mercoledì 11 marzo 2020 11:09

We introduced the concept of area of a parametric surface: $\mathcal{M} = \vec{\Phi}(D) \subset \mathbb{R}^3$ where

$$\vec{\Phi} : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

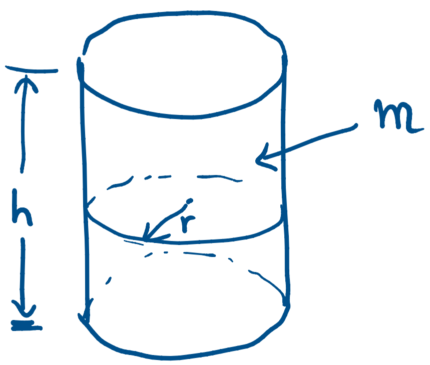
s.t. $\boxed{\partial_u \vec{\Phi},\ \partial_v \vec{\Phi} \text{ be lin indep } \forall (u,v) \in D}$

$\vec{\Phi}$ is <u>immersive</u> / or an <u>immersion</u>

Then

$$\text{Area}(\mathcal{M}) = \int_D \| \partial_u \vec{\Phi} \wedge \partial_v \vec{\Phi} \| \, du\, dv.$$

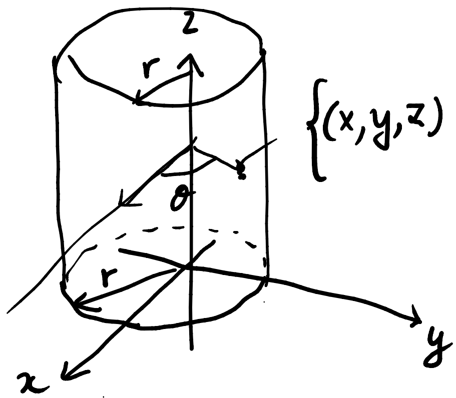
<u>Example</u>: (cylinder radius $r > 0$, height $h > 0$)



Guess: $\text{Area} = 2\pi r \cdot h$.

Let's prove this simple formula.

First we need a parametrization of the cylinder.



$$\{(x,y,z) : \underbrace{x^2+y^2=r^2}, \ \underbrace{0 \leq z \leq h}\}$$

$$\begin{cases} x = r\cos\theta = \Phi_x \\ y = r\sin\theta = \Phi_y = \vec{\Phi}(\theta, z) \\ z = z \qquad\quad = \Phi_z \end{cases}$$

$\vec{\Phi} = \vec{\Phi}(\theta, z) : D = [0, 2\pi] \times [0, h] \longrightarrow \mathbb{R}^3$

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$$= \vec{\Phi}(D).$$

Now,

$$\partial_\theta \vec{\Phi} \wedge \partial_z \vec{\Phi} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_\theta \phi_x & \partial_\theta \phi_y & \partial_\theta \phi_z \\ \partial_z \phi_x & \partial_z \phi_y & \partial_z \phi_z \end{bmatrix}$$

$$= \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= 1 \cdot \det \begin{bmatrix} \vec{i} & \vec{j} \\ -r\sin\theta & r\cos\theta \end{bmatrix} = r\cos\theta\,\vec{i} + r\sin\theta\,\vec{j} + 0\vec{k}$$

$$\| \partial_\theta \phi \wedge \partial_z \phi \| = \sqrt{(r\cos\theta)^2 + (r\sin\theta)^2 + 0^2}$$

$$= \sqrt{r^2} = r \qquad (r > 0)$$

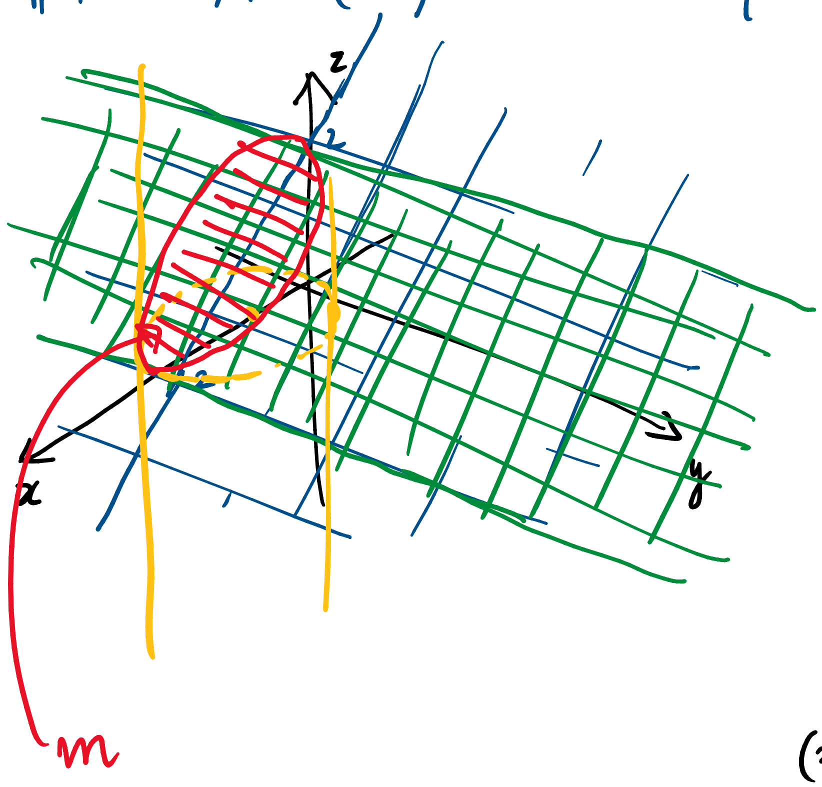
$$\Rightarrow \quad \text{Area}(\text{Cyl}) = \int_{\substack{0 \le \theta \le 2\pi \\ 0 \le z \le h}} r \; d\theta\, dz = r \lambda_2([0,2\pi] \times [0,h])$$

$$= r \cdot 2\pi \cdot h \quad \blacksquare$$

Ex 5.7.1

#1 Area (M) $\quad M = \{(x,y,z) \in \mathbb{R}^3 : x \geq 0, z \geq 0$

#1 Area(m) $\quad m = \{ (x,y,z) \in \mathbb{R}^3 : x \geq 0, z \geq 0, \; \boxed{x+z=2}, \; x^2+y^2-2x \leq 0 \}$



