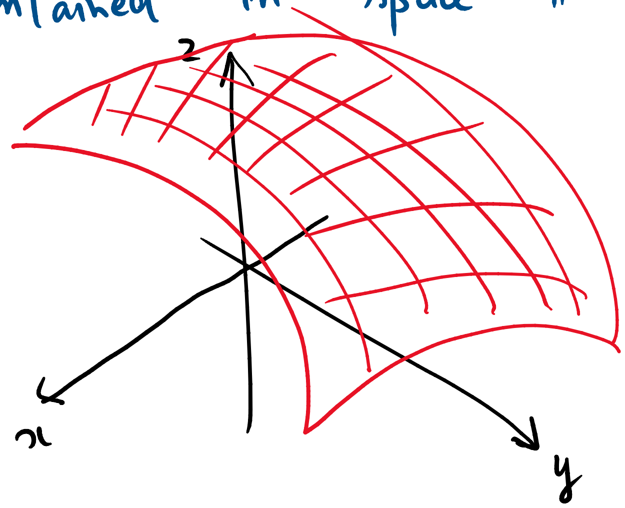
martedì 10 marzo 2020 10:53

Surface Integrals

Pb: Compute the area of a surface $\mathcal{M}$ contained in space $\mathbb{R}^3$:

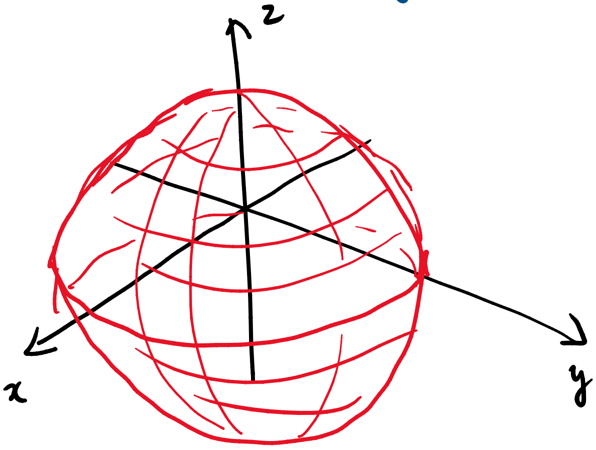


Let's start with this: how can we define a surface in $\mathbb{R}^3$?

There're two principal ways:

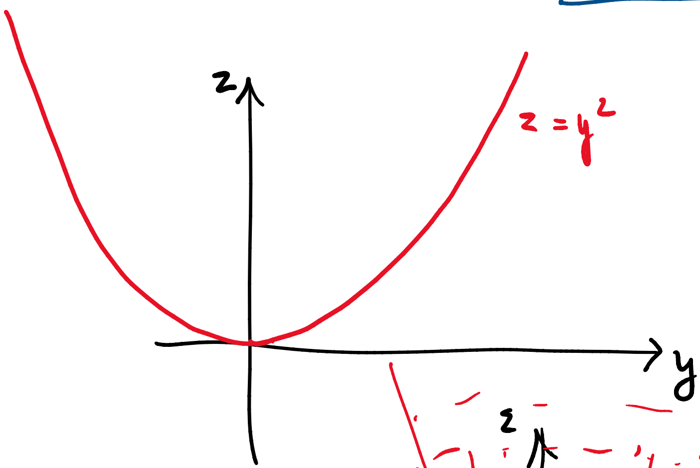
I) through equations

$\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = r^2\}$ (sphere centered at $\vec{0}$, radius r)



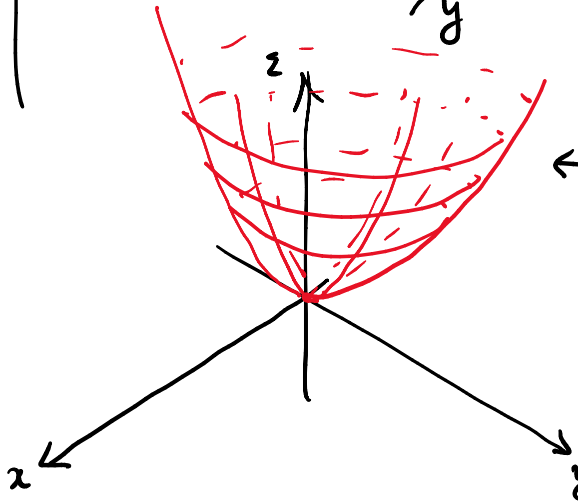
$\{(x,y,z) \in \mathbb{R}^3 : z = x^2 + y^2\}$ this is a set

$\{(x,y,z) \in \mathbb{R}^3 : z = x^2 + y^2\}$ this is a set invariant for rots around z-axis.



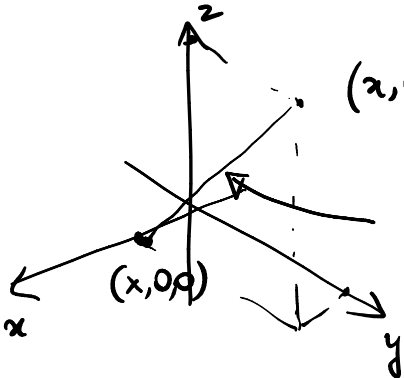
$x = 0$

$z = y^2$



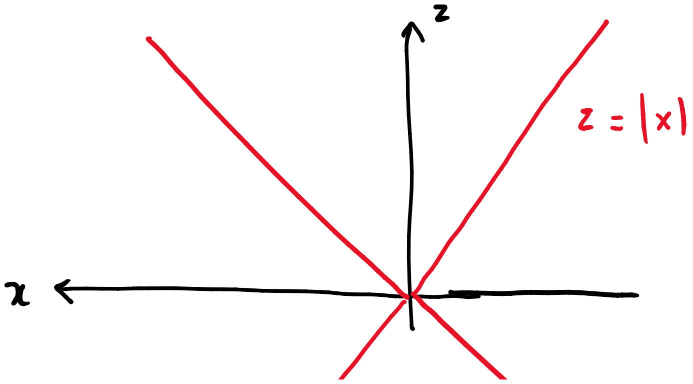
← paraboloid.

