martedì 17 marzo 2020 08:55

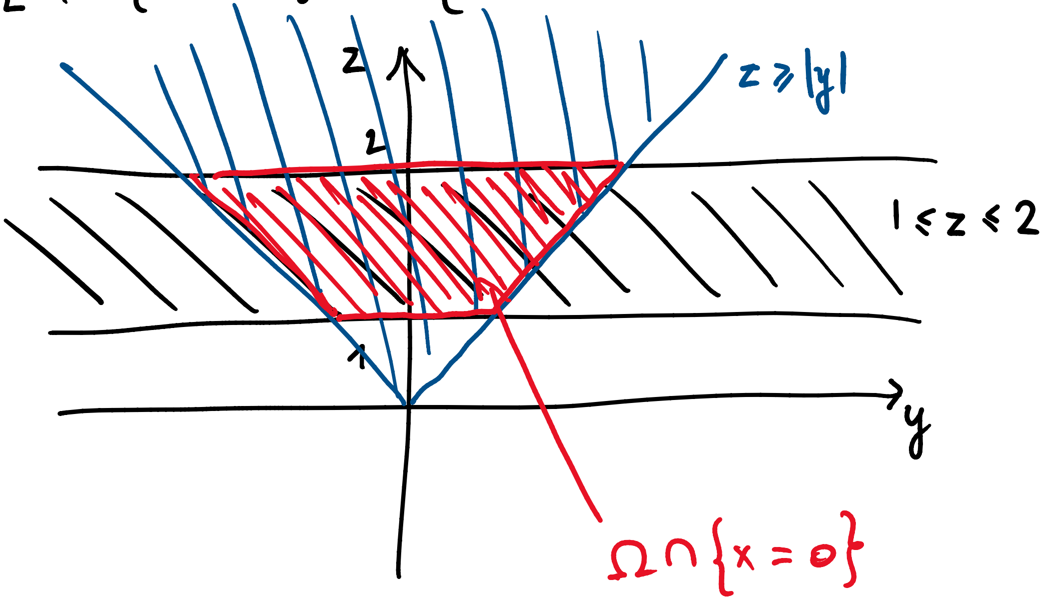
Exercises on Divergence Thm

Ex 5.7.3 $\Omega = \{ (x,y,z) \in \mathbb{R}^3 : 1 \le z \le 2,\ z \ge \sqrt{x^2+y^2} \}$

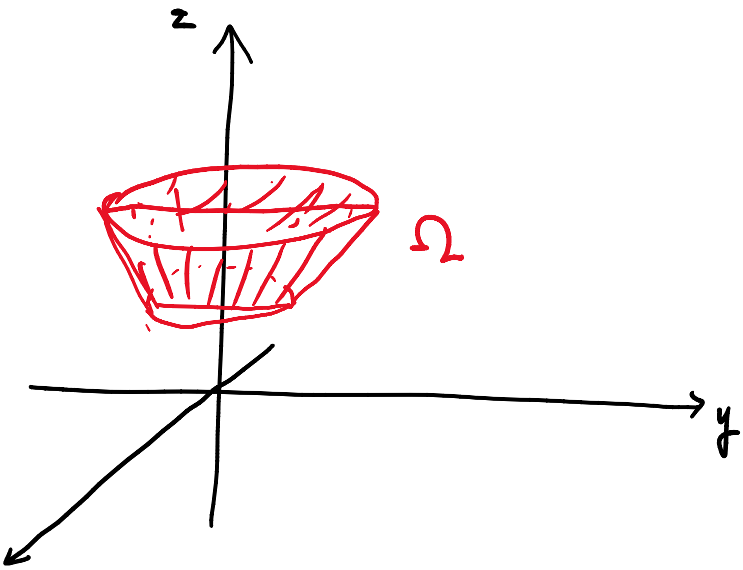
Draw Ω and compute outward flux of $\vec{F} = (x^3, 0, z^2)$ by Ω.

