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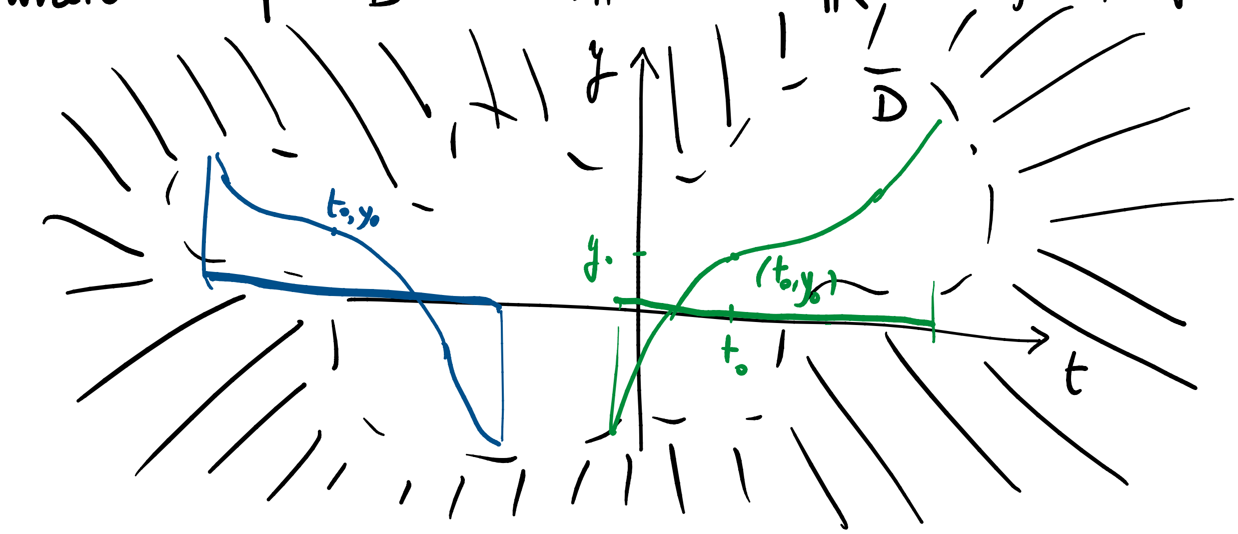
Let's consider a CP

$$CP(t_0,y_0) \quad \begin{cases} y'(t) = f(t,y(t)) \\ y(t_0) = y_0 \end{cases}$$

where $f : D \subset \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$, $f = f(t,y)$

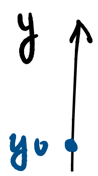


We may notice that in general the time int. of def depends on the initial cond

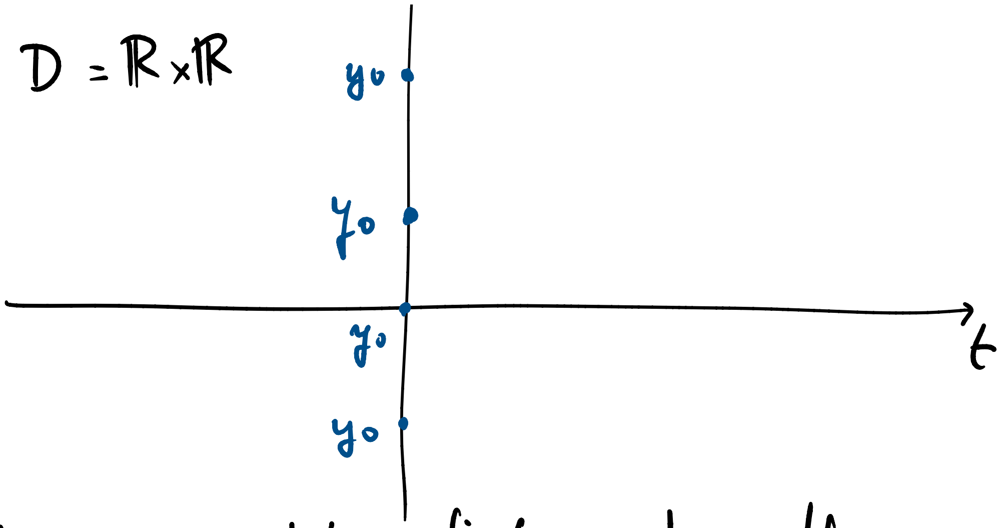
Ex.
$$\begin{cases} y' = y^2 = f(t,y) \\ y(0) = y_0 \end{cases}$$