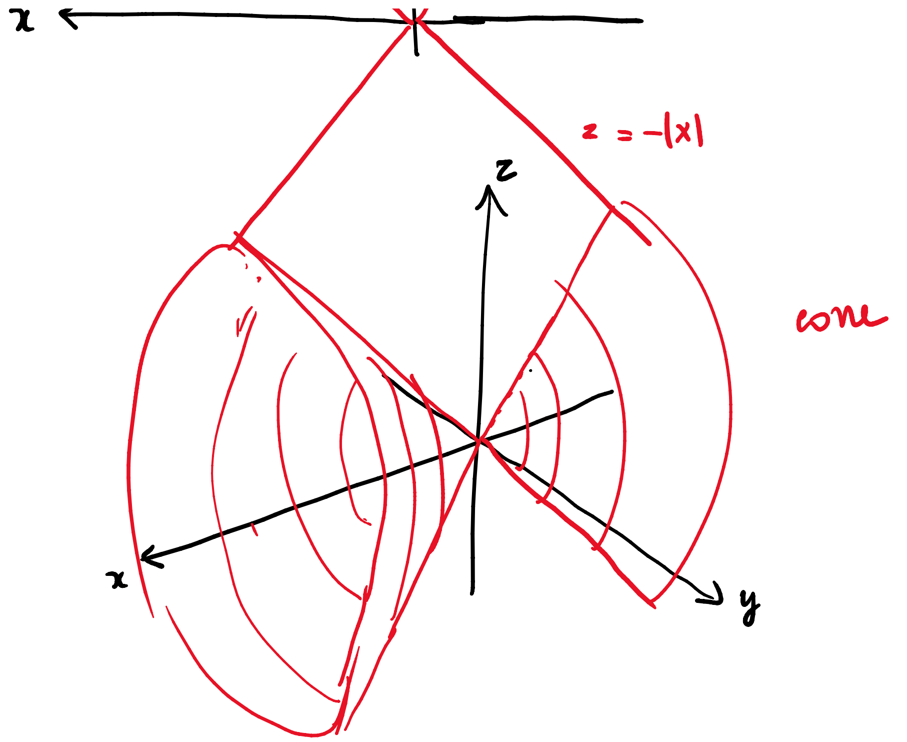
$S = \{(x,y,z) \in \mathbb{R}^3 : x^2 = y^2 + z^2\}$ is rots inv resp x-axis.



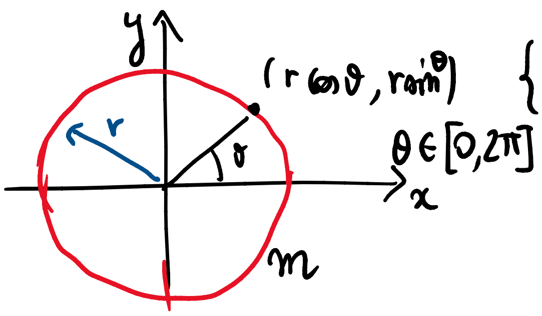
dist $= \sqrt{y^2 + z^2}$

Let's see $S \cap \{y = 0\} = \{x^2 = z^2\} = \{|x| = |z|\}$
$= \{z = \pm|x|\}$





II) through parametrization



$$\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = r^2\}$$

$$|||$$

$$\{(r\cos\theta, r\sin\theta) : \theta \in [0,2\pi]\}$$

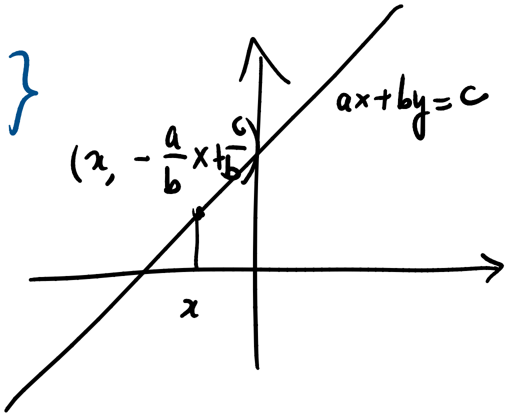
The map $\qquad$ parametrization

$$\underset{\substack{\uparrow \\ \text{parameter}}}{\mathbb{R} \ni \theta} \xmapsto{\ \Phi\ } (r\cos\theta, r\sin\theta) \in \mathbb{R}^2$$

$$\Rightarrow \quad m = \{(r\cos\theta, r\sin\theta) =: \phi(\theta) : \theta \in [0,2\pi]\}$$

$$= \Phi(\underset{\substack{\uparrow \\ \text{domain of parameteers.}}}{[0,2\pi]})$$