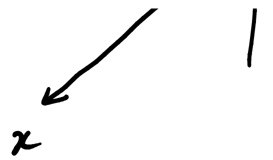
Sol: Ω is rotation invariant respect to z-axis.
We may plot first $\Omega \cap$ plane $yz = \Omega \cap \{x=0\}$ then rotate this plane figure around z-axis.

$$\Omega \cap \{x=0\} = \{ 1 \le z \le 2,\quad z \ge \sqrt{y^2} = |y| \}$$



therefore





To compute the outward flux of $\vec{F}$, we apply divergence thm:

$$\int_{\partial\Omega} \vec{F}\cdot\vec{n}_e \, d\sigma = \int_{\Omega} \operatorname{div}\vec{F} \; dx\,dy\,dz$$

being $\operatorname{div}\vec{F} = \nabla\cdot\vec{F} \equiv (\partial_x, \partial_y, \partial_z)\cdot(x^3, 0, z^3)$

$$= \partial_x x^3 + \partial_y 0 + \partial_z z^3$$

$$= 3(x^2+y^2)$$

$$= \int_{\Omega} 3(x^2+y^2)\, dx\,dy\,dz$$

$$\underset{\text{cyl coords}}{=} 3\int_{\substack{1\le z\le 2\\ z\ge\rho\\ 0\le\theta\le 2\pi}} \rho^2\cdot\rho \; d\rho\, d\theta\, dz$$

$$\overset{RF}{=} 6\pi \int_{\substack{1\le z\le 2\\ 0\le\rho\le z}} \rho^3 \, d\rho\, dz$$

$$\overset{RF}{=} 6\pi \int_1^2 \left(\int_0^z \rho^3\, d\rho\right) dz$$

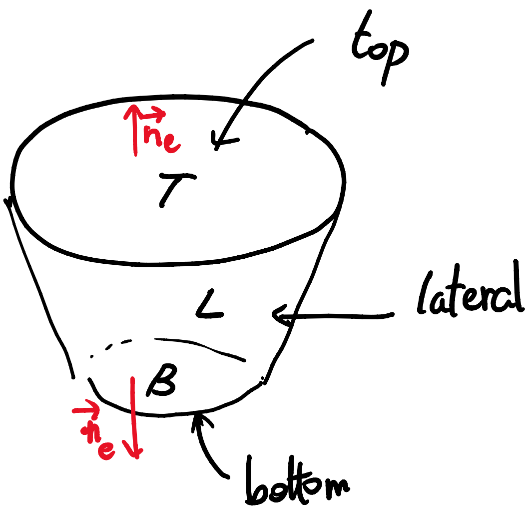
$$\rho^4 \,\Big|^z \qquad z^4$$

$$\frac{\rho^4}{4}\Big|_0^z = \frac{z^4}{4}$$

$$= \frac{3}{2}\pi \int_1^2 z^4\,dz = \frac{3}{2}\pi \, \frac{z^5}{5}\Big|_1^2$$

$$= \frac{3}{10}\pi(32-1) = \frac{93}{10}\pi.$$

<u>Extra:</u> To compute the components of outward flux on parts of $\partial\Omega$ we may notice that



$$\int_{\partial\Omega} \vec{F}\cdot\vec{n}_e = \int_T \vec{F}\cdot\vec{n}_e + \int_B \vec{F}\cdot\vec{n}_e + \int_L \vec{F}\cdot\vec{n}_e$$

$$\int_T \vec{F}\cdot\vec{n}_e\,d\sigma = \int_{\{(x,y,2):\ x^2+y^2\le 4\}} \underbrace{\underbrace{(x^3,0,2^3)}_{\vec{F}(x,y,2)}\cdot(0,0,1)}_{=\,2^3=8}\,d\sigma$$

$$= 8\int_{x^2+y^2\le 4} dx\,dy = 8\pi\cdot 4 = 32\pi.$$

$\|\ \ \| = 1$ (easy)

$$\int_B \vec{F}\cdot\vec{n}_e\,d\sigma = \int_{\{(x,y,1):\ x^2+y^2\le 1\}} (x^3,0,1^3)\cdot(0,0,-1)\,d\sigma$$

$$\int_B \vec F \cdot \vec n_e \, d\sigma = \int_{\{(x,y,1):\ x^2+y^2\le 1\}} \vec F(x,y,1) \ldots$$

$$= -\int_{x^2+y^2\le 1} dx\,dy = -\pi$$

$$\Rightarrow \int_L \vec F\cdot \vec n_e \, d\sigma = \int_{\partial\Omega} \vec F\cdot \vec n_e - \int_B - \int_T$$