Here $f : D \subset \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ $\quad D = \{(t,y)\in\mathbb{R}^2\} = \mathbb{R}^2$

$D = \mathbb{R}\times\mathbb{R}$



$D = \mathbb{R} \times \mathbb{R}$



To det sol let's first solve the eqn

$$y' = y^2 \underset{y \neq 0}{\iff} \frac{y'}{y^2} = 1$$

$$\left(-y^{-1}\right)' = -(-1)\, y^{-2} y' = + \frac{y'}{y^2}$$

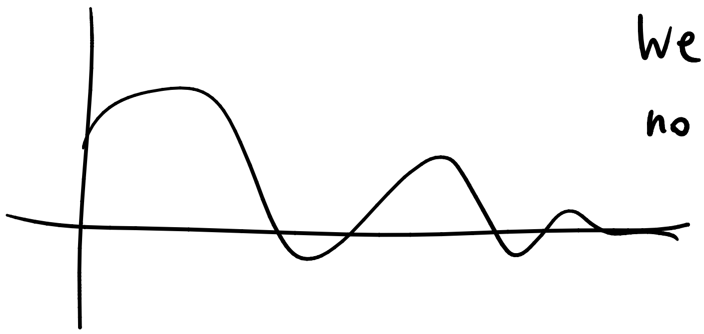
$$\iff \left(-\frac{1}{y}\right)' = 1$$

$$\iff -\frac{1}{y} = \int 1\, dt + c = t + c$$

$$\iff \frac{1}{y} = -t - c$$

$$\iff \boxed{\boxed{y = \frac{1}{-t-c}} \qquad c \in \mathbb{R}} \quad \text{(non zero sols)}$$

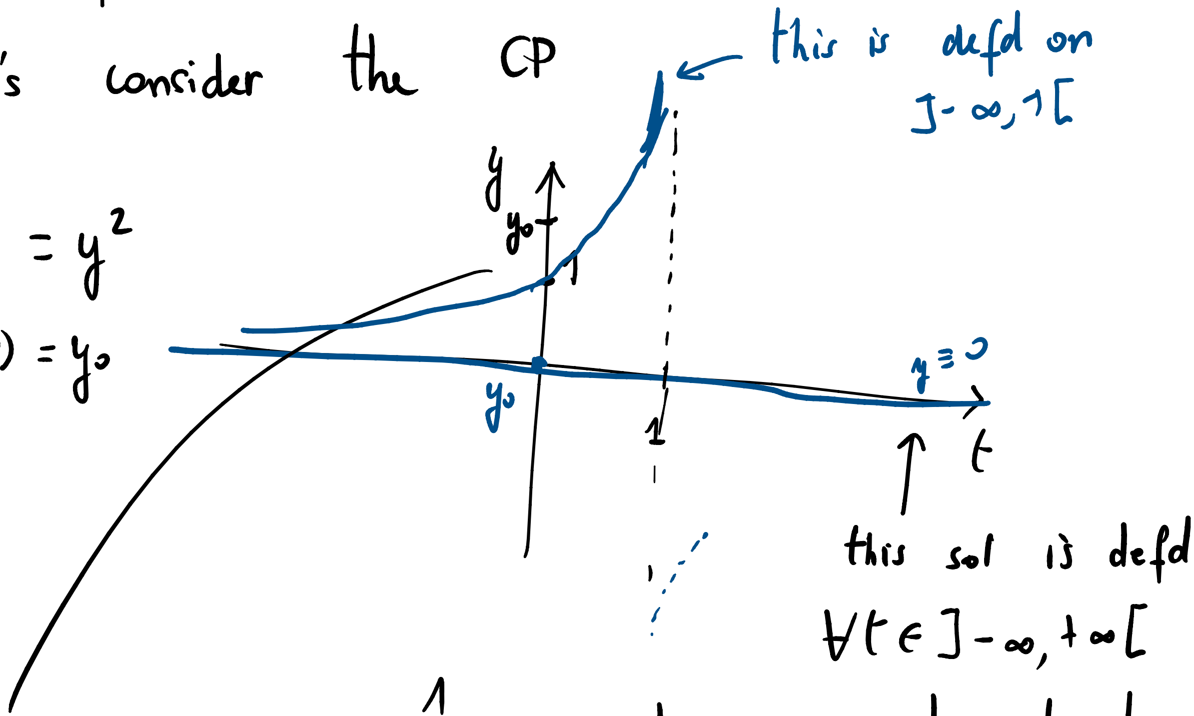
We may notice also that $\boxed{y \equiv 0}$ is a sol.