$$\{ (x,y) \in \mathbb{R}^2 : \quad ax + by = c \}$$



$b \neq 0 \qquad y = -\frac{a}{b}x + \frac{c}{b}$

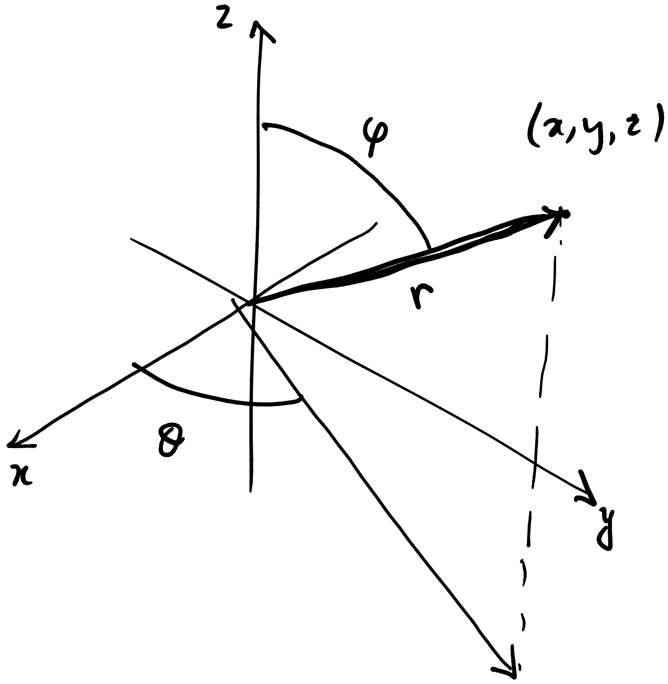
$$\equiv \Big\{ \underbrace{\Big(x, -\frac{a}{b}x + \frac{c}{b} \Big)}_{\vec{\Phi}(x)} \quad : \quad x \in \mathbb{R} \Big\}$$

Here, if we def

$$\mathbb{R} \ni x \stackrel{\vec{\Phi}}{\longmapsto} \Big(x, -\frac{a}{b}x + \frac{c}{b} \Big) \in \mathbb{R}^2$$

$$=: \vec{\phi}(\mathbb{R})$$

$$\{ (x,y,z) : \quad x^2 + y^2 + z^2 = r^2 \} \qquad r > 0 \quad \text{fixed.}$$



$$\begin{cases} x = r \sin\varphi \cos\theta \\ y = r \sin\varphi \sin\theta \\ z = r \cos\varphi \end{cases}$$

$\{ (r \sin\varphi \cos\theta, \; r \sin\varphi \sin\theta, \; r \cos\varphi) : \quad \varphi \in [0, \pi], \; \}$

$$= \left\{ (r\sin\varphi\cos\theta,\ r\sin\varphi\sin\theta,\ r\cos\varphi) : \begin{matrix}\varphi\in[0,\pi], \\ \theta\in[0,2\pi]\end{matrix} \right\}$$

$$\Big\uparrow \vec{\Phi}$$

$$(\varphi,\theta)$$

$$\vec{\Phi} : [0,\pi]\times[0,2\pi] \longrightarrow \mathbb{R}^3$$

$$\text{Sphere} = \vec{\Phi}\big([0,\pi]\times[0,2\pi]\big).$$

<u>Def:</u> Given a function

$$\vec{\Phi} : D\subset\mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

$\vec{\phi} = \vec{\phi}(u,v)$ $\quad (u,v)\in D$, $\quad \vec{\Phi}\in C^1$ $\quad \big(\vec{\phi}\in C,$

$(\vec{\phi})'\in C$ $\quad \vec{\phi}(u,v) = \big(\phi_x(u,v), \phi_y(u,v), \phi_z(u,v)\big)$

$\uparrow$
jac. matrix

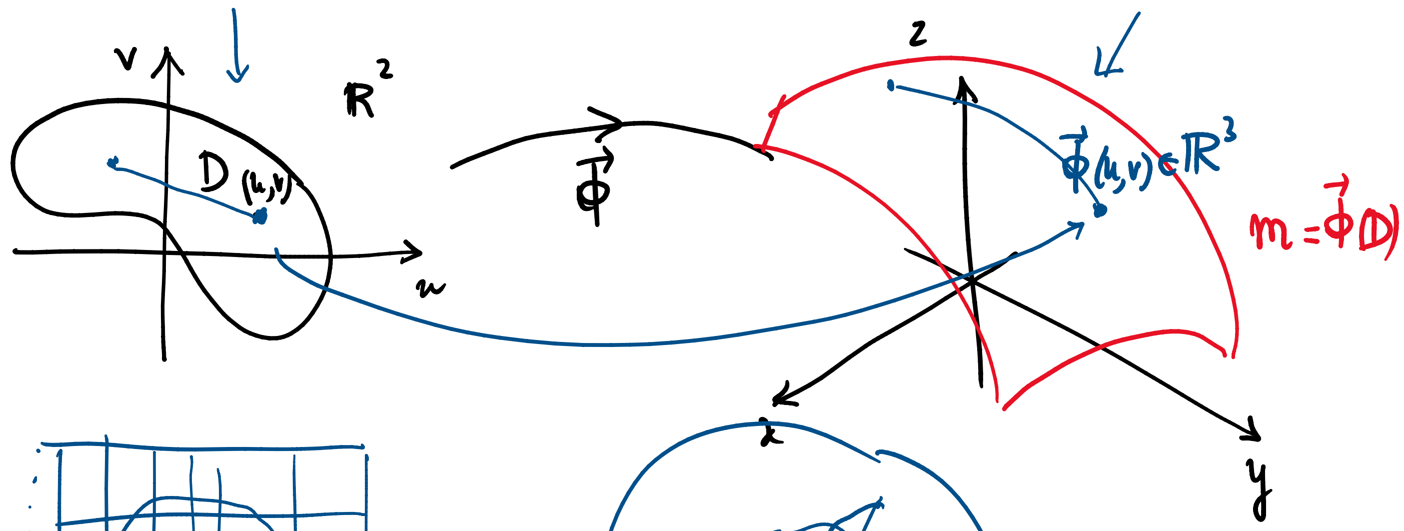
$$\vec{\phi}'(u,v) = \begin{array}{cc} \partial_u & \partial_v \\ \left[\begin{matrix} \partial_u\phi_x & \partial_v\phi_x \\ \partial_u\phi_y & \partial_v\phi_y \\ \partial_u\phi_z & \partial_v\phi_z \end{matrix}\right. & \left.\right] \end{array} \quad \begin{matrix}3\times 2\\ \text{matrix}\end{matrix}$$