$$= \frac{93}{10}\pi - 32\pi + \pi = -\pi\,\frac{310-93}{10}$$

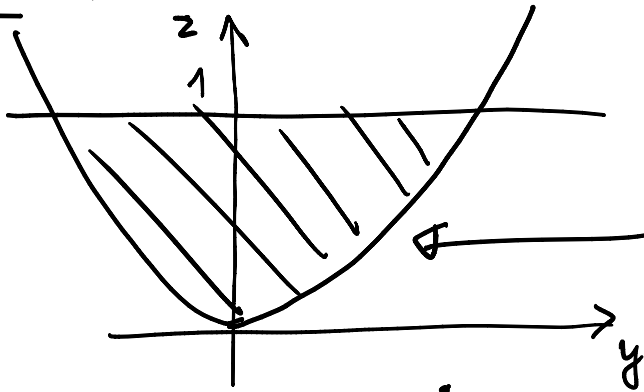
5.7.5 $\Omega = \{(x,y,z)\in\mathbb{R}^3 :\ x^2+y^2 \le z \le 1\}$

Draw Ω

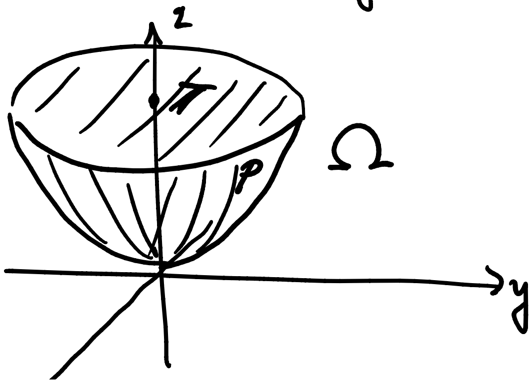
Area of $\partial\Omega$

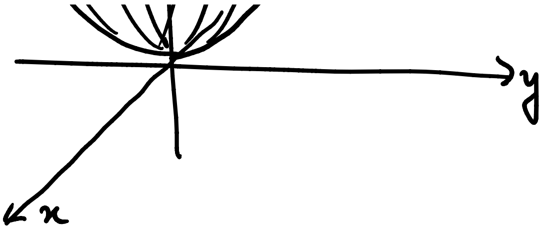
Outward flux of $\vec F = (x,y,z)$ and components.

Sol: Ω is rotation invariant resp. z-axis.



$\Omega \cap \{x=0\} = \{y^2 \le z \le 1\}$





$\partial\Omega = T \cup P$ $\qquad$ T = disk centred at $(0,0,1)$
radius $= 1$

$\text{Area}(T) = \pi \cdot 1^2 = \pi$

$P = \{ z = \underset{\substack{\uparrow \\ f(x,y)}}{x^2 + y^2} : x^2+y^2 \leq 1 \} = \text{Graph}(f)$

$$\Rightarrow \text{Area}(P) = \int_{x^2+y^2 \leq 1} \sqrt{1 + \|\nabla f\|^2}\, dx\, dy$$

$\nabla f = (2x, 2y) \rightarrow \|\nabla f\|^2 = 4x^2 + 4y^2$

$$\Rightarrow \text{Area}(P) = \int_{x^2+y^2 \leq 1} \sqrt{1 + 4(x^2+y^2)}\, dx\, dy$$

$$\underset{\text{pol. coords}}{=} \int_{\substack{0 \leq \rho \leq 1 \\ 0 \leq \theta \leq 2\pi}} \sqrt{1 + 4\rho^2}\, \rho\, d\rho\, d\theta$$

$$\overset{RF}{=} \cancel{2}\pi \int_0^1 (1+4\rho^2)^{1/2}\, 2\rho\, d\rho$$

$$\partial_\rho (1+4\rho^2)^{3/2} = \frac{3}{\cancel{2}} (1+4\rho^2)^{1/2}\, \cancel{4}8\rho$$

$$= \frac{\pi}{6} \int_0^1 (1+4\rho^2)^{1/2} \, 12\rho \, d\rho$$

$$= \frac{\pi}{6} \left[(1+4\rho^2)^{3/2} \right]_0^1$$

$$= \frac{\pi}{6} \left[5^{3/2} - 1 \right].$$

$$\Rightarrow \text{Area } \partial\Omega = \frac{\pi}{6}\left(5^{3/2} - 1\right) + \pi.$$

<u>Flux:</u> We apply the divergence thm:

$$\int_{\partial\Omega} \vec{F} \cdot \vec{n}_e = \int_{\Omega} \operatorname{div} \vec{F} \, dx \, dy \, dz = 3 \int_{\Omega} 1 \, dx \, dy \, dz$$

