1})'(u,v)$$

$$\Rightarrow \quad \det(\phi^{-1})'(u,v) = \frac{1}{\det\left[\phi'\left(\phi^{-1}(u,v)\right)\right]}$$

$$\Rightarrow \quad \det(\phi^{-1})'(u,v) = \frac{}{\det\left[\phi'(\phi^{-1}(u,v))\right]}$$

In practice, in the case of the example

$$\phi(x,y) = \left(xy, \frac{y}{x} \right) =: (u,v)$$

$$\phi' = \begin{array}{cc} \partial_x & \partial_y \end{array}$$
$$\phi' = \begin{bmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{bmatrix} \Rightarrow \det[\phi'] = \frac{y}{x} + \frac{y}{x^2}x$$
$$= 2\, \underbrace{y/x}_{=v}$$

$$\Rightarrow \quad \left| \det \phi'(\phi^{-1}(u,v)) \right| = |2v| = 2|v|$$

$$\Rightarrow \quad \left| \det (\phi^{-1})'(u,v) \right| = \frac{1}{2|v|}.$$

$$\Rightarrow \quad \int_D xy \, dx\,dy = \int_{\substack{1 \le u \le 3 \\ 1 \le v \le 3}} u \cdot \frac{1}{2|v|} \, du\,dv$$

$$(x,y) \in D \iff \begin{cases} 1 \le xy \le 3 & \quad 1 \le u \le 3 \\ x \le y \le 3x \ \underset{\text{on } D}{\overset{\iff}{x>0}} & \quad 1 \le \underbrace{\frac{y}{x}}_{=v} \le 3 \end{cases}$$

$u = xy$

$v = y/x$

$$\overset{\text{RF}}{=} \frac{1}{2} \int_1^3 \int_1^3 \frac{u}{v} \, du \, dv$$

$$= \frac{1}{2} \int_1^3 \frac{1}{v} \cdot \int_1^3 u \, du \, dv.$$

$$= \frac{1}{2} \int_1 \frac{1}{v} \cdot \int_1 u \, du \quad dv$$

$$\frac{u^2}{2} \Big|_1^3 = \frac{1}{2} 8 = 4$$

$$= 2 \left[\log v \right]_1^3$$

$$= 2 \log 3. \quad \blacksquare$$

Do 4.5.8 #2, #3

4.5.9

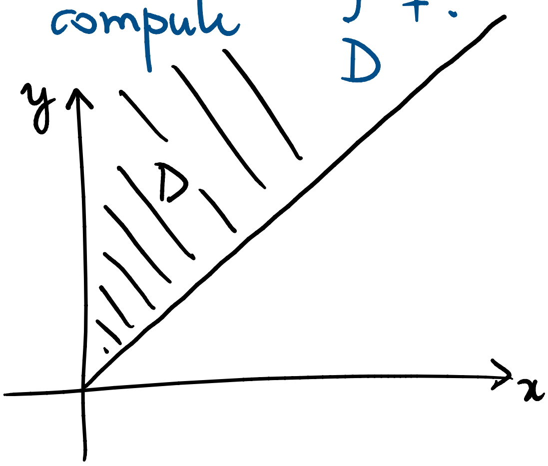
4.5.10 (*)

$$D = \{ (x,y) \in \mathbb{R}^2 : \quad 0 \leq x \leq y \}$$

$$f(x,y) := \frac{x^{3/2}}{\sqrt{y-x}} e^{-(xy)^{3/2}} \qquad (x,y) \in D$$

By using ch. of vars $\begin{cases} u = xy \geq \\ v = x/y \geq 0 \end{cases} : \Phi$

compute $\int_D f$.



