giovedì 12 marzo 2020 11:10

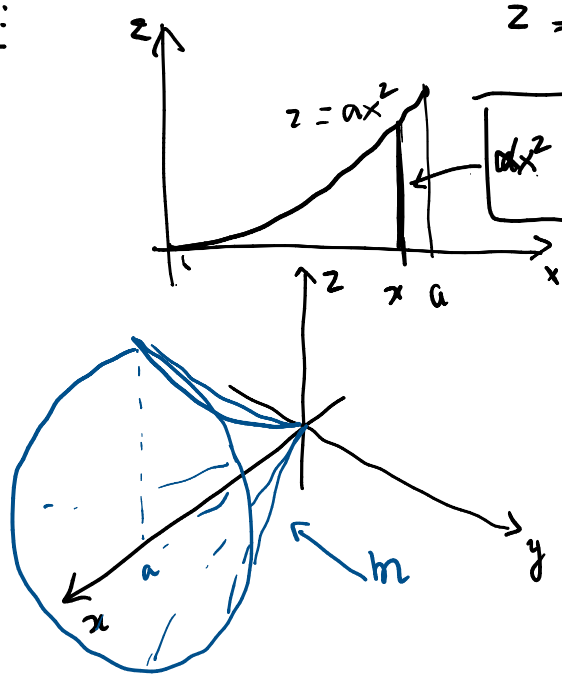
Ex 5.7.1

\#3 compute Area of $\mathcal{M}$, rotation surface around x axis of the graph of

$f(x) = \alpha x^2 \quad x \in [0,a] \quad \alpha > 0$

Sol:

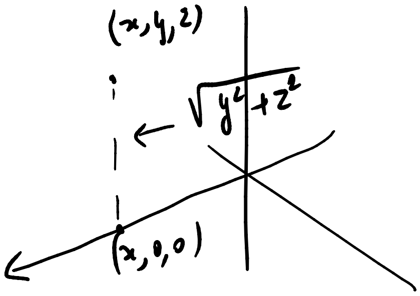
$z = \alpha x^2 \qquad x \in [0,a]$



$\boxed{\alpha x^2 = \sqrt{y^2+z^2}}$

$y^2 + z^2 = \alpha^2 x^4$

dist to x axis



$$\mathcal{M} = \{ (x,y,z) : y^2 + z^2 = \alpha^2 x^4 \}.$$

$$(x,y,z) = (x, \alpha x^2 \cos\theta, \alpha x^2 \sin\theta)$$

$$= \Phi(x,\theta) \quad \text{on } D = \{ 0 \le x \le a, 0 \le \theta \le 2\pi \}$$

$$\Rightarrow \text{Area}(\mathcal{M}) = \int_D \| \partial_x \phi \wedge \partial_\theta \phi \| \, dx \, d\theta$$

$$\partial_x \phi \wedge \partial_\theta \phi = \det \begin{bmatrix} i & j & k \\ 1 & 2\alpha x \cos\theta & 2\alpha x \sin\theta \end{bmatrix}$$

$$\partial_x \Phi \wedge \partial_\theta \Phi = \det \begin{bmatrix} 1 & 2\alpha x \cos\theta & 2\alpha x \sin\theta \\ 0 & -\alpha x^2 \sin\theta & \alpha x^2 \cos\theta \end{bmatrix}$$

$$= \left(\det \begin{bmatrix} 2\alpha x \cos\theta & 2\alpha x \sin\theta \\ -\alpha x^2 \sin\theta & \alpha x^2 \cos\theta \end{bmatrix}, -\det \begin{bmatrix} 1 & 2\alpha x \sin\theta \\ 0 & \alpha x^2 \cos\theta \end{bmatrix}, \det \begin{bmatrix} 1 & 2\alpha x \cos\theta \\ 0 & -\alpha x^2 \sin\theta \end{bmatrix} \right)$$

$$= \left(2\alpha^2 x^3 \underbrace{\det \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}}_{= 1}, -\alpha x^2 \cos\theta, -\alpha x^2 \sin\theta \right)$$

$$= \left(2\alpha^2 x^3, -\alpha x^2 \cos\theta, -\alpha x^2 \sin\theta \right)$$