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \partial_x x + \partial_y y + \partial_z z = 3$$

$$= 3 \int_{x^2+y^2 \leq z \leq 1} dx \, dy \, dz$$

$$\underset{\text{cyl coords}}{=} 3 \int_{\substack{\rho^2 \leq z \leq 1 \\ 0 \leq \theta \leq 2\pi}} \rho \, d\rho \, d\theta \, dz$$

$$\overset{RF}{=} 6\pi \int_{\rho^2 \leq z \leq 1} \rho \, d\rho \, dz$$

$$\rho \leq 1$$

$$\overset{RF}{=} 6\pi \int_0^1 \left(\int_{\rho^2}^{1} \rho \, dz \right) d\rho \qquad \rho \leq 1$$

$$= 6\pi \int_0^1 \underbrace{\rho\,(1-\rho^2)}_{\rho - \rho^3} \, d\rho$$

$$= 6\pi \left[\underbrace{\frac{\rho^2}{2}\Big|_0^1}_{=\frac{1}{2}} - \underbrace{\frac{\rho^4}{4}\Big|_0^1}_{\frac{1}{4}} \right] = \frac{3}{2}\pi.$$

<u>Components:</u> $\displaystyle\int_{\partial\Omega} \vec{F}\cdot\vec{n}_e = \int_T \vec{F}\cdot\vec{n}_e + \int_P \vec{F}\cdot\vec{n}_e$

$$\int_T \vec{F}\cdot\vec{n}_e \, d\sigma = \int_{\{(x,y,1)\,:\, x^2+y^2\leq 1\}} \underbrace{(x,y,1)}_{\vec{F}(x,y,1)} \cdot (0,0,1) \, d\sigma$$

$$= \int_{x^2+y^2\leq 1} 1 \cdot \| \quad \| \, dx\,dy = \pi. \qquad \| \quad \| = 1$$

$$\int_P \vec{F}\cdot\vec{n}_e \, d\sigma = \frac{3}{2}\pi - \pi = \frac{\pi}{2} \qquad \blacksquare$$

<u>Rmk:</u> What if we want to compute directly

$$\int_P \vec{F} \cdot \vec{n_e} \, d\sigma \quad ?$$

Because $P = \{ z = x^2 + y^2 : x^2 + y^2 \leq 1 \}$, a standard param. would be

$$\phi(x,y) = (x, y, x^2 + y^2)$$

$$\Rightarrow \int_P \vec{F} \cdot \vec{n}_\phi = \int_{x^2+y^2 \leq 1} \det \begin{bmatrix} \vec{F}(\phi(x,y)) \\ \partial_x \phi \\ \partial_y \phi \end{bmatrix} dx\,dy$$

$$= \int_{x^2+y^2 \leq 1} \det \begin{bmatrix} x & y & x^2+y^2 \\ 1 & 0 & 2x \\ 0 & 1 & 2y \end{bmatrix} dx\,dy$$

$$= \int_{x^2+y^2 \leq 1} x \det \begin{bmatrix} 0 & 2x \\ 1 & 2y \end{bmatrix} - 1 \cdot \det \begin{bmatrix} y & (x^2+y^2) \\ 1 & 2y \end{bmatrix}$$

$$= \int_{x^2+y^2 \leq 1} x(-2x) - (\cancel{2}y^2 - (x^2 + \cancel{y}^2)) \, dx\,dy$$

$$- x^2 - y^2$$

$$= - \int (x^2 + y^2) \, dx\,dy$$

$$= - \int_{x^2+y^2\le 1} (x^2+y^2)\, dx\, dy$$

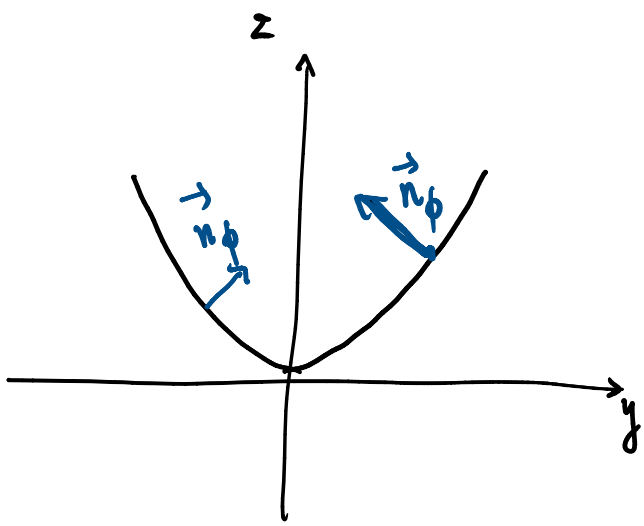
$$\underset{\text{pol coords}}{=} - \int_{\substack{0\le\rho\le 1\\ 0\le\theta\le 2\pi}} \rho^2\cdot\rho \; d\rho\, d\theta$$