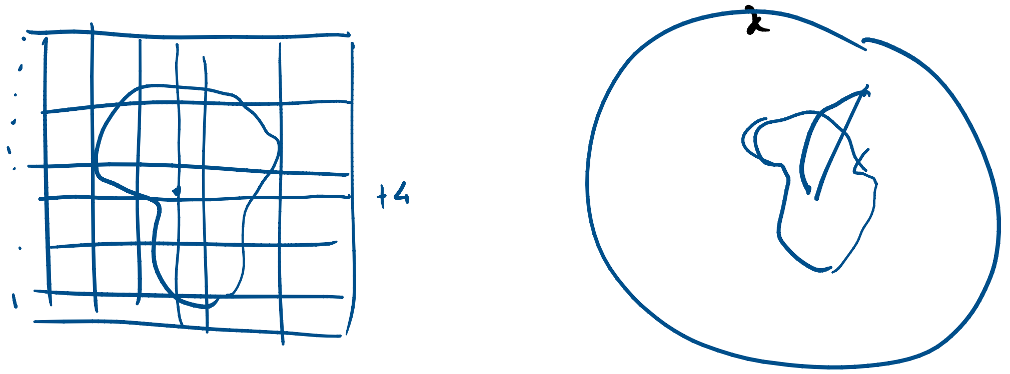
$$= \left[\partial_u\vec{\phi} \quad \partial_v\vec{\phi}\right] \Big)$$

we call <u>parametric surface</u> defd by $\vec{\phi}$

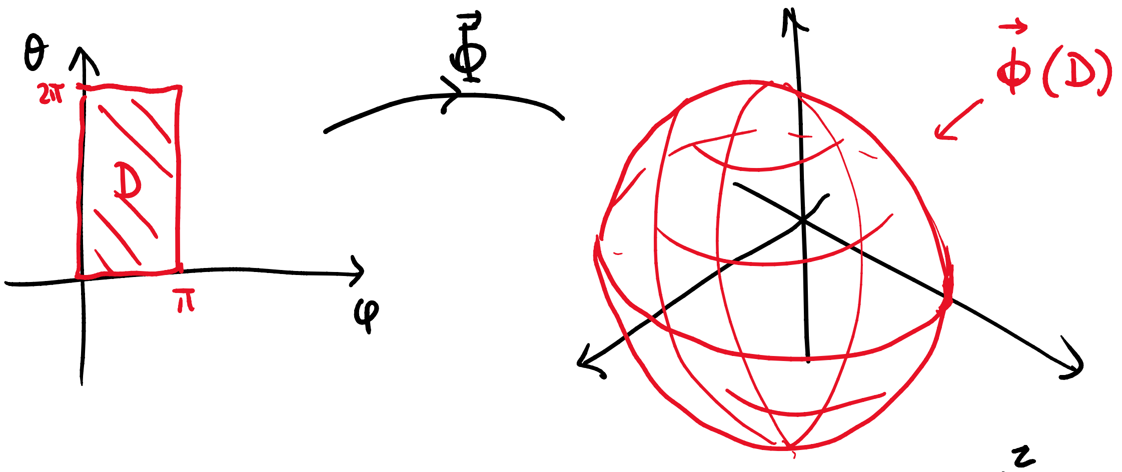
We call <u>parametric surface</u> acta by T
the set

$$\mathcal{M} = \vec{\Phi}(D) = \{ \vec{\Phi}(u,v) : (u,v) \in D \}$$





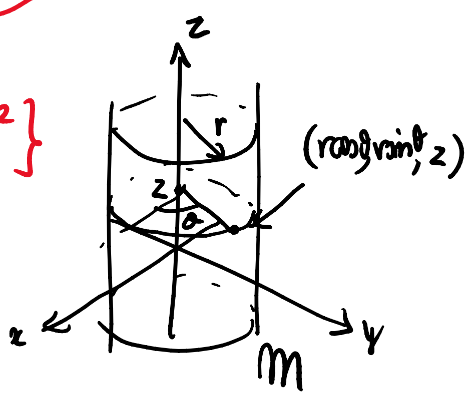
Example: Sphere $= \vec{\Phi}([0,\pi] \times [0,2\pi])$



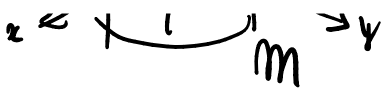
Cylinder $= \{ (x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = r^2 \}$