$$\| \partial_x \Phi \wedge \partial_\theta \Phi \| = \sqrt{4\alpha^4 x^6 + \underbrace{\alpha^2 x^4 (\cos\theta)^2 + \alpha^2 x^4 (\sin\theta)^2}}$$

$$= \alpha x^2 \sqrt{1 + 4\alpha^2 x^2}$$

Now $\displaystyle \text{Area}(M) = \int_{\substack{0 \leq x \leq a \\ 0 \leq \theta \leq 2\pi}} \alpha x^2 \sqrt{1 + 4\alpha^2 x^2} \, dx \, d\theta$

$$\overset{RF}{=} \int_0^a \left(\alpha x^2 \sqrt{1 + 4\alpha^2 x^2} \underbrace{\int_0^{2\pi} 1 \, d\theta}_{= 2\pi} \right) dx$$

$$= 2\pi\alpha \int_0^a x^2 \underbrace{\sqrt{1 + 4\alpha^2 x^2}}_{1 + (2\alpha x)^2} \, dx \quad \leftarrow$$

$1 + (\quad)^2 = (\quad)^2 \qquad 2\alpha x = \operatorname{Sh} u$

$$Ch^2 - Sh^2 = 1$$
$$Ch^2 = 1 + Sh^2$$

$$x = \frac{1}{2\alpha} Sh\, u \qquad dx = \frac{1}{2\alpha} Ch\, u\, du$$

$$Ch\, u = |Ch\, u| = \sqrt{(Ch\, u)^2}$$

$$\int x^2 \sqrt{1 + 4\alpha^2 x^2}\, dx = \int \frac{1}{4\alpha^2} (Sh\, u)^2 \sqrt{(Ch\, u)^2} \cdot \frac{1}{2\alpha} Ch\, u\, du$$

$$= \frac{1}{8\alpha^3} \int \underbrace{(Sh\, u)^2 (Ch\, u)^2}_{\frac{1}{4}(Sh(2u))^2} du, \qquad (Ch\, u)^2 = 1 + (Sh\, u)^2$$

$$\int (Sh\, u)^2 du \qquad \qquad \int (Sh\, u)^4 du$$

$$\int (Sh\, u)^2 du = \int Sh\, u\, (Ch\, u)' du = Sh\, u\, Ch\, u - \int (Ch\, u)^2 du$$

$$(Ch\, u)^2 = 1 + (Sh\, u)^2$$

$$= Sh\, u\, Ch\, u - u - \int (Sh\, u)^2 du$$

$$\Rightarrow \cancel{2} \int (Sh\, u)^2 du = \frac{1}{2}(Sh\, u\, Ch\, u - u)$$

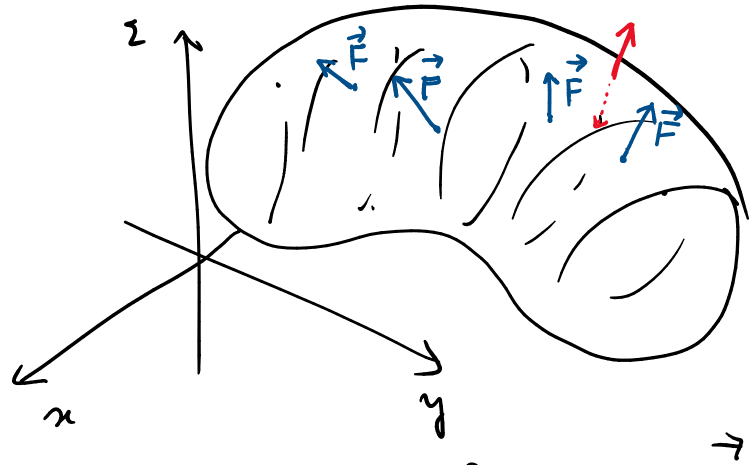
$$\int (Sh\, u)^4 du =$$

You finish.

Divergence Thm

Outward flux of a vector field $\vec{F}$ by $\partial\Omega$ Ω open set.