$z = 2 - x$

$x+z=2$

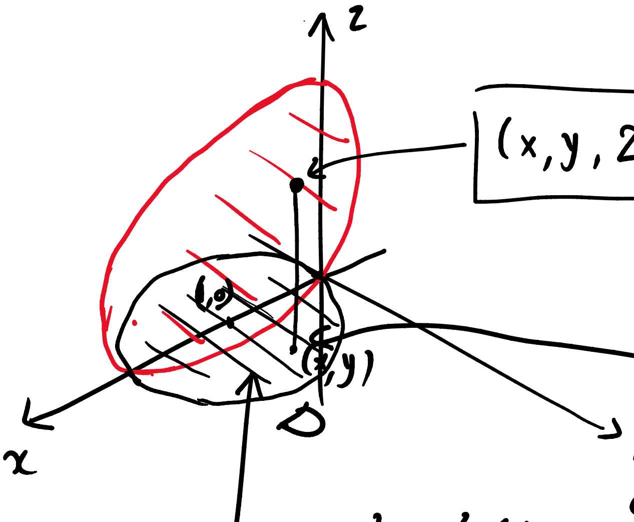
$x \geq 0, \quad z \geq 0$

$$\underbrace{x^2+y^2-2x}_{(x-1)^2-1} \leq 0$$

$$(x-1)^2 - 1 + y^2 \leq 0$$

$$(x-1)^2 + y^2 \leq 1$$

Area(m). We need first a parametrization:



$\boxed{(x,y,2-x)} = \vec{\Phi}(x,y)$

$(x,y) \in D$

$\Downarrow$

$m = \vec{\Phi}(D)$

$(x-1)^2+y^2 \leq 1$

$\Downarrow$

$x \geq 0$

Therefore $\quad$ Area$(m) = \displaystyle\int_D \| \partial_x \vec{\Phi} \wedge \partial_y \vec{\Phi} \| \, dx\,dy$

Therefore $\quad \text{Area}(m) = \iint_D \| \partial_x \vec{\phi} \wedge \partial_y \vec{\phi} \| \, dx\, dy$

$$\partial_x \vec{\phi} \wedge \partial_y \vec{\phi} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= -1 \cdot \det \begin{bmatrix} \vec{i} & \vec{k} \\ 1 & -1 \end{bmatrix} = -\left(-\vec{i} - \vec{k} \right)$$

$$= \vec{i} + \vec{k}$$

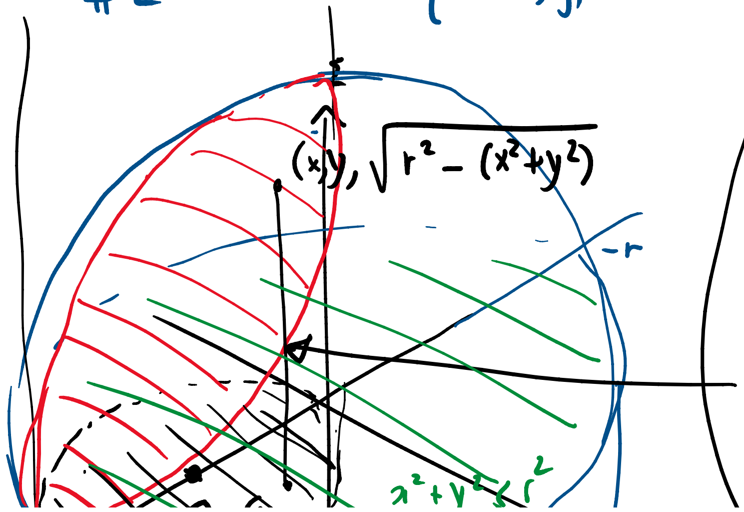
$$\| \partial_x \vec{\phi} \wedge \partial_y \vec{\phi} \| = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$$

$$\Rightarrow \text{Area}(m) = \iint_{(x-1)^2 + y^2 \leq 1} \sqrt{2} \, dx\, dy$$

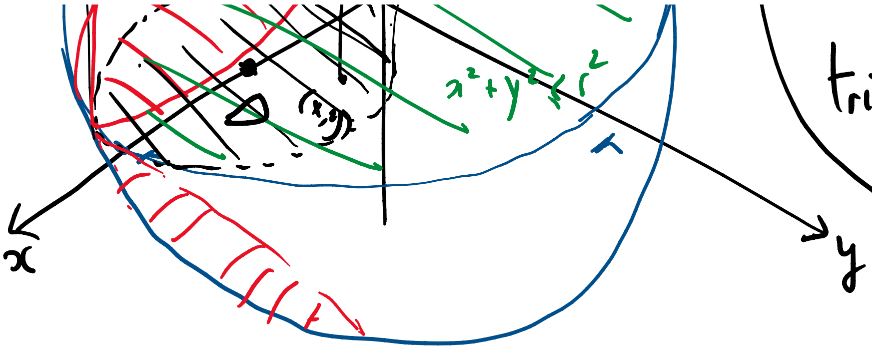
$$= \sqrt{2} \; \lambda_2(\text{Disk radius} = 1) = \sqrt{2}\,\pi \cdot 1^2 = \sqrt{2}\,\pi. \quad \blacksquare$$

5.7.1

#2 $\quad m = \left\{ (x,y,z) \in \mathbb{R}^3 : \boxed{x^2 + y^2 + z^2 = r^2}, \; \left(x - \frac{r}{2}\right)^2 + y^2 \leq \frac{r^2}{4} \right\}$