We accept there are no other sols.

Now let's consider the CP

$$\begin{cases} y' = y^2 \\ y(0) = y_0 \end{cases}$$



here sol is $y = \frac{1}{-t-c}$ where c has to be determined imposing $y(0) = 1$

$$-\frac{1}{c} = 1 \quad \Rightarrow \quad c = -1$$

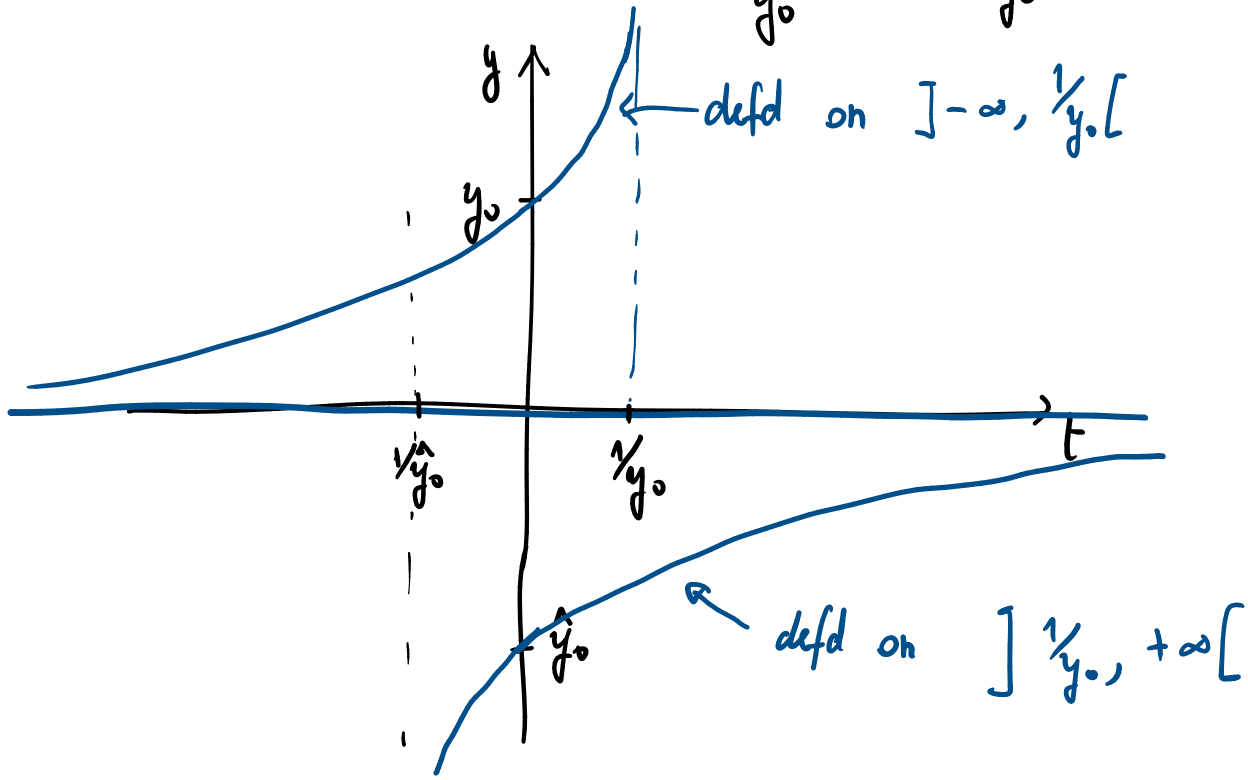
$$\Rightarrow \quad y = \frac{1}{-t-(-1)} = \frac{1}{1-t}.$$