Let's compute Φ^{-1} : $\begin{matrix} x = \dots \\ y = \end{matrix} \begin{pmatrix} u \\ v \end{pmatrix}$

$$\begin{aligned} u &= xy \\ v &= \frac{x}{y} \end{aligned} \implies \begin{aligned} & x^2 = uv \Rightarrow x = \sqrt{uv} \\ & y^2 = \frac{u}{v} \underset{y \geq 0}{\Rightarrow} \end{aligned}$$

$$y = \sqrt{\frac{u}{v}}$$

$$y = \sqrt{\frac{u}{v}}$$

$$\Rightarrow \phi^{-1} \begin{cases} x = \sqrt{uv} \\ y = \sqrt{\frac{u}{v}} \end{cases}$$

$$\Rightarrow \det\left(\phi^{-1}\right)' = \det \begin{array}{cc} \partial_u & \partial_v \\ \multicolumn{2}{c}{\begin{bmatrix} \frac{\sqrt{v}}{2\sqrt{u}} & \frac{\sqrt{u}}{2\sqrt{v}} \\ \frac{1}{2\sqrt{uv}} & -\frac{1}{2}\frac{\sqrt{u}}{v^{3/2}} \end{bmatrix}} \end{array}$$

$$= \det \begin{bmatrix} \frac{v^{1/2}}{2u^{1/2}} & \frac{u^{1/2}}{2v^{1/2}} \\ \frac{1}{2u^{1/2}v^{1/2}} & -\frac{u^{1/2}}{2v^{3/2}} \end{bmatrix} =$$

$$= -\frac{1}{4}\frac{1}{v} - \frac{1}{4}\frac{1}{v} = -\frac{1}{2v}$$

$$\left(\text{by the way: } \det \phi' = \det \begin{bmatrix} y & x \\ 1/y & -x/y^2 \end{bmatrix} = -\frac{x}{y} - \frac{x}{y}\right.$$

$$\left. = -2\frac{x}{y} = -2v \right)$$

$$\Rightarrow \det\left(\phi^{-1}\right)' = \frac{1}{-2v}$$

$$\Rightarrow \int_D f(x,y)\,dx\,dy = \int_D \frac{x^{3/2}}{\sqrt{y-x}} e^{-(xy)^{3/2}}\,dx\,dy$$

$\sqrt{v} = x^{1/2}$ $x = \sqrt{uv}$

$\cancel{x^{3/2}}$ $-u^{3/2}$ $\left|\frac{1}{2v}\right|$

$$= \int \frac{x^{3/2}}{y^{1/2}\sqrt{1-\frac{x}{y}}} e^{-u^{3/2}} \left|\frac{1}{2v}\right| du\,dv$$

$u = xy$

$v = x/y$

$$= \int_{\substack{0\le v\le 1\\ 0\le u<+\infty}} \sqrt{v}\cdot\frac{\sqrt{uv}}{\sqrt{1-v}} e^{-u^{3/2}} \left|\frac{1}{2v}\right| du\,dv$$

$(x,y)\in D \iff$

$0 \le x \le y \iff 0 \le \frac{x}{y} \le 1$

$$= \frac{1}{2}\int_{\substack{0\le v\le 1\\ 0\le u<+\infty}} \frac{\sqrt{u}}{\sqrt{1-v}} e^{-u^{3/2}} du\,dv$$

$$\overset{RE}{=} \frac{1}{2}\int_0^{+\infty} u^{1/2} e^{-u^{3/2}} \int^1 \frac{-1}{\sqrt{1-v}}\,dv\,du$$

$$-\frac{1}{2}(1-v)^{-1/2} \overset{?}{=} \partial_v (1-v)^{1/2}$$

$$= -\int_0^{+\infty} u^{1/2} e^{-u^{3/2}} \left[(1-v)^{1/2}\right]_0^1 du$$

$$= -\frac{2}{3}\int_0^{+\infty} -\frac{3}{2} u^{1/2} e^{-u^{3/2}} du$$

$$\partial_u e^{-u^{3/2}} = e^{-u^{3/2}}\left(-\frac{3}{2}u^{1/2}\right)$$

$u^{3/2} \to +\infty$

$$= -\frac{2}{3}\left[\, e^{-u^{3/2}} \atop -1 \,\right]_0^{+\infty} = \frac{2}{3} \quad ■$$