To compute area of m we need to parametrize m:



trize m :

$$z = \pm\sqrt{r^2 - (x^2+y^2)}$$

$$\vec{\Phi} = \vec{\Phi}(x,y) = \left(x, y, \sqrt{r^2 - (x^2+y^2)}\right)$$

$$(x,y) \in D = \left\{ \left(x - \tfrac{r}{2}\right)^2 + y^2 \leq \frac{r^2}{4} \right\}$$

$$\Rightarrow \quad \text{Area}(m) = \int_D \| \partial_x \vec{\Phi} \wedge \partial_y \vec{\Phi} \| \, dx\,dy$$

$$\partial_x \vec{\Phi} \wedge \partial_y \vec{\Phi} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \dfrac{-\cancel{2}x}{\cancel{2}\sqrt{r^2-(x^2+y^2)}} \\ 0 & 1 & -\dfrac{y}{\sqrt{}} \end{bmatrix}$$

$$= \vec{i} \det \begin{bmatrix} 0 & -\dfrac{x}{\sqrt{}} \\ 1 & -\dfrac{y}{\sqrt{}} \end{bmatrix} - 1 \det \begin{bmatrix} \vec{j} & \vec{k} \\ 1 & -\dfrac{y}{\sqrt{}} \end{bmatrix}$$

$$= \vec{i}\left(\frac{x}{\sqrt{}}\right) - 1 \cdot \left(-\frac{y}{\sqrt{}}\vec{j} - \vec{k}\right)$$

$$= \left(\frac{x}{\sqrt{}} \,,\, \frac{y}{\sqrt{}} \,,\, 1 \right)$$

$$\Rightarrow \| \partial_x \vec{\Phi} \wedge \partial_y \vec{\Phi} \| = \sqrt{ \frac{x^2}{r^2-(x^2+y^2)} + \frac{y^2}{r^2-(x^2+y^2)} + 1 }$$

$$= \sqrt{ \frac{\cancel{x^2+y^2} + r^2 - \cancel{(x^2+y^2)}}{r^2 - (x^2+y^2)} }$$

$$= r \frac{1}{\sqrt{r^2-(x^2+y^2)}} .$$

$$\Rightarrow \text{Area}(m) = \int_{(x-\frac{r}{2})^2 + y^2 \leq \frac{r^2}{4}} \frac{r}{\sqrt{r^2-(x^2+y^2)}} \, dx\, dy$$

$$\begin{cases} x = \rho \cos\theta \\ y = \rho \sin\theta \end{cases}$$

$$\begin{cases} x - \frac{r}{2} = \rho\cos\theta \\ y = \rho \sin\theta \end{cases}$$

$$= \int_{\substack{-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \\ \rho \leq r\cos\theta}} \frac{r}{\sqrt{r^2-\rho^2}} \rho \, d\rho \, d\theta$$

$$x^2 - rx + \cancel{\frac{r^2}{4}} + y^2 \leq \cancel{\frac{r^2}{4}}$$

$$x^2+y^2 - rx \leq 0$$

$\pi/2$

$\Updownarrow$ 2

$$x^2+y^2-rx \le 0$$

$$\Updownarrow \quad \rho^2 - r\rho\cos\theta \le 0$$

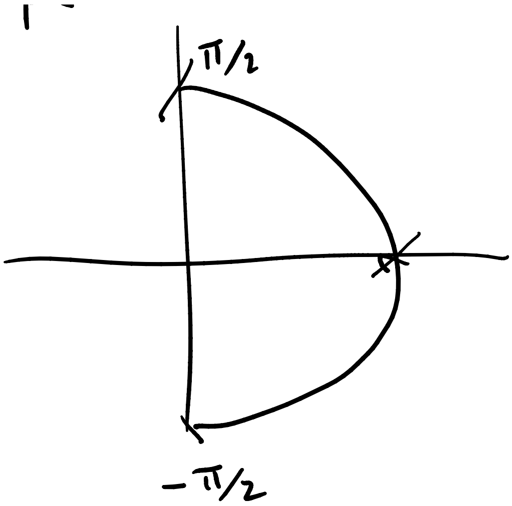
$$\rho \ge 0 \ \Updownarrow \quad \rho - r\cos\theta \le 0$$

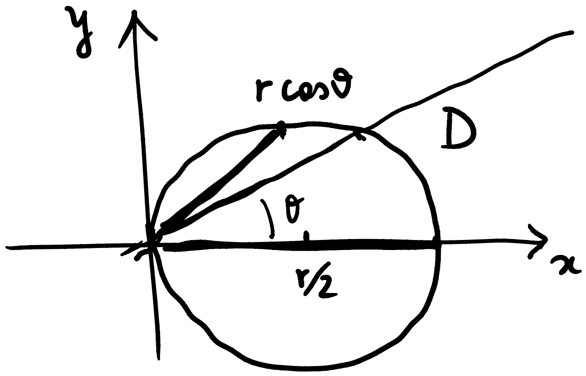
$$\Updownarrow$$

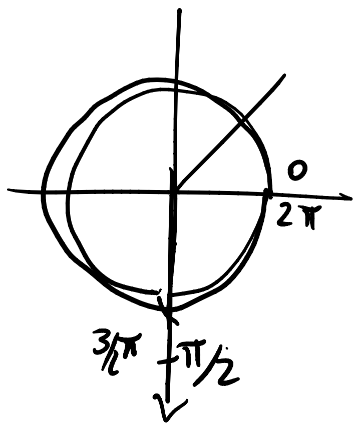
$$0 \le \rho \le r\cos\theta$$

$$\Downarrow$$

$$\cos\theta \ge 0 \iff -\tfrac{\pi}{2} \le \theta \le \tfrac{\pi}{2}$$







$$\overset{RF}{=} \int_{-\pi/2}^{\pi/2} \int_0^{r\cos\theta} \frac{r\rho}{\sqrt{r^2-\rho^2}}\, d\rho \, d\theta$$