$\mathcal{M} = \vec{\Phi}(D)$ where

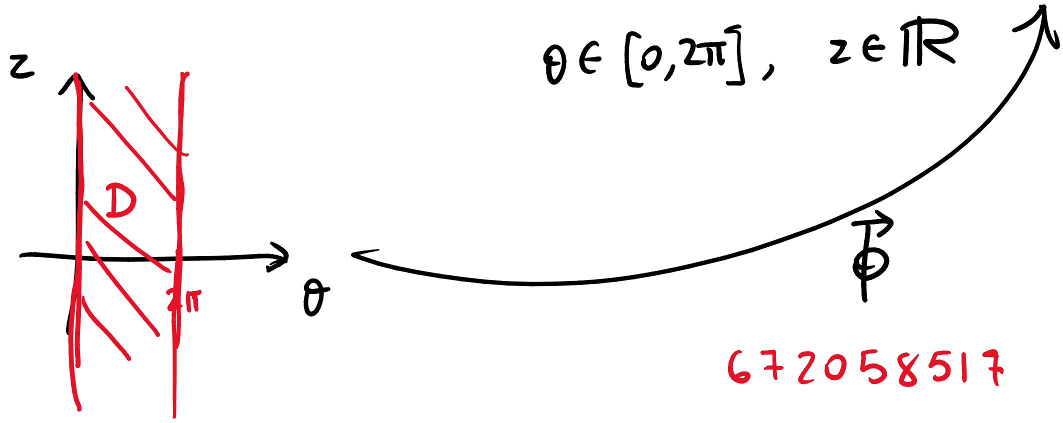
$\vec{\Phi}(\theta, z) = (r\cos\theta, r\sin\theta, z)$



$$\vec{\Phi}(\theta, z) = (r\cos\theta,\ r\sin\theta,\ z)$$



$\theta \in [0, 2\pi]$, $z \in \mathbb{R}$



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<u>Question:</u> Are we really sure that

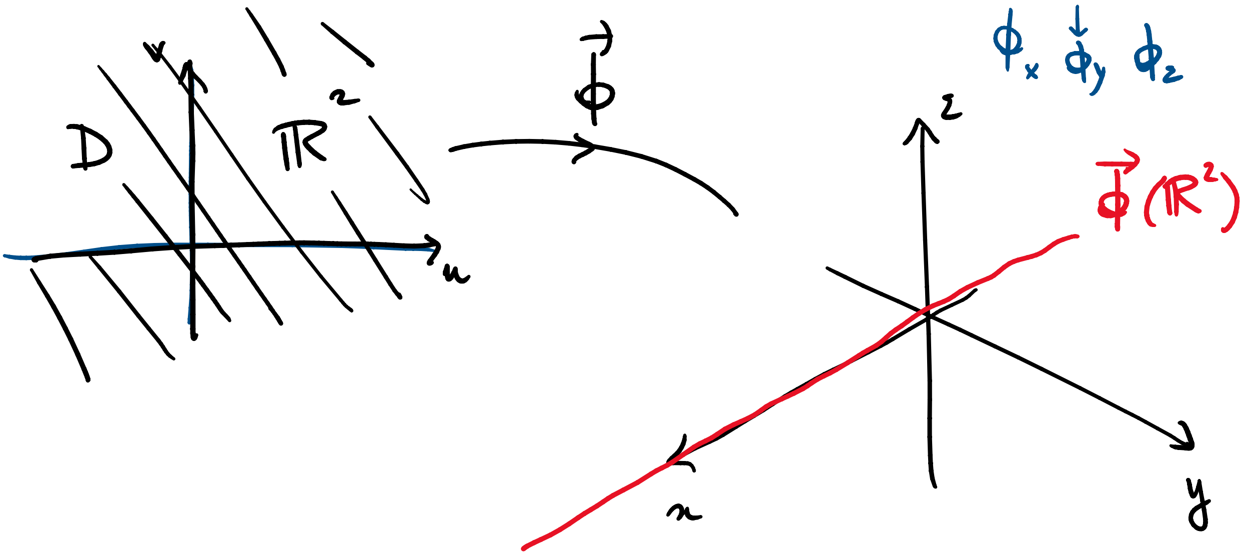
$$M = \vec{\Phi}(D)$$

(with $\vec{\Phi} : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, $\vec{\Phi} \in \mathcal{C}^1$) is always a surface in $\mathbb{R}^3$?

<u>Example:</u> $\vec{\Phi} = \vec{\Phi}(u,v) : D = \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