Let $\Omega \subset \mathbb{R}^3$ be an open subset of $\mathbb{R}^3$



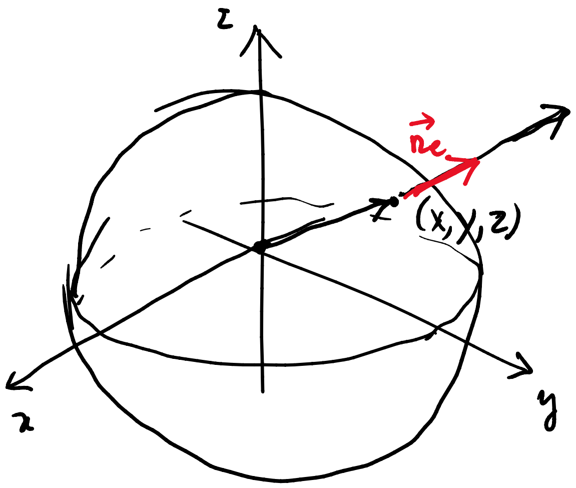
We want to def the flux of $\vec{F}$ through the surface $\mathcal{M} = \partial\Omega$
$\uparrow$
boundary of Ω

Assuming that $\mathcal{M} = \partial\Omega$ be a parametric surface, $\mathcal{M} = \Phi(D)$, the flux through $\mathcal{M}$ is given by

$$\int_{\partial\Omega} \vec{F} \cdot \vec{n} \, d\sigma$$

We will denote by $\vec{n}_e$ the normal unit vector pointing outward resp to $\partial\Omega$.
$\uparrow$
exterior

<u>Example</u>: $\mathcal{M} = \{ x^2 + y^2 + z^2 = r^2 \}$



$$\vec{n}_e = \frac{(x,y,z)}{\|(x,y,z)\|} = \frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}} = \frac{1}{r}(x,y,z)$$

<u>Def</u>: We call outward flux of $\vec{F}$ through $\partial\Omega$

$$\int_{\partial\Omega} \vec{F}\cdot\vec{n}_e \, d\sigma.$$

<u>Thm</u> (divergence thm)

$$\int_{\partial\Omega} \vec{F}\cdot\vec{n}_e \, d\sigma = \int_{\substack{\Omega \\ \cap \\ \mathbb{R}^3}} \operatorname{div}(\vec{F}) \, dx\,dy\,dz$$

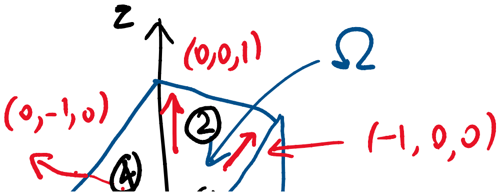
where $\operatorname{div}\vec{F} = \partial_x f + \partial_y g + \partial_z h = \underbrace{\nabla}_{(\partial_x, \partial_y, \partial_z)} \cdot \vec{F}$

$\vec{F} = (f, g, h)$ (divergence of $\vec{F}$)

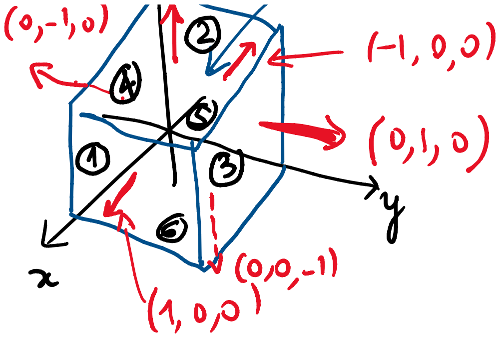
$\vec{F}(x,y,z) = (f(x,y,z), g(x,y,z), h(x,y,z))$

Idea of why div thm holds.