$$= -r \int_{-\pi/2}^{\pi/2} \int_0^{r\cos\theta} -(r^2-\rho^2)^{-1/2}\rho \, d\rho \, d\theta$$

$$\partial_\rho (r^2-\rho^2)^{1/2} = \frac{1}{\cancel{2}}(r^2-\rho^2)^{-1/2}(-\cancel{2}\rho)$$

$$= -r \int_{-\pi/2}^{\pi/2} \left[(r^2-\rho^2)^{1/2} \right]_{\rho=0}^{\rho = r\cos\vartheta} d\vartheta$$

$$\sqrt{r^2 - r^2(\cos\theta)^2} - \sqrt{r^2}$$

$$= r\sqrt{1-(\cos\theta)^2} - r = r\left(|\sin\vartheta| - 1\right)$$

$$= -r \int_{-\pi/2}^{\pi/2} r \left(|\sin\theta| - 1 \right) d\vartheta$$

$$\|$$

$$= +r^2\, 2 \int_0^{\pi/2} (-\sin\vartheta + 1)\, d\vartheta$$

$$\|$$

$$\cos\vartheta \Big|_0^{\pi/2} + \pi/2$$

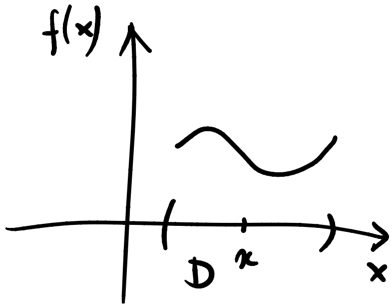
$$\|$$

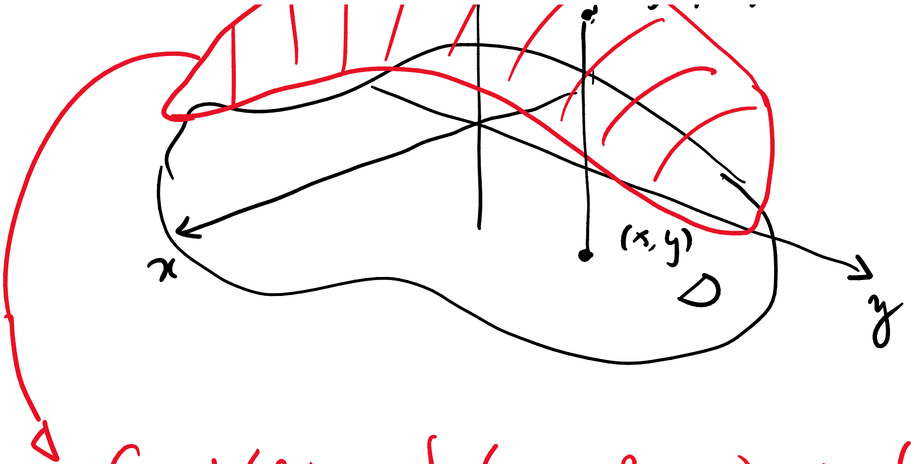
$$0 - 1$$

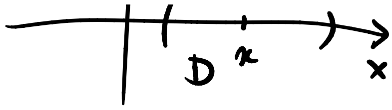
$$= r^2 \cdot 2 \cdot \left(\frac{\pi}{2} - 1 \right) = r^2 (\pi - 2). \qquad \boxtimes$$

Parametric Surface Graph of a Function

Let $f = f(x,y) : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$







$$\mathrm{Graph}(f) = \{ (x, y, f(x,y)) \ : \ (x,y) \in D \}$$

If $f \in \mathcal{C}^1$ we have that Graph(f) may be seen as parametric surface.

Indeed if we def $\vec{\Phi} : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$(x,y) \longmapsto (x, y, f(x,y))$$

$\Rightarrow$ $\vec{\Phi}(D) = \mathrm{Graph}(f)$. To check that $\vec{\Phi}$ defines a surface we need to check if

$$\mathrm{rk}\, [\partial_x \phi \quad \partial_y \phi] = 2$$

$$\partial_x \phi = (1, 0, \partial_x f)$$

$$\partial_y \phi = (\underbrace{0, 1}_{}, \partial_y f)$$

$$\partial_x \phi \overset{?}{=} \lambda \partial_y \phi$$

$$\begin{pmatrix} 1 \\ 0 \\ \square \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ 1 \\ \diamond \end{pmatrix}$$ impossible

$\Downarrow$

$\partial_x \phi, \partial_y \phi$ are lin indep

$\Rightarrow$ ϕ is an immersion $\blacksquare$

Let's see what happens in this case to area formula:

$$\text{Area}(\text{Graph}(f)) = \int_D \| \partial_x \phi \wedge \partial_y \phi \| \, dx\, dy$$

$$\partial_x \phi \wedge \partial_y \phi = \det \begin{bmatrix} i & j & k \\ 1 & 0 & \partial_x f \\ 0 & 1 & \partial_y f \end{bmatrix}$$

$$= i \det \begin{bmatrix} 0 & \partial_x f \\ 1 & \partial_y f \end{bmatrix} \ominus j \det \begin{bmatrix} 1 & \partial_x f \\ 0 & \partial_y f \end{bmatrix}^{= \partial_y f}$$

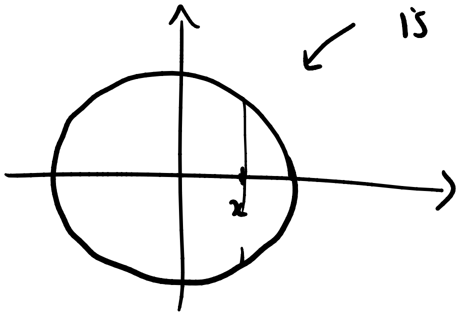
$$+ k \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= (-\partial_x f \,,\, -\partial_y f \,,\, 1)$$