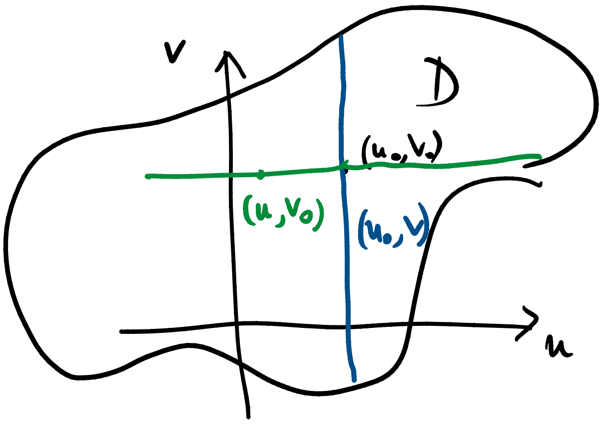
$$(u,v) \longmapsto (u, 0, 0)$$

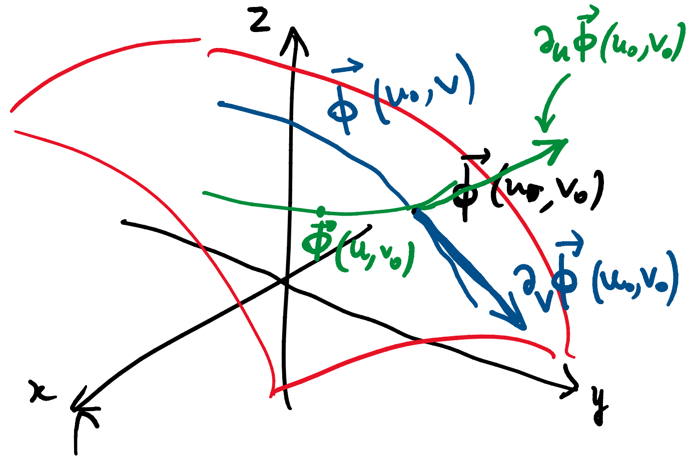
$$\phi_x \quad \phi_y \quad \phi_z$$



We wonder under which condition a parametriza-

We wonder under which condition a parametrization $\vec{\phi}$ defines a surface (2-dim) in $\mathbb{R}^3$





$$u \mapsto \vec{\phi}(u,v_0) \in \mathbb{R}^3 \quad \Rightarrow \quad \text{velocity} = \partial_u \vec{\phi}(u_0,v_0)$$
$\in \mathbb{R}$
$$v \mapsto \vec{\phi}(u_0,v) \in \mathbb{R}^3 \quad \Rightarrow \quad \text{velocity} = \partial_v \vec{\phi}(u_0,v_0)$$

<u>Def:</u> We say that $\vec{\phi} : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is <u>immersive</u> (or an <u>immersion</u>) if

$$\partial_u \vec{\phi}, \ \partial_v \vec{\phi} \text{ are lin. } \underline{\text{indep}} \quad \forall (u,v) \in D$$

$$\Leftrightarrow \quad \vec{\phi}' = \left[\partial_u \vec{\phi} \quad \partial_v \vec{\phi} \right] \text{ has rank} = 2.$$

<u>Example:</u> $\vec{\phi}(\varphi,\theta) = (r \sin\varphi \cos\theta, \ r \sin\varphi \sin\theta, \ r\cos\varphi)$

$(\varphi,\vartheta) \in [0,\pi] \times [0,2\pi]$.

$$\partial_\varphi \vec{\Phi} = \left(\partial_\varphi (r \sin\varphi \cos\vartheta), \ \partial_\varphi (r \sin\varphi \sin\vartheta), \ \partial_\varphi (r\cos\varphi) \right)$$

$$= (r\cos\varphi\cos\theta,\ r\cos\varphi\sin\theta,\ -r\sin\varphi)$$

$$\partial_\theta \vec{\Phi} = (-r\sin\varphi\sin\theta,\ r\sin\varphi\cos\theta,\ 0)$$

$\partial_\varphi \vec{\Phi}$ and $\partial_\theta \vec{\Phi}$ are not lin indep

$\Updownarrow$

are lin dep

$$\Longleftrightarrow \exists \lambda : \quad \partial_\varphi \vec{\Phi} = \lambda\, \partial_\theta \vec{\Phi}$$

$$\begin{pmatrix} r\cos\varphi\cos\theta \\ r\cos\varphi\sin\theta \\ -r\sin\varphi \end{pmatrix} = \lambda \begin{pmatrix} -r\sin\varphi\sin\theta \\ r\sin\varphi\cos\theta \\ 0 \end{pmatrix}$$

$$\Longleftrightarrow \begin{cases} r\cos\varphi\cos\theta = -\lambda r\sin\varphi\sin\theta \\ r\cos\varphi\sin\theta = \lambda r\sin\varphi\cos\theta \\ -\cancel{r}\sin\varphi = 0 \end{cases} \qquad \varphi \in [0,\pi]$$

$$\Longleftrightarrow \begin{cases} \varphi = 0, \pi \\ = \\ = \end{cases}$$

$$\Longleftrightarrow \begin{cases} \varphi = 0 \\ \cancel{r}\cos\theta = 0 \\ \cancel{r}\sin\theta = 0 \end{cases} \quad \vee \quad \begin{cases} \varphi = \pi \\ -\cancel{r}\cos\theta = 0 \\ -\cancel{r}\sin\theta = 0 \end{cases}$$

imposs $\qquad\qquad$ imposs

$\Rightarrow \quad \partial_\varphi \vec{\Phi}$ and $\partial_\theta \vec{\Phi}$ are lin indep

$\Rightarrow \quad \vec{\Phi}$ immersive. $\boxtimes$

Rmk: To check $\vec{\Phi}$ immersive we need to check if

$$\operatorname{rk}\left[\partial_u \vec{\Phi} \quad \partial_v \vec{\Phi}\right] = 2$$