$\Omega = \text{cube} = [0,1]^3$



Ω = cube = $[0,1]$



$$\int_{\partial\Omega} \vec{F}\cdot\vec{n}_e = \int_{①} + \int_{②}$$

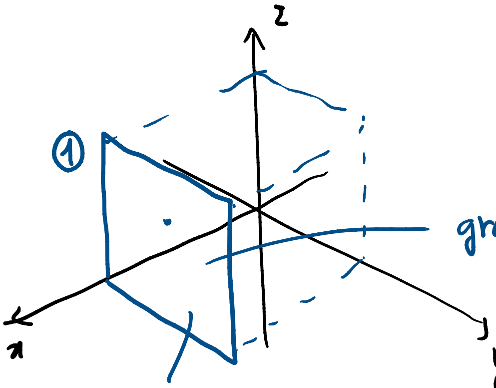
$$+ \int_{③} + \int_{④}$$

$$+ \int_{⑤} + \int_{⑥}$$

$$\int_{①} + \int_{②} = \int_{①} \underbrace{\vec{F}\cdot(1,0,0)}_{f} + \int_{②} \underbrace{\vec{F}\cdot(-1,0,0)}_{-f}$$

$\vec{F} = (f,g,h)$

$$= \int_{①} f + \int_{②} -f$$



graph of $\varphi(y,z) \equiv 1$

$\Phi(y,z) = (1,y,z)$ $\quad (y,z) \in [0,1]^2$

$\partial_y\Phi \wedge \partial_z\Phi$
$= (0,1,0)\wedge(0,0,1) = (1,0,0)$

$$\int_{①} = \int_{[0,1]^2} f(1,y,z)\cdot 1 \, dy\,dz$$

$$\int_{②} = \int_{[0,1]^2} f(0,y,z)\, dy\,dz$$

$$\Rightarrow \int_{①} f - \int_{②} f = \int_{[0,1]^2} \left(f(1,y,z) - f(0,y,z) \right) dy\,dz$$

$$\left(\int_0^1 \partial_x f \, dx \right)$$

$$\varphi(b) - \varphi(a) = \int_a^b \varphi'(t)\,dt$$

$$= \int_{[0,1]^3} \partial_x f \, dx\,dy\,dz \quad +$$

Similarly

$$\int_{③} + \int_{④} = \int_{[0,1]^3} \partial_y g \, dx\,dy\,dz \quad +$$

$$\int_{⑤} + \int_{⑥} = \int_{[0,1]^3} \partial_z h \, dx\,dy\,dz \quad +$$

$$\int_{\partial\Omega} \vec{F} \cdot \vec{n}_e = \int_{\Omega} \underbrace{\partial_x f + \partial_y g + \partial_z h}_{\operatorname{div} \vec{F}.} \, dx\,dy\,dz$$

Exercise 5.4.3

Let $\Omega = \{ (x,y,z) \in \mathbb{R}^3 : \; x^2+y^2 < z < 1 + \sqrt{1-(x^2+y^2)} \}$

Compute the outward flux by Ω for

$$\vec{F} = (x, y, x^2+y^2)$$

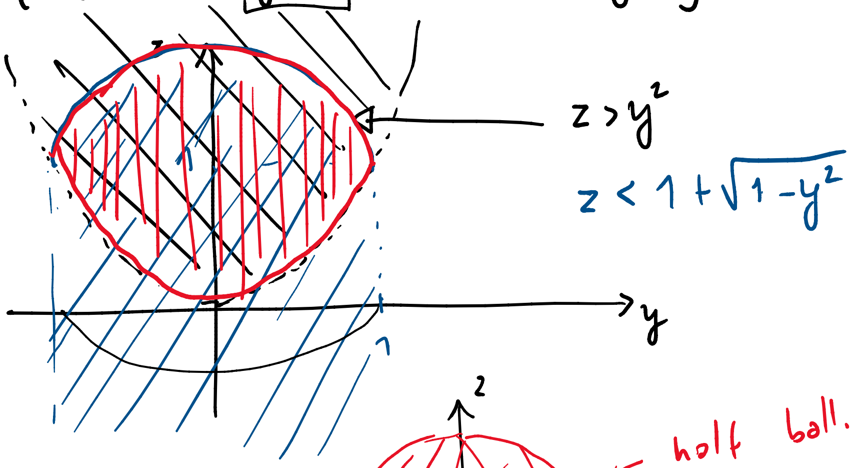
determining also its components of different parts of $\partial\Omega$.