$$\overset{RF}{=} -2\pi \underbrace{\int_0^1 \rho^3\, d\rho}_{=\frac{\rho^4}{4}\Big|_0^1} = -2\pi\cdot\frac{1}{4} = -\pi/2 .$$

Finally, to check if this is the right component, we should check if $\vec{n}_\phi = \vec{n}_e$ or $-\vec{n}_e$

Notice that, apart for a $\oplus$ scaling factor,

$$\vec{n}_\phi \text{ “=” } \partial_x\phi \wedge \partial_y\phi = \det\begin{bmatrix} i & j & k \\ 1 & 0 & 2x \\ 0 & 1 & 2y \end{bmatrix}$$



$$= (-2x, -4y, 1)$$

For example, on plane $x=0$

$$\vec{n}_\phi = (0, -2y, 1)$$

by which we see that

$$\vec{n}_\phi = -\vec{n}_e .$$

∎

$n_\phi = - \dots e\,\cdot$

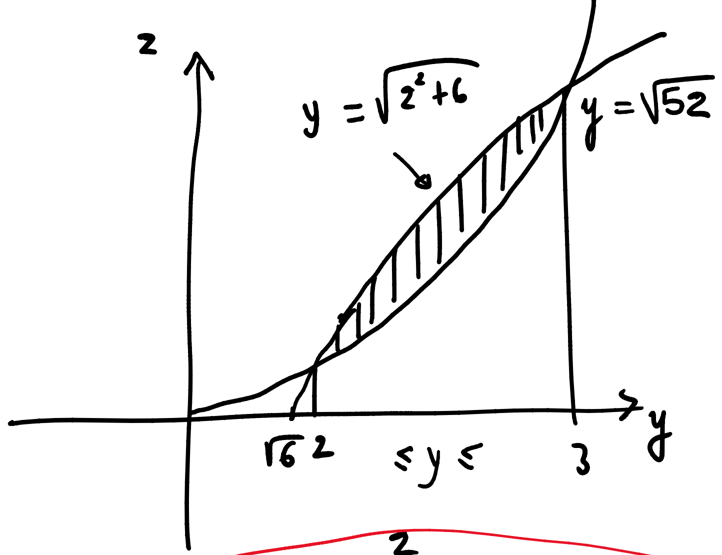
Ex 5.7.10 $\Omega = \{(x,y,z) : z^2+6 \le x^2+y^2 \le 5z\}$

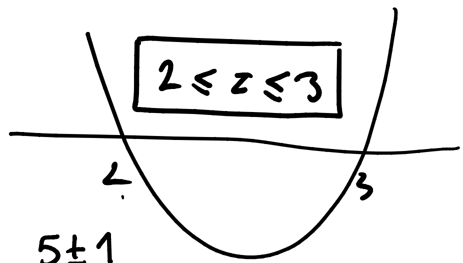
- Is Ω a rotation solid? Draw Ω
- Vol Ω
- Outward flux of $\vec{F} = (x^2, y, z^2)$ and components.

Sol: Yes, Ω is a rotation solid resp. to z-axis because eqns defining Ω depends on (x,y) through x^2+y^2. Section on plane yz: $\Omega \cap \{x=0\} =$

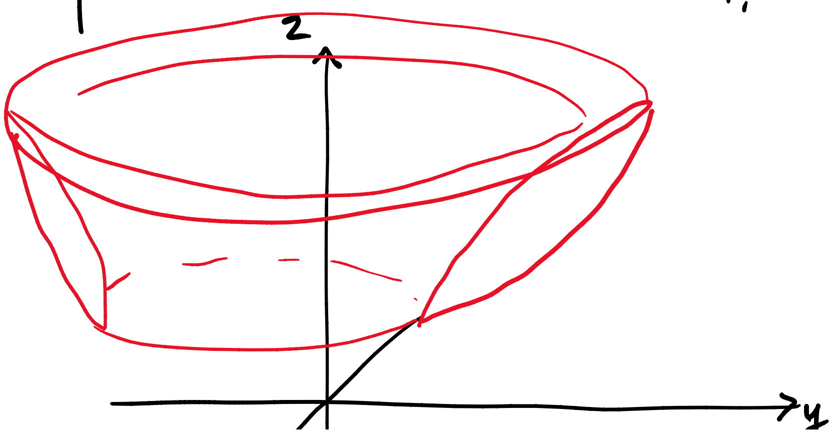
$$= \{ z^2+6 \le \underbrace{y^2 \le 5z}_{\Downarrow \; z \ge 0} \}$$