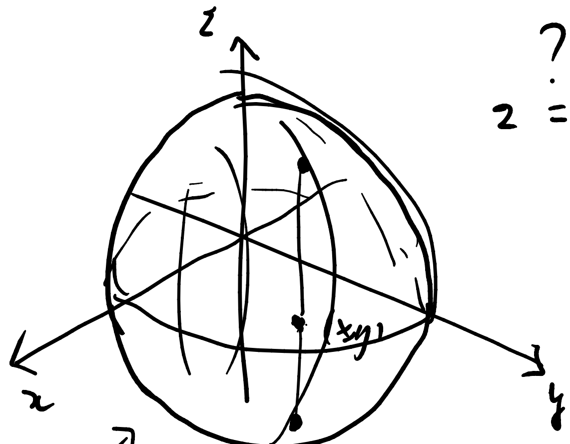
$$\Rightarrow \| \partial_x \phi \wedge \partial_y \phi \| = \sqrt{\underbrace{(\partial_x f)^2 + (\partial_y f)^2}_{\substack{\| \\ \| (\partial_x f, \partial_y f) \|^2 \\ \| \\ \| \nabla f \|^2}} + 1}$$

$$\Rightarrow \quad \text{Area}(\text{Graph}(f)) = \int_D \sqrt{1+\|\nabla f\|^2}\,dx\,dy.$$

Ex 5.1.5 Area of a sphere of radius r.



is not a graph $y = f(x)$



$$z \overset{?}{=} f(x,y)$$

is not a graph

but an half sphere

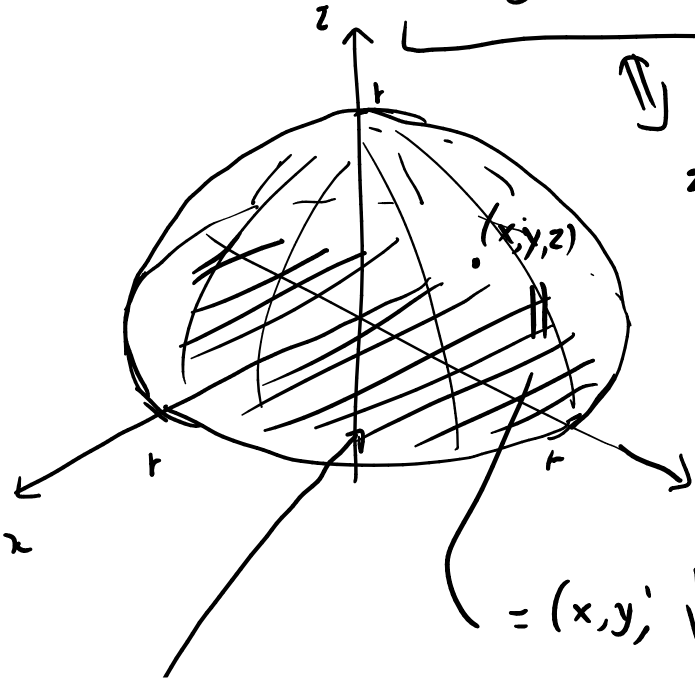
$$\Big\{(x,y,z)\in\mathbb{R}^3 : \underbrace{x^2+y^2+z^2=r^2, \quad z\geq 0}\Big\}$$

$$\Updownarrow$$

$$z^2 = r^2-(x^2+y^2)$$

$$\Downarrow \; z\geq 0$$

$$z = \sqrt{r^2-(x^2+y^2)}$$



$$= \left(x,y, \sqrt{r^2-(x^2+y^2)}\right)$$

$$= \left(x, y, \underbrace{\sqrt{r^2 - (x^2+y^2)}}_{\overset{!!}{f(x,y)}}\right)$$

def'd on $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq r^2\} =: D$

$$\Rightarrow \text{Area}\left(\tfrac{1}{2}\text{ Sphere}\right) = \int_{x^2+y^2 \leq r^2} \sqrt{1 + \|\nabla f\|^2}\, dx\,dy$$

$$f(x,y) = \sqrt{r^2 - (x^2+y^2)}$$

$$\nabla f = (\partial_x f, \partial_y f) = \left(\frac{-\cancel{2}x}{\cancel{2}\sqrt{r^2-(x^2+y^2)}}, \frac{-y}{\sqrt{}}\right)$$

$$\Rightarrow \|\nabla f\|^2 = \frac{x^2}{r^2-(x^2+y^2)} + \frac{y^2}{r^2-(x^2+y^2)}$$

$$= \frac{x^2+y^2}{r^2-(x^2+y^2)}$$

$$\Rightarrow \text{Area}\left(\tfrac{1}{2}\text{ Sph}\right) = \int_{x^2+y^2 \leq r^2} \sqrt{1 + \frac{x^2+y^2}{r^2-(x^2+y^2)}}\, dx\,dy$$

$$= \int_{x^2+y^2 \leq r^2} \sqrt{\frac{r^2 - \cancel{(x^2+y^2)} + \cancel{(x^2+y^2)}}{r^2 - (x^2+y^2)}}\, dx\,dy$$

$$= r \int_{x^2+y^2 \leq r^2} \left(r^2 - (x^2+y^2)\right)^{-1/2} \, dx\,dy$$

$$\underset{\text{pol coords}}{=} r \int_{\substack{\rho \leq r \\ 0 \leq \theta \leq 2\pi}} (r^2-\rho^2)^{-1/2} \, \rho \; d\rho \, d\theta$$

$$\overset{RF}{=} -r \cdot 2\pi \int_0^r -(r^2-\rho^2)^{-1/2} \rho \; d\rho$$

$$\parallel$$

$$\partial_\rho (r^2-\rho^2)^{1/2} = \cancel{1/2}(r^2-\rho^2)^{-1/2}(-\cancel{2}\rho)$$