For general $y(0) = y_0 \neq 0$, sol is $y = \frac{1}{-t-c}$

$$\Rightarrow \quad y_0 = \frac{1}{-c} \quad \Rightarrow \quad c = -1/y_0$$

$$\Rightarrow \quad y(t) = \frac{1}{\quad} = \frac{1}{\quad}.$$

$$\Rightarrow \quad y(t) = \frac{1}{-t + 1/y_0} = \frac{1}{1/y_0 - t}.$$



Example

$$\begin{cases} y' = 1 + y^2 = f(t,y) \\ y(0) = y_0 \end{cases}$$

$f : D = \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$

$f \in \mathcal{C}(D)$ but

$\partial_y f = 2y$ is not bounded on D

$\Downarrow$

Gl. $\exists!$ doesn't apply.

This eqn. can be solved

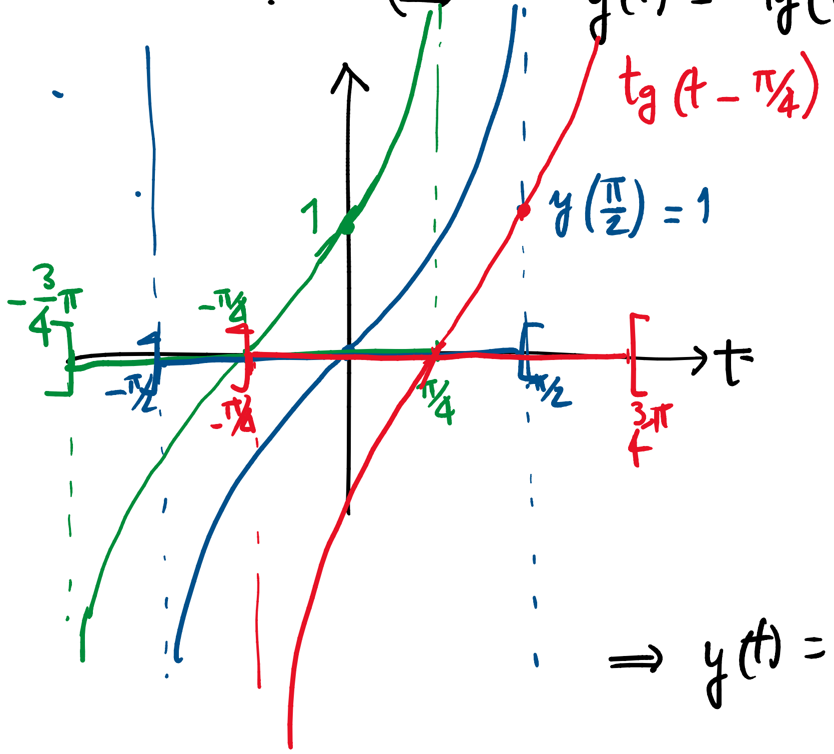
$$y' = 1 + y^2$$

$$\Updownarrow$$

$$\frac{y'}{1+y^2} = 1 \iff (\operatorname{arctg} y)' = 1$$

$$\Longleftrightarrow \quad \operatorname{arctg} y(t) = t + c$$

$$\Longleftrightarrow \quad y(t) = \operatorname{tg}(t+c)$$



$$\begin{cases} y' = 1 + y^2 \\ y(0) = 0 \end{cases}$$

$$\operatorname{tg}(t+c)\big|_{t=0} = \operatorname{tg} c = 0$$

$$\Longleftrightarrow \quad c = k\pi$$

$$\Rightarrow \quad y(t) = \operatorname{tg}(t + k\pi) = \operatorname{tg} t$$

$$\begin{cases} y' = 1 + y^2 \\ y(0) = 1 \end{cases}$$ Imposing $\operatorname{tg}(t+c)\big|_{t=0} = 1 \Longleftrightarrow \operatorname{tg} c = 1$

$$c = \pi/4 + k\pi$$

$$\Rightarrow \quad y(t) = \operatorname{tg}\left(t + \frac{\pi}{4} + k\pi\right) = \operatorname{tg}\left(t + \pi/4\right)$$