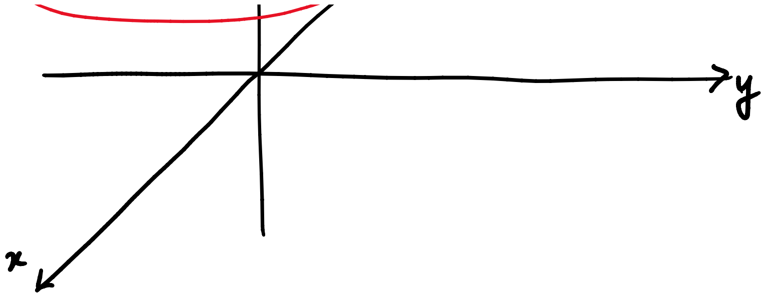
$$z^2+6 \le 5z$$
$$\Updownarrow$$
$$z^2-5z+6 \le 0$$





$$z_{1,2} = \frac{5 \pm 1}{2} = 2, 3$$





$$\mathrm{Vol}\,\Omega = \int_{\Omega} 1 \, dx\, dy\, dz$$

$$\underset{\text{cyl coords}}{=} \int_{\substack{z^2+6 \le \rho^2 \le 5z \\ 0 \le \theta \le 2\pi}} \rho \, d\rho \, d\theta \, dz$$

$$\overset{RF}{=} 2\pi \int_{z^2+6 \le \rho^2 \le 5z} \rho \, d\rho \, dz$$

$$\overset{RF}{=} 2\pi \int_2^3 \left(\int_{\sqrt{z^2+6}}^{\sqrt{5z}} \rho \, d\rho \right) dz$$

$$= \cancel{2}\pi \int_2^3 \frac{\rho^2}{\cancel{2}} \Big|_{\sqrt{z^2+6}}^{\sqrt{5z}} dz$$

$$= \pi \int_2^3 \left(5z - (z^2+6) \right) dz$$

$$= \pi \left[5 \, \frac{z^2}{2} \Big|_2^3 - \frac{z^3}{3} \Big|_2^3 - 6 \cdot 1 \right]$$

$$= \pi \left[\frac{5}{2} (3^2 - 2^2) - \frac{1}{3} (3^3 - 2^3) - 6 \right]$$

$$= \pi \Big[\tfrac{5}{2}\underbrace{(3^2-2^2)}_{\substack{9-4\\ \shortparallel \\ 5}} - \tfrac{1}{3}\underbrace{(3^3-2^3)}_{\substack{27-8\\ 19}} - 6 \Big]$$

$$= \pi \left[\frac{25}{2} - \frac{19}{3} - 6 \right] = \pi \, \frac{75 - 38 - 36}{6}$$

$$= \frac{\pi}{6}.$$

<u>Outward flux:</u>

$$\int_{\partial\Omega} \vec{F}\cdot\vec{n}_e \underset{\substack{\uparrow \\ \text{divergence thm}}}{=} \int_{\Omega} \nabla\cdot\vec{F} \; dx\,dy\,dz$$

$$= \int_{\Omega} \underbrace{\left(\partial_x (x^2) + \partial_y y + \partial_z z^2\right)}_{2x + 1 + 2z} dx\,dy\,dz$$

$$= \underbrace{\mathrm{Vol}\,\Omega}_{\substack{\shortparallel \\ \frac{\pi}{6}}} + 2 \int_{\Omega} (x+z) \; dx\,dy\,dz$$

$$\underset{\substack{\text{cyl.} \\ \text{coords}}}{=} \frac{\pi}{6} + 2 \int_{\substack{z^2+6 \le \rho^2 \le 5z \\ 0 \le \theta \le 2\pi}} (\rho\cos\theta + z)\, d\rho\, dz\, d\theta$$