$$= -2\pi r \left[(r^2-\rho^2)^{1/2} \right]_{\rho=0}^{\rho=r} = 2\pi r^2$$

$$0 - (r^2)^{1/2} = -r$$

$$\Rightarrow \text{Area}(\text{Sph.}) = 4\pi r^2 \quad \blacksquare$$

Surface Integrals

We may imagine that

$$\underset{\substack{\parallel \\ \phi(D)}}{\text{Area}(m)} = \int_m 1 \, d\sigma \qquad \Bigg| \qquad \lambda_2(S) = \int_S 1$$

$$= \int_D 1 \, \boxed{\|\partial_u \phi \wedge \partial_v \phi\| \, du\,dv}$$

Let $g = g(x,y,z) : \mathbb{R}^3 \longrightarrow \mathbb{R}$. We want to

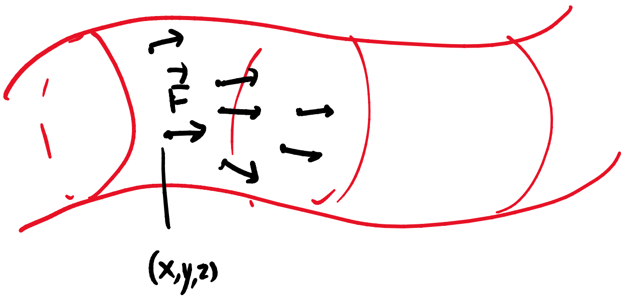
Let $g = g(x,y,z) : \mathbb{R}^3 \longrightarrow \mathbb{R}$. We want to give a meaning to

$$\boxed{\int_{\mathcal{M}} g \, d\sigma := \int_{D} g(\phi(u,v)) \, \| \partial_u \phi \wedge \partial_v \phi \| \, du\, dv}$$

Flux of a vector field

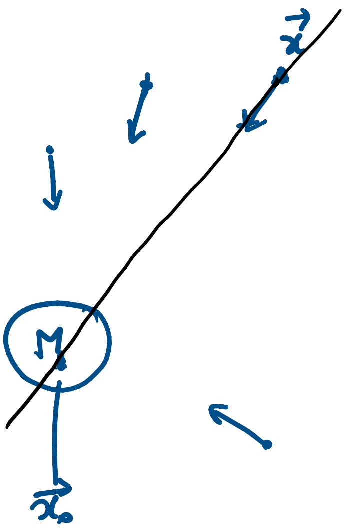
Def: A vector field is a function

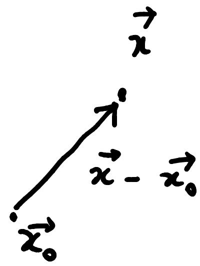
$$\vec{F} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$



$$\vec{F}(x,y,z) = \left(F_1(x,y,z),\ F_2(x,y,z),\ F_3(x,y,z) \right)$$

Example: (gravitational field)

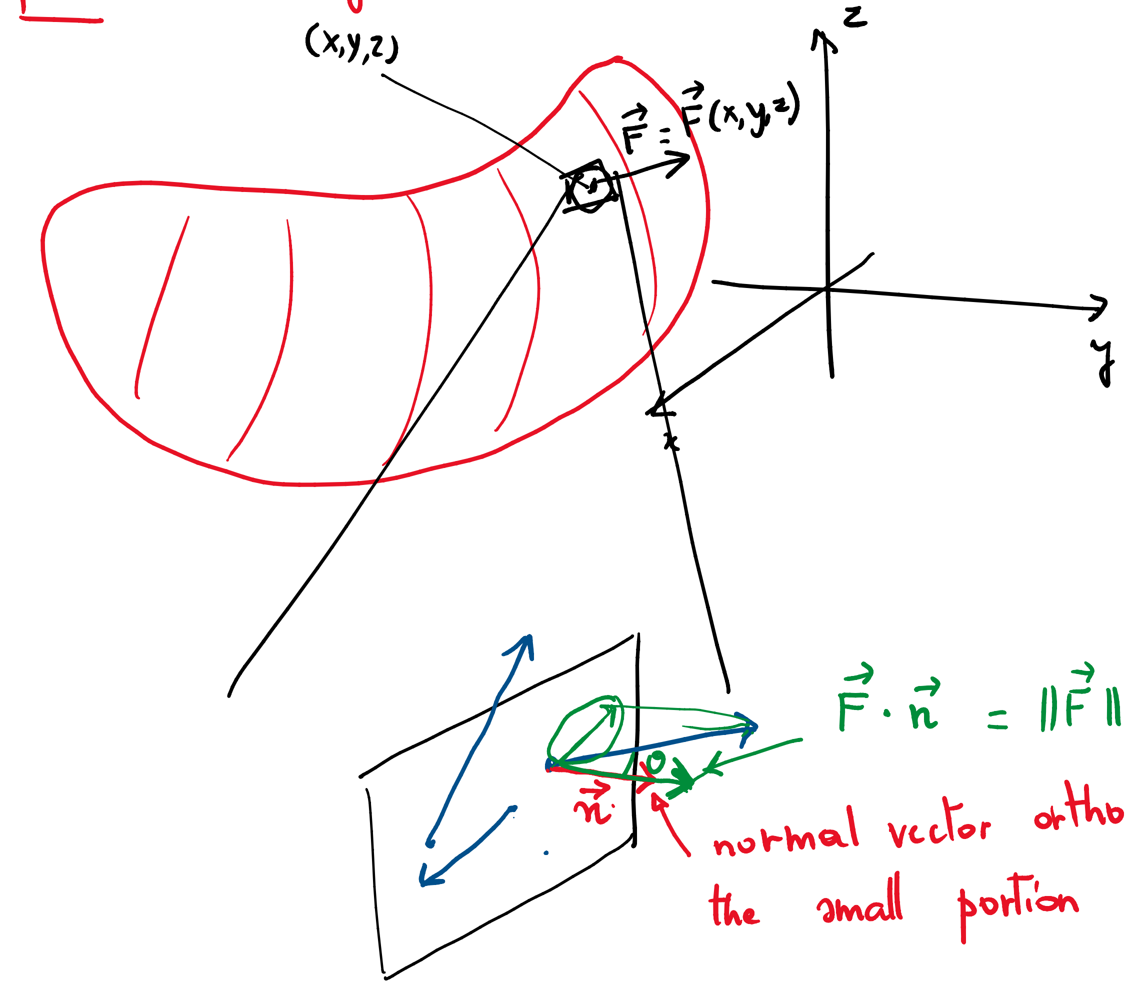




$$\vec{F}(\vec{x}) = -\frac{kmM(\vec{x} - \vec{x}_0)}{\|\vec{x} - \vec{x}_0\|^3}$$

applications of surface integrals is

One of major applications of surface integrals is to the problem of computing a flux of a vector field through a surface.