$$\begin{cases} y' = 1 + y^2 \\ y(\frac{\pi}{2}) = 1 \end{cases} \qquad \operatorname{tg}(t+c)\big|_{t=\pi/2} = 1$$

$$\parallel$$

$$\operatorname{tg}\left(\frac{\pi}{2} + c\right) = 1$$

$$\parallel$$

$$-\operatorname{tg} c = 1 \qquad c = -\frac{\pi}{4} + k\pi$$

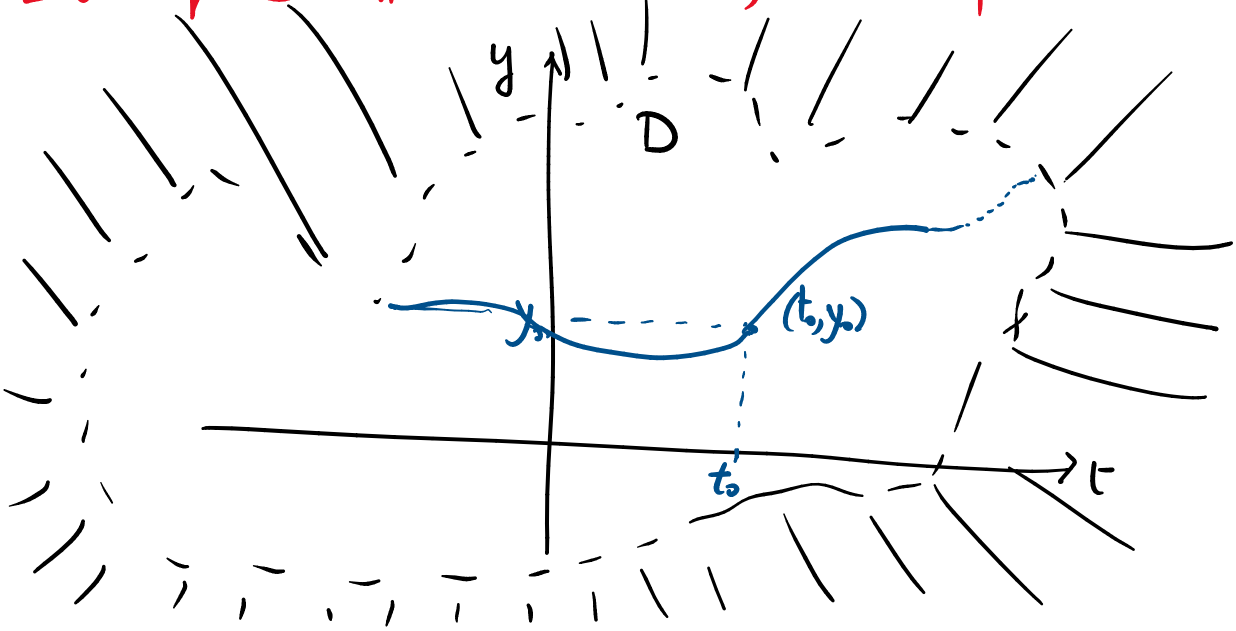
$$\begin{array}{c} \| \\ -\operatorname{tg} c = 1 \end{array} \qquad c = -\frac{\pi}{4} + k\pi$$

$$\Rightarrow \quad y(t) = \operatorname{tg}\left(t - \tfrac{\pi}{4}\right)$$

Let's return to the gen discussion.

Thm: (Local Existence and Uniqueness thm)

Let $f : D \subset \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, D open