$$\overset{\text{RF}}{=} \frac{\pi}{6} + 2 \int_{z^2+6 \le \rho^2 \le 5z} \Big(\rho \underbrace{\int_0^{2\pi} \cos\theta \, d\theta}_{\substack{\shortparallel \\ 0}} \Big) d\rho\, dz$$

$$+ \; 2 \int_{z^2+6 \leq \rho^2 \leq 5z} \Big(\rho \underbrace{\int_0^{2\pi} 1 \, d\theta}_{= 2\pi} \Big) \, d\rho \, dz$$

$$= \frac{\pi}{6} + 4\pi \underbrace{\int_{z^2+6 \leq \rho^2 \leq 5z} \rho \, d\rho \, dz}_{\text{already computed } = 1/12}$$

$$= \frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{2} .$$

Components We compute the component of $\int_{\partial\Omega} \vec{F} \cdot \vec{n}_e$ on $5z = x^2 + y^2 \iff z = \frac{x^2+y^2}{5}$ with

$$4 \leq x^2 + y^2 \leq 9$$

(recall figure at beginning)

Thus: $P = \left\{ \left(x, y, \frac{x^2+y^2}{5} \right) =: \phi(x,y) : \; 4 \leq x^2+y^2 \leq 9 \right\}$

$$\int_P \vec{F} \cdot \vec{n}_\phi = \int_{4 \leq x^2+y^2 \leq 9} \det \begin{bmatrix} \vec{F} \\ \partial_x \phi \\ \partial_y \phi \end{bmatrix} dx\,dy$$

$$= \int_{4 \leq x^2+y^2 \leq 9} \det \begin{bmatrix} x^2 & y & \left(\frac{x^2+y^2}{5}\right)^2 \\ 1 & 0 & \frac{2}{5}x \\ 0 & 1 & \frac{2}{5}y \end{bmatrix} dx\,dy$$

$$4 \leq x^2+y^2 \leq 9 \qquad \begin{matrix} & & \\ 0 & 1 & \frac{2}{5}x \\ & & \frac{2}{5}y \end{matrix} \Big]$$

$$= \int_{4 \leq x^2+y^2 \leq 9} x^2 \cdot \left(-\frac{2}{5}x\right) - 1 \cdot \left(\frac{2}{5}y^2 - \left(\frac{x^2+y^2}{5}\right)^2\right) dxdy$$

$$\left(-\frac{2}{5}(x^2+y^2) + \frac{1}{25}(x^2+y^2)^2\right) dxdy$$

$$\underset{\text{pol coords}}{=} \int_{\substack{2 \leq \rho \leq 3 \\ 0 \leq \theta \leq 2\pi}} \left(-\frac{2}{5}\rho^2 + \frac{1}{25}\rho^4\right) \rho \, d\rho \, d\theta$$

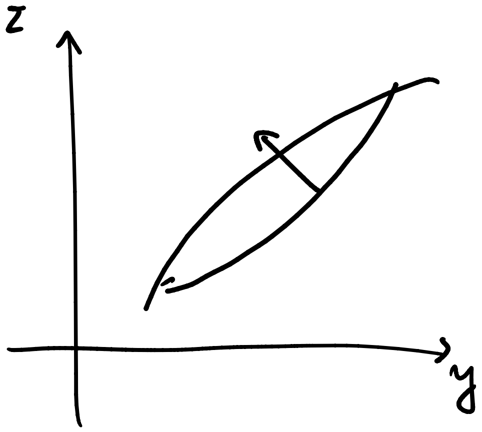
$$\overset{RF}{=} 2\pi \left[-\frac{2}{5} \underbrace{\int_2^3 \rho^3 \, d\rho}_{\frac{\rho^4}{4}\Big|_2^3} + \frac{1}{25} \underbrace{\int_2^3 \rho^5 \, d\rho}_{\frac{\rho^6}{6}\Big|_2^3} \right]$$

$$= 2\pi \left[-\frac{1}{10}(81-16) + \frac{1}{150}(729-64) \right]$$

To check if this is outward or inward we've to check $\vec{n}_\phi \underset{?}{=} \pm \vec{n}_e$?

$$\partial_x \phi \wedge \partial_y \phi = \det \begin{bmatrix} i & j & k \\ 1 & 0 & \frac{2}{5}x \\ 0 & 1 & \frac{2}{5}y \end{bmatrix}$$

$$= \left(-\frac{2}{5}x, \ -\frac{2}{5}y, \ 1\right)$$



<u>inward</u>.

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