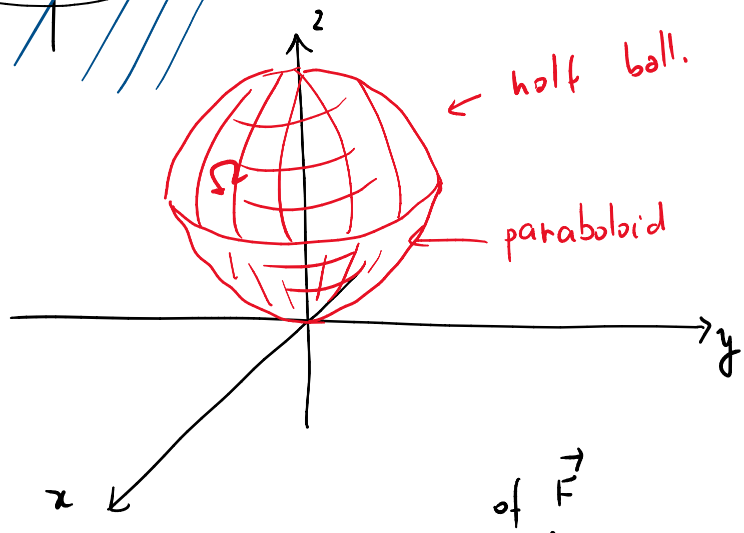
Sol: Let's first give a look to Ω.

Ω is a solid invariant by rotations around

Ω is a solid invariant by rotations around z axis. Let's plot $\Omega \cap \{x=0\}$

$$= \{ (y,z) : y^2 < z < 1 + \sqrt{1-y^2} \}$$





To compute the outward flux of $\vec{F}$ by Ω we use the div thm:

$$\int_{\partial\Omega} \vec{F} \cdot \vec{n}_e = \int_{\Omega} \operatorname{div} \vec{F} \, dx\,dy\,dz$$

$$\vec{F} = (x, y, x^2+y^2)$$

$$\operatorname{div} = \nabla \cdot \vec{F} = (\partial_x, \partial_y, \partial_z)(x, y, x^2+y^2)$$

$$= \partial_x x + \partial_y y + \partial_z (x^2+y^2)$$

$$= 1 + 1 + 0 = 2$$

$$= \int_\Omega 2 \, dx\,dy\,dz = 2 \int_\Omega dx\,dy\,dz$$

$$\underset{\text{cyl coords}}{=} 2 \int_{\substack{\rho^2 \leq z \leq 1+\sqrt{1-\rho^2} \\ 0 \leq \theta \leq 2\pi}} \rho \; d\rho\, d\theta\, dz$$

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z \end{cases}$$

$$\overset{RF}{=} 2 \int_{\rho^2 \leq z \leq 1+\sqrt{1-\rho^2}} \int_0^{2\pi} \rho \; d\theta \; d\rho\, dz$$

$$= 4\pi \int_{\rho^2 \leq z \leq 1+\sqrt{1-\rho^2}} \rho \; d\rho\, dz$$

$$= 4\pi \int_0^1 \rho \int_{\rho^2}^{1+\sqrt{1-\rho^2}} 1 \; dz \; d\rho$$

$$\rho^2 \leq 1 + \sqrt{1-\rho^2} \iff \underset{\ominus}{\rho^2 - 1} \leq \underset{\oplus}{\sqrt{1-\rho^2}}$$

$0 \leq \rho \leq 1$ true

$$= 4\pi \int_0^1 \rho \left(1 + \sqrt{1-\rho^2} - \rho^2\right) d\rho$$

$$= 4\pi \Big[\int_0^1 \rho - \rho^3 + \rho\sqrt{1-\rho^2} \; d\rho$$

$$= \frac{\rho^2}{2}\Big|_0^1 - \frac{\rho^4}{4}\Big|_0^1 + -\frac{1}{3}\int_0^1 -3(1-\rho^2)^{1/2} \rho \, d\rho \Big]$$

$$\partial_\rho (1-\rho^2)^{3/2} = \frac{3}{2}(1-\rho^2)^{1/2}(-2\rho)$$

$$= 4\pi \left[\frac{1}{2} - \frac{1}{4} - \frac{1}{3} \left[(1-\rho^2)^{3/2} \right]_0^1 \right]$$