be such that

1. $f \in \mathscr{C}(D)$
2. $\partial_y f \in \mathscr{C}(D)$

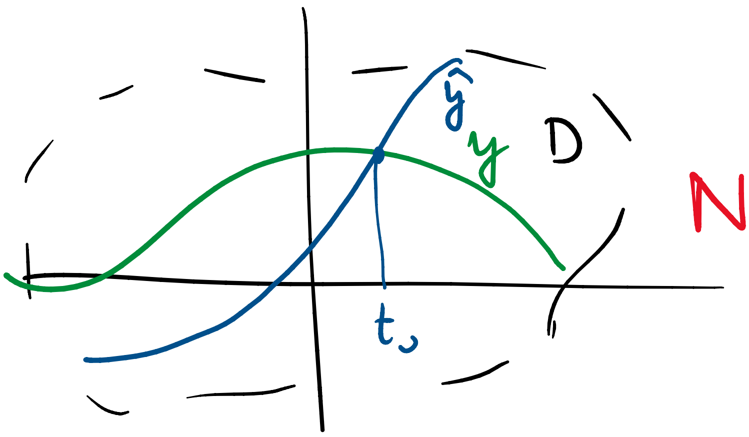
Then

(Existence) $\forall (t_0, y_0) \in D \ \exists y :]\alpha, \beta[\longrightarrow \mathbb{R}$ sol of CP s.t. $\nexists \underbrace{J \supset]\alpha, \beta[}_{\neq}$ on which y can s.t. $y : J =]\alpha, \beta[$

be defined.

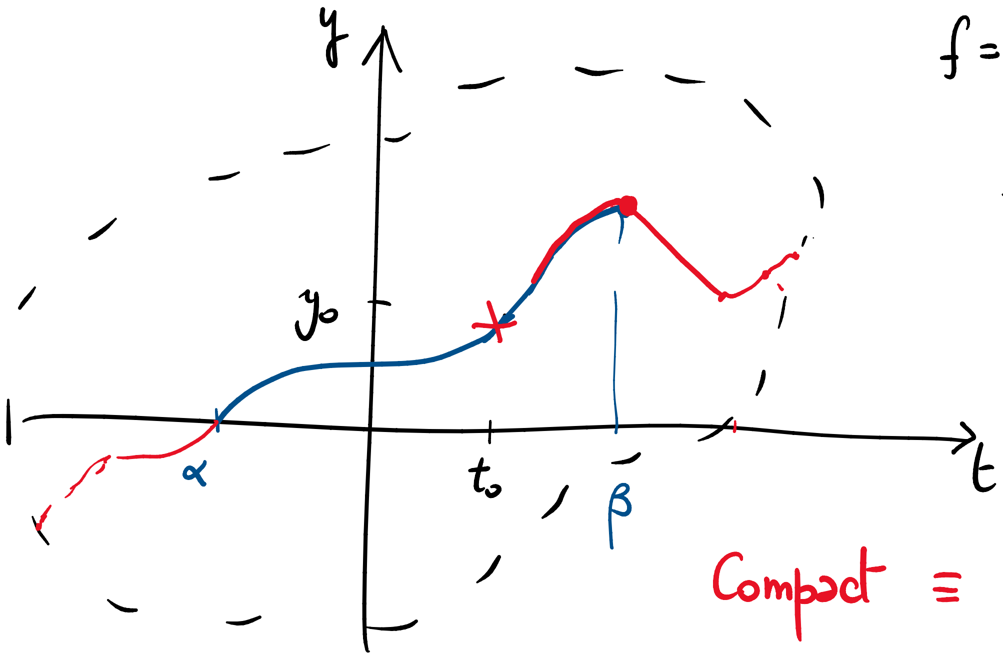
maximal time interval of definition for y.

(Uniqueness) If $y, \hat{y}$ are solutions of the eqn and $y(t_0) = \hat{y}(t_0) \Rightarrow y = \hat{y}$ where both defined



NO!

Exit from compact sets



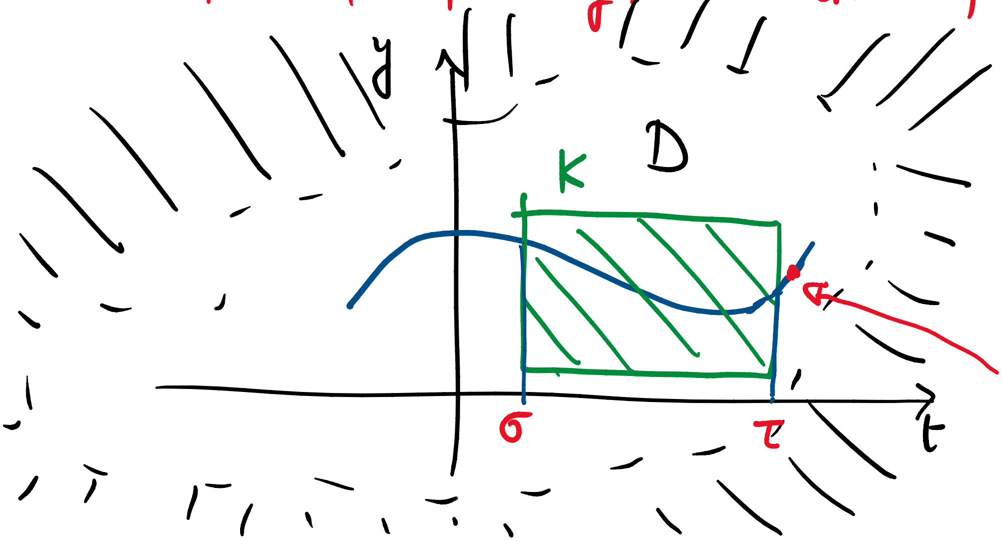
$f = f(t,y) : D \subset \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