$\Uparrow$ non true

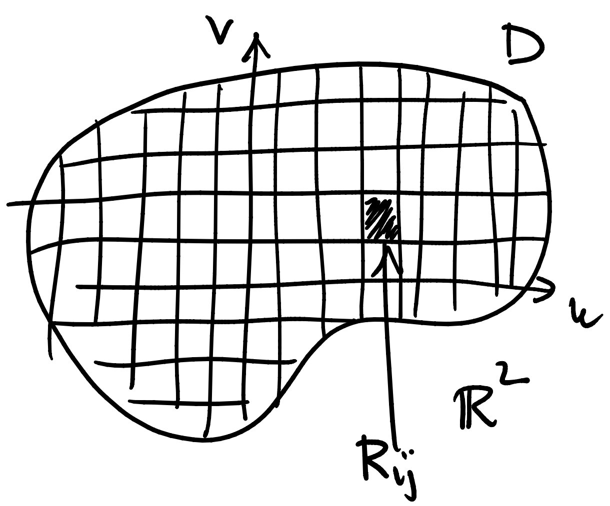
Now

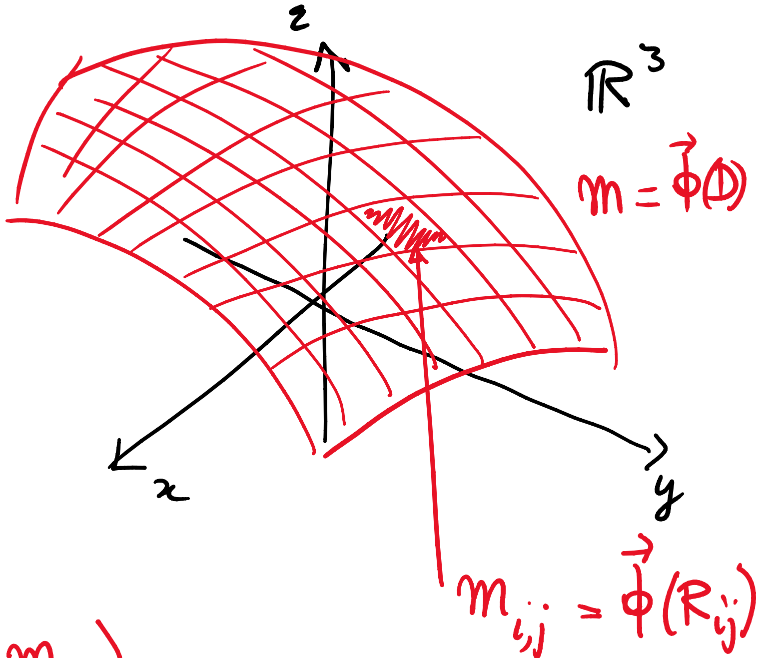
$$\operatorname{rk}\Big[\overbrace{\partial_u \vec{\Phi} \quad \partial_v \vec{\Phi}}^{\phi'}\Big] < 2$$

$\Leftrightarrow$ if all 2×2 sub dets of ϕ' are $= 0$ $\boxtimes$

Since now we consider $m = \vec{\Phi}(D)$, $\vec{\Phi}$ immersive. Let's tackle the pb of computing

Area (m)



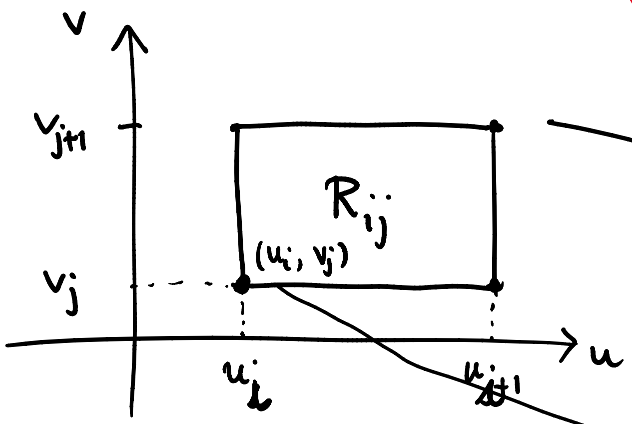


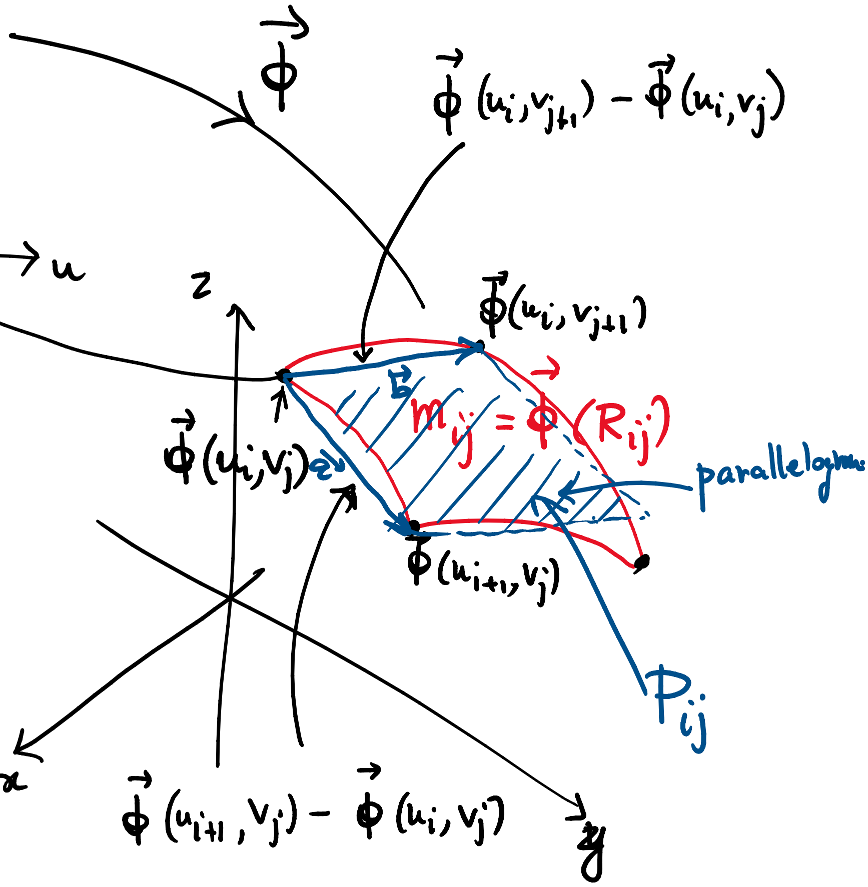
$\text{Area}(m) = \sum \text{Area}(m_{ij})$