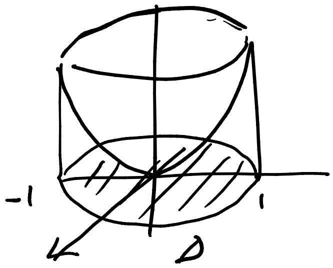
0 - 1

$$= 4\pi\left[\frac{1}{2} - \frac{1}{4} + \frac{1}{3}\right].$$

Now because $\partial\Omega$ = paraboloid $\cup$ ½ sphere

$$\Rightarrow \underbrace{\int_{\partial\Omega} \vec{F}\cdot\vec{n}_e}_{\text{known}} = \int_{\text{parab}} + \int_{\text{½ sphere}}$$

$$\int_{\text{parab}} \vec{F}\cdot\vec{n}_e \, d\sigma = \int_D \det\begin{bmatrix} \vec{F} \\ \partial_u\phi \\ \partial_v\phi \end{bmatrix} du\,dv \qquad z = x^2+y^2$$



$$\phi(x,y) = \underline{(x, y, x^2+y^2)}$$

on $D = \{x^2+y^2 \leq 1\}$

$$= \int_{x^2+y^2\leq 1} \det\begin{bmatrix} x & y & x^2+y^2 \\ 1 & 0 & 2x \\ 0 & 1 & 2y \end{bmatrix} dx\,dy$$

$$= \int_{x^2+y^2\leq 1} x \det\begin{bmatrix} 0 & 2x \\ 1 & 2y \end{bmatrix} - 1 \det\begin{bmatrix} y & x^2+y^2 \\ 1 & 2y \end{bmatrix} dx\,dy$$

$$= \int_{x^2+y^2\leq 1} \overset{-2x^2}{-x\,2x} - \left(2y^2 - (x^2+y^2)\right) \, dx\,dy$$

$$-2(x^2+y^2) + (x^2+y^2) = -(x^2+y^2)$$

$$= -\int_{x^2+y^2\leq 1} x^2+y^2 \, dx\,dy$$

$$= \quad . \quad - \int \rho^2\cdot\rho \, d\rho\, d\theta \overset{RF}{=} -2\pi\int_0^1 \rho^3 d\rho$$

$$\shortparallel$$

$$\overset{=}{\text{pol coords}} \; - \int_{\substack{0\le\rho\le1\\0\le\theta\le2\pi}} \rho^2\cdot\rho \, d\rho \, d\theta \overset{\text{FV}}{=} -2\pi\int_0 \rho \, d\rho$$

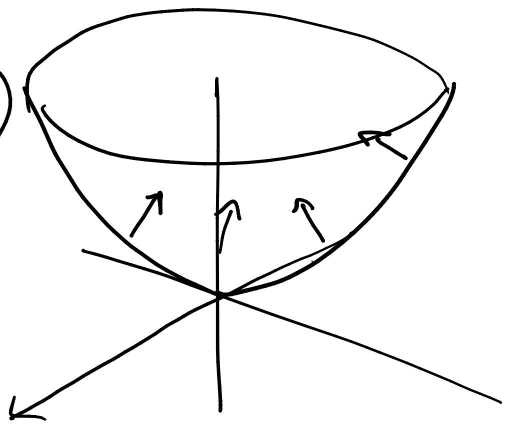
$$= -\frac{2\pi}{4} \qquad \frac{\rho^4}{4}\Big|_0^1$$

$$= -\frac{\pi}{2}.$$

To be sure that $\vec{n} = \vec{n}_e$ we need to check if $\partial_x\phi \wedge \partial_y\phi$ is pointing inward or outward on D.

$$\partial_x\phi\wedge\partial_y\phi = \det\begin{bmatrix} i & j & k \\ 1 & 0 & 2x \\ 0 & 1 & 2y \end{bmatrix}$$

$$= (-2x, -2y, 1)$$



$$\vec{n} = \frac{(-2x,-2y,1)}{\sqrt{1+4(x^2+y^2)}}$$

so $\displaystyle\int_{\text{parab}} \vec{F}\cdot\vec{n}_e = -\left(-\pi/2\right) = +\pi/2$

The other component can be derived by diff. $\boxed{\square}$

Do Ex 5.7.2 $\longrightarrow$ 5.7.13