$$\vec{F}\cdot\vec{n} = \|\vec{F}\|\,\underbrace{\|\vec{n}\|}_{=1}\cos\theta$$

normal vector orthogonal to the small portion of surface

We may say that the intensity of $\vec{F}$ on small portion of surface is

$$\vec{F}\cdot\vec{n}\cdot\text{area}(\text{surface})$$

$\Rightarrow$ Total contribution $\sum \vec{F}\cdot\vec{n}\cdot\text{area}(\text{surface})$

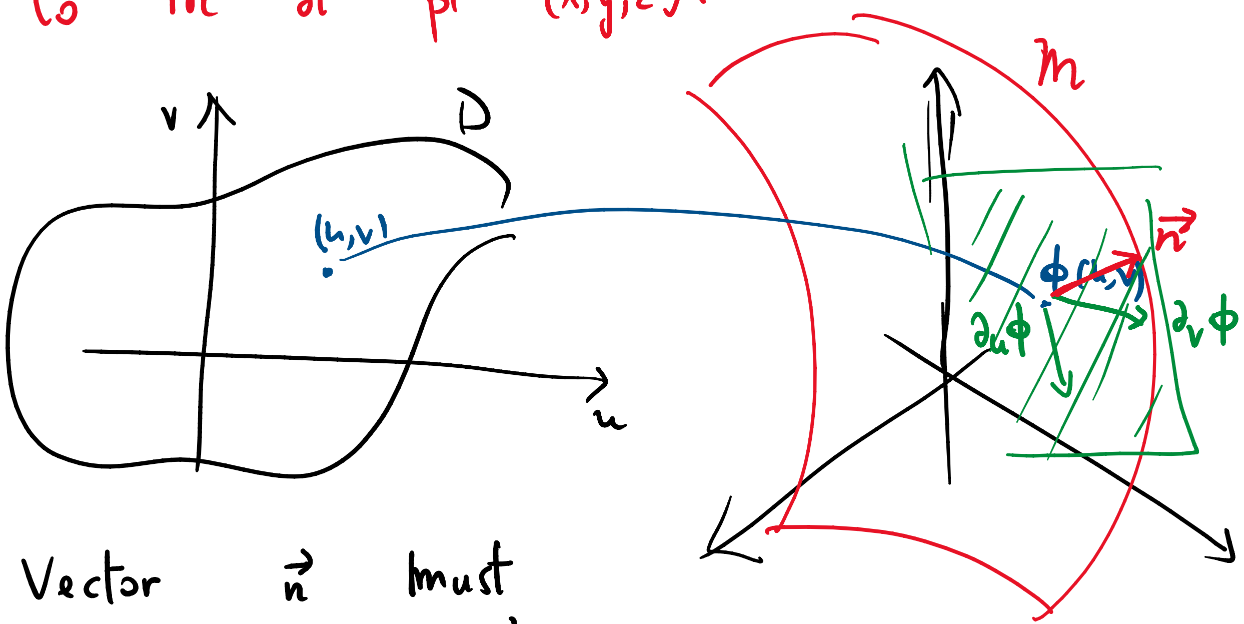
$$\sim \int \vec{F}\cdot\vec{n}\,d\sigma$$

$\mathcal{M}$

Def: We call flux of $\vec{F}$ through $\mathcal{M}$ the quantity

$$\int_{\mathcal{M}} \vec{F}\cdot\vec{n}\, d\sigma$$

where $\vec{n} = \vec{n}(x,y,z)$ is the unit normal vector ($\| \ \| = 1$, perpendicular) to $\mathcal{M}$ at pt (x,y,z).



Vector $\vec{n}$ must be $\perp$ $\partial_u\phi$, $\partial_v\phi$

This vector is

$$\boxed{\vec{n} := \frac{\partial_u\phi\wedge\partial_v\phi}{\|\partial_u\phi\wedge\partial_v\phi\|}}$$

In part, if $\mathcal{M} = \phi(D)$

$$\int \vec{F}\cdot\vec{n}\, d\sigma = \int \vec{F}\cdot\vec{n}\cdot\|\partial_u\phi\wedge\partial_v\phi\|\, du\, dv$$

$\mathcal{M}$

D

$$= \int_D \vec{F} \cdot \frac{\partial_u \phi \wedge \partial_v \phi}{\cancel{\|\partial_u \phi \wedge \partial_v \phi\|}} \cancel{\|\partial_u \phi \wedge \partial_v \phi\|} \, du\, dv$$

$$\int_{\mathcal{M}} \vec{F} \vec{n} \, d\sigma = \int_D \vec{F} \cdot \underbrace{\partial_u \phi \wedge \partial_v \phi} \; \underbrace{du\, dv}$$

$$(\square, \lozenge, \heartsuit)$$

$$\parallel$$

$$\left(\vec{F} = \vec{F}(x,y,z) \right) \qquad \vec{F}(\phi(u,v)) \qquad \square i + \lozenge j + \heartsuit k$$

$$\vec{F} \cdot \partial_u \phi \wedge \partial_v \phi = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} \cdot \det \begin{bmatrix} i & j & k \\ \bigcirc & \partial_u \phi & \bigcirc \\ \bigcirc & \partial_v \phi & \bigcirc \end{bmatrix}$$