$$m_{i,j} = \phi(R_{ij})$$

$$\text{Area}(m) = \sum \text{Area}(m_{ij})$$

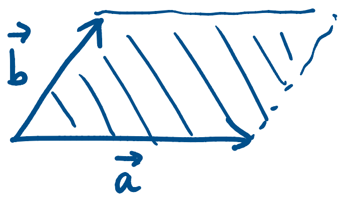
$$\| \quad \text{Area}(\vec{\phi}(R_{ij}))$$





$$\text{Area}(m_{ij})$$

$$\approx \text{Area}(P_{ij})$$



$$= \| \vec{a} \wedge \vec{b} \|$$

$$\vec{a} = \left(\vec{\phi}(u_{i+1}, v_j) - \vec{\phi}(u_i, v_j) \right)$$

$$= \underbrace{\partial_u \vec{\phi}(u_i, v_j)}_{\text{vect}} \underbrace{(u_{i+1} - u_i)}_{\text{scalar}}$$

Lagrange

$$\left(f(b) - f(a) = f'(c)(b-a) \right.$$

$$\vec{b} = \partial_v \vec{\phi}(u_i, v_j)(v_{j+1} - v_j)$$

$$\Rightarrow \| \vec{a} \wedge \vec{b} \| = \| \underbrace{(u_{i+1} - u_i)} \underbrace{(v_{j+1} - v_j)} \partial_u \vec{\phi} \wedge \partial_v \vec{\phi} \|$$

$$= \underbrace{\overset{du}{(u_{i+1}-u_i)}\,\overset{dv}{(v_{j+1}-v_j)}}\,\|\partial_u\vec{\phi}\wedge\partial_v\vec{\phi}\|$$

$$= \text{Area}\,(R_{ij})\,\|\partial_u\phi\wedge\partial_v\phi\|$$

$$\Rightarrow \text{Area}\,(M) \approx \sum \text{Area}\,(M_{ij})$$

$$= \sum \|\partial_u\phi\wedge\partial_v\phi\|\;du\,dv$$

$$= \int_D \|\partial_u\vec{\phi}\wedge\partial_v\vec{\phi}\,(u,v)\|\;du\,dv$$

$$\boxed{\text{Area}\,(\vec{\phi}(D)) = \int_D \|\partial_u\phi\wedge\partial_v\phi\|\;du\,dv}$$

area element.

Example: In the case of the sphere:

$$\vec{\phi}(\varphi,\vartheta) = (r\sin\varphi\cos\vartheta,\ r\sin\varphi\sin\vartheta,\ r\cos\varphi)$$

we have

$$\partial_u\vec{\phi}\wedge\partial_v\vec{\phi} = \det\begin{bmatrix}\vec{i} & \vec{j} & \vec{k}\\ \partial_u\phi_x & \partial_u\phi_y & \partial_u\phi_z\\ \partial_v\phi_x & \partial_v\phi_y & \partial_v\phi_z\end{bmatrix}$$

$$\begin{bmatrix} \partial_v \phi_x & \partial_v \phi_y & \partial_v \phi_z \end{bmatrix}$$

$$= \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ r\cos\varphi\cos\theta & r\cos\varphi\sin\theta & -r\sin\varphi \\ -r\sin\varphi\sin\theta & r\sin\varphi\cos\theta & 0 \end{bmatrix}$$

$$= \vec{i}\left(+ r^2(\sin\varphi)^2\cos\theta\right) + \vec{j}\left(+ r^2(\sin\varphi)^2\sin\theta\right)$$

$$+ \vec{k}\,\underbrace{\left(r^2\cos\varphi\sin\varphi(\cos\theta)^2 + r^2\cos\varphi\sin\varphi(\sin\theta)^2\right)}_{= r^2\cos\varphi\sin\varphi}$$

$$\|\partial_\varphi \vec{\Phi} \wedge \partial_\theta \vec{\Phi}\| = \sqrt{r^4(\sin\varphi)^4(\cos\theta)^2 + r^4(\sin\varphi)^4(\sin\theta)^2 + r^4(\cos\varphi)^2(\sin\varphi)^2}$$

$$= r^2\sqrt{(\sin\varphi)^2\underbrace{\left((\sin\varphi)^2 + (\cos\varphi)^2\right)}_{=1}} = r^2\sqrt{(\sin\varphi)^2}$$

$$= r^2\,|\sin\varphi| \underset{\varphi\in[0,\pi]}{=} r^2\sin\varphi$$

$$\Rightarrow \text{Area}(\text{Sphere}(r)) = \int_{\substack{0\leq\varphi\leq\pi \\ 0\leq\theta\leq 2\pi}} r^2\sin\varphi \; d\varphi\, d\theta$$

RF π 2π

RF

$$= r^2 \int_0^{\pi} \int_0^{2\pi} \sin\varphi \, d\theta \, d\varphi$$

$$= r^2 \, 2\pi \int_0^{\pi} \sin\varphi \, d\varphi = 4\pi r^2 . \quad \blacksquare$$

$$\left[-\cos\varphi\right]_0^{\pi}$$

Do Ex 5.7.1 #1.