D open

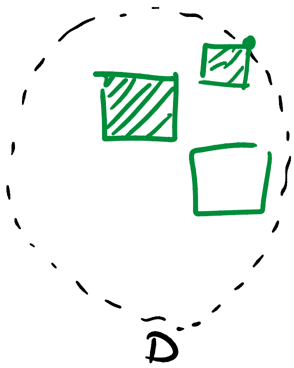
Compact $\equiv$ closed & bdd

Thm: Assume f verifies Local $\exists!$ conds

($f\in\mathcal{C}(D)$, $\partial_y f\in\mathcal{C}(D)$, D open). Let $y:]\alpha,\beta[\longrightarrow \mathbb{R}$ be a sol of a CP with $]\alpha,\beta[$ maximal time int of def for y. Then for every compact set $K\subset D$



$\exists\, \alpha<\sigma<\tau<\beta$: $\forall t<\sigma$, $\forall t>\tau$ $(t,y(t))\notin K$



Qualitative Study of Diff Eqns

Ex 7.3.1

Consider the CP

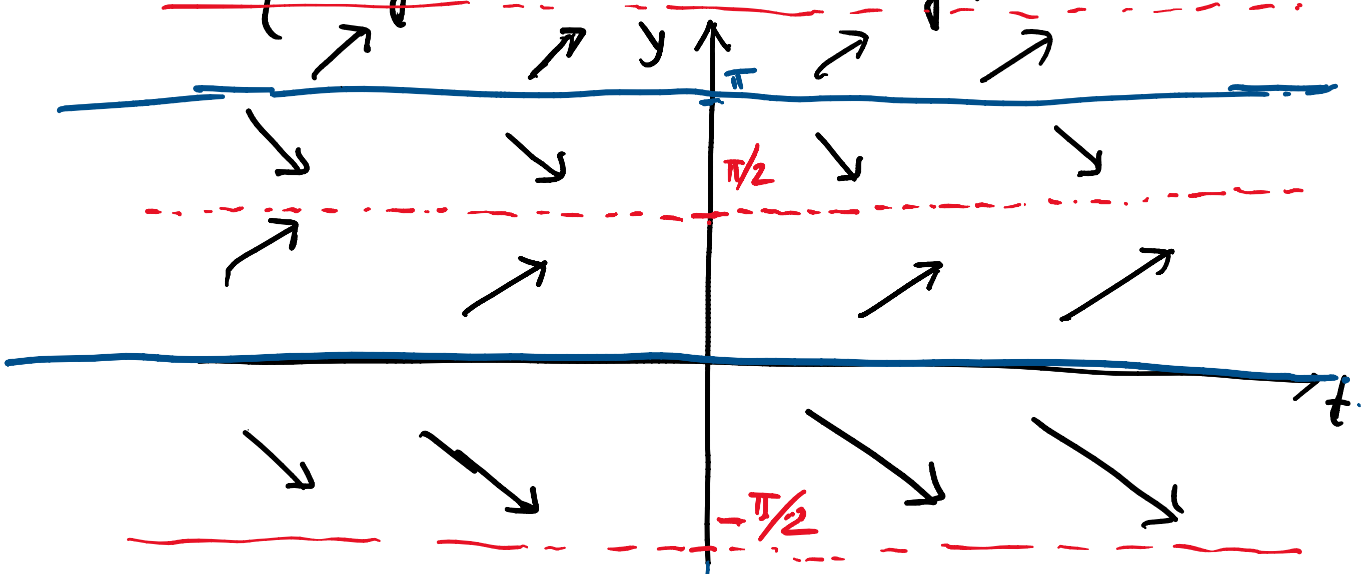
$$\begin{cases} y' = \dfrac{tg\, y}{1+y^2} = f(t,y) \\ y(0) = y_0 \end{cases}$$

i) Show that local $\exists$ and uniqueness holds.