$$= F_1 \cdot \det \begin{bmatrix} \partial_u \phi_2 & \partial_u \phi_3 \\ \partial_v \phi_2 & \partial_v \phi_3 \end{bmatrix} + F_2 \left[- \det \begin{bmatrix} \partial_u \phi_1 & \partial_u \phi_3 \\ \partial_v \phi_1 & \partial_v \phi_3 \end{bmatrix} \right]$$

$$+ F_3 \det \begin{bmatrix} \partial_u \phi_1 & \partial_u \phi_2 \\ \partial_v \phi_1 & \partial_v \phi_2 \end{bmatrix}$$

$$= \det \begin{bmatrix} F_1 & F_2 & F_3 \\ \partial_u \phi_1 & \partial_u \phi_2 & \partial_u \phi_3 \\ \partial_v \phi_1 & \partial_v \phi_2 & \partial_v \phi_3 \end{bmatrix}$$

We proved that

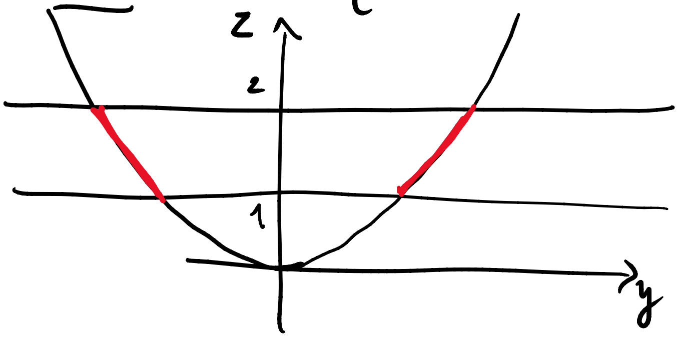
$$\int_{\mathcal{M}} \vec{F} \cdot \vec{n} \, d\sigma = \int_D \det \begin{bmatrix} \vec{F} \\ \partial_u \phi \\ \partial_v \phi \end{bmatrix} du\, dv.$$

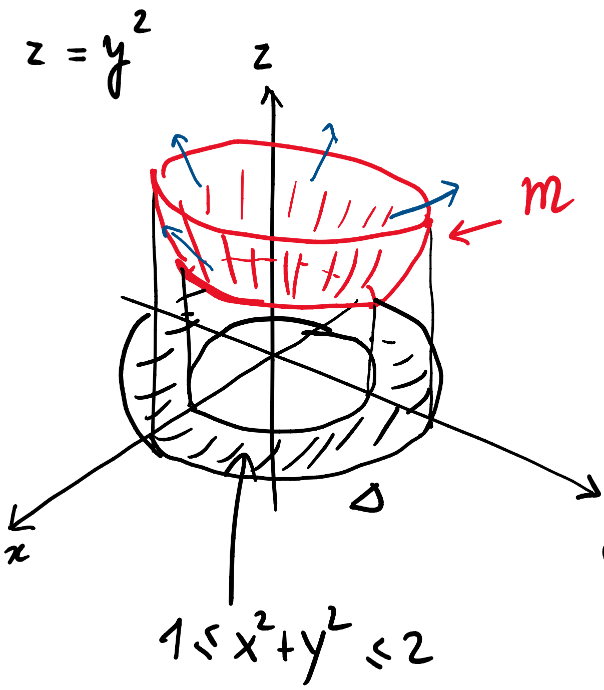
Ex 5.2.2 Compute the flux of

$$\vec{F} = \vec{F}(x,y,z) := (y, z, x)$$

through paraboloid $z = x^2+y^2$ between $1 \leq z \leq 2$.

Sol: $\mathcal{M} = \{(x,y,z) \in \mathbb{R}^3 \; ; \; z = x^2+y^2, \; 1 \leq z \leq 2\}$





We have to compute

$$\int_{\mathcal{M}} \vec{F}\cdot\vec{n}\, d\sigma$$

$$= \int_D \det \begin{bmatrix} \vec{F} \\ \partial_u \phi \\ \partial_v \phi \end{bmatrix} du\, dv.$$

$\phi : D = \{ 1 \leq x^2+y^2 \leq 2\}$

Let's det a parametrization. $\phi(x,y) = (x, y, x^2+y^2)$

$$\vec{F}(\phi(x,y)) = F(x, y, x^2+y^2)$$

$$= \int_{1 \leq x^2+y^2 \leq 2} \det \begin{array}{cl} \begin{bmatrix} y & x^2+y^2 & x \\ 1 & 0 & 2x \\ 0 & 1 & 2y \end{bmatrix} & \begin{array}{l} \\ \leftarrow \partial_x \phi \\ \leftarrow \partial_y \phi \end{array} \end{array}$$

$$= \int_{1 \le x^2+y^2 \le 2} -y \cdot 2x \; - 1 \cdot \left(2y(x^2+y^2) - x\right) + 0 \ldots \, dx\,dy$$

$$= \int_{1 \le x^2+y^2 \le 2} \left(-2xy - 2y(x^2+y^2) + x\right) dx\,dy$$

$$\underset{\text{pol. coords}}{=} \int_{\substack{1 \le \rho^2 \le 2 \\ 0 \le \theta \le 2\pi}} \left(-2\rho^2 \cos\theta \sin\theta - 2\rho^3 \sin\theta + \rho\cos\theta\right) \rho \, d\rho\, d\theta$$

$$\overset{\text{RF}}{=} \int_1^{\sqrt{2}} -\cancel{2}\rho^3 \underbrace{\int_0^{2\pi} \sin(2\theta)\, d\theta}_{=0} - 2\rho^4 \underbrace{\int_0^{2\pi} \sin\theta}_{=0} + \rho^2 \underbrace{\int_0^{2\pi} \cos\theta\, d\theta}_{=0}$$

$$= 0. \; \blacksquare$$

Do 5.7.1 # 3,4

5.7.2