i) Show that local $\exists$ and uniqueness holds. Find constant sols. and regions of domain D where sols are increasing/decreasing

Sol: Here $f(t,y) = \dfrac{tg\, y}{1+y^2}$ defined on

$$D = \left\{ (t,y) \in \mathbb{R}^2 : t \in \mathbb{R},\ y \neq \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z} \right\}$$



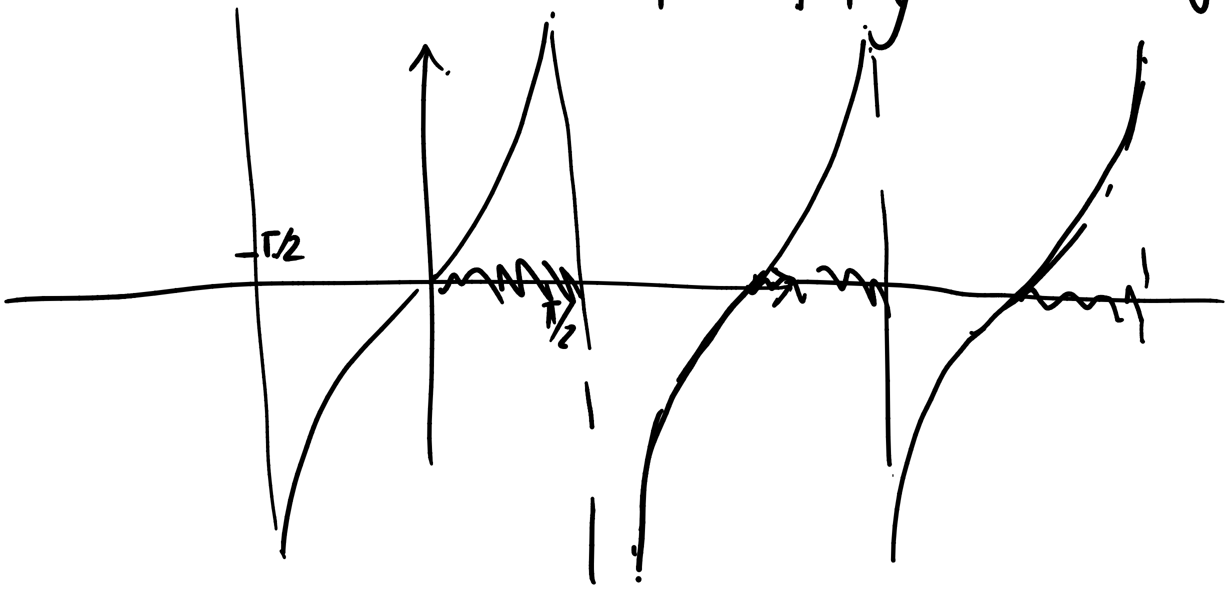
Clearly D is open, $f \in \mathcal{C}(D)$, $\partial_y f \in \mathcal{C}(D)$

$\Rightarrow$ Local existence and uniqueness holds.

$y \equiv C$ is a sol. $\underset{y'=0}{\Leftrightarrow}$ $0 = \dfrac{tg\, C}{1+C^2}$

$\Leftrightarrow tg\, C = 0 \Leftrightarrow C = k\pi \quad k \in \mathbb{Z}$

Now: $y \nearrow \iff 0 \le y' \underset{\text{eqn}}{=} \dfrac{\operatorname{tg} y}{1+y^2} \iff \operatorname{tg} y \geq 0$



$\iff k\pi \le y < \frac{\pi}{2} + k\pi \quad k \in \mathbb{Z}$

. . .

ii) Consider CP $y(0) = y_0 \in \left]0, \frac{\pi}{2}\right[$