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Qualitative Study of Diff Eqns

Ex 7.3.1

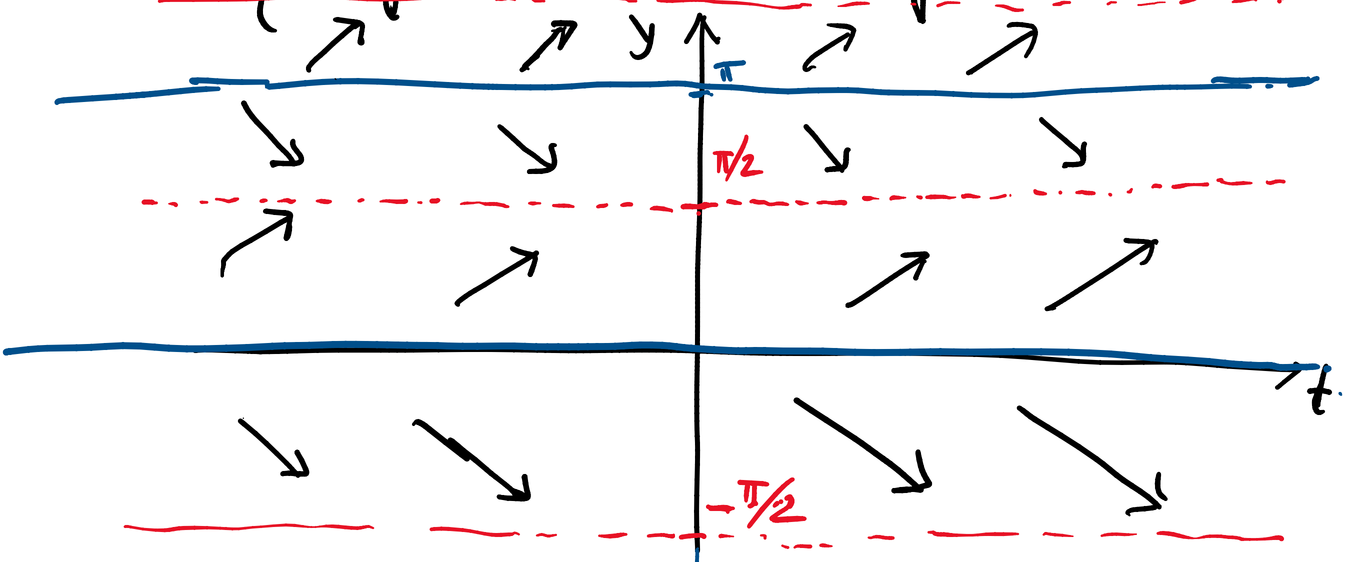
Consider the CP

$$\begin{cases} y' = \dfrac{\operatorname{tg} y}{1+y^2} = f(t,y) \\ y(0) = y_0 \end{cases}$$

i) Show that local $\exists$ and uniqueness holds.
Find constant sols. and regions of domain D where sols are increasing/decreasing

Sol: Here $f(t,y) = \dfrac{\operatorname{tg} y}{1+y^2}$ defined on

$$D = \left\{ (t,y) \in \mathbb{R}^2 : t \in \mathbb{R},\ y \neq \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z} \right\}$$



Clearly D is open $f \in \mathcal{C}(D)$. $\partial_y f \in \mathcal{C}(D)$

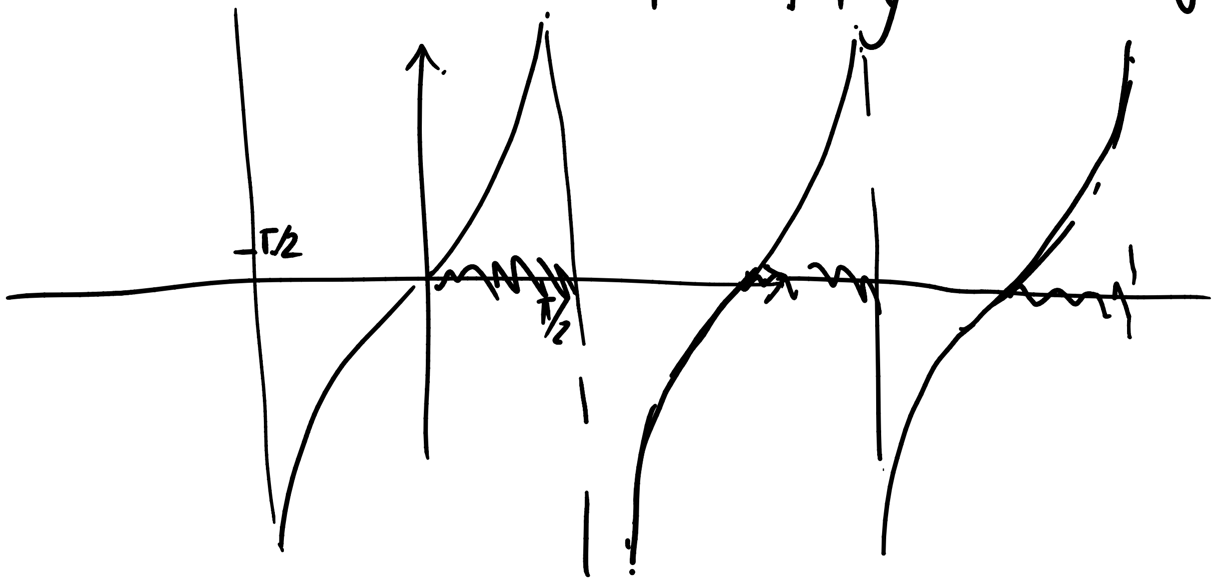
Clearly D is open, $f \in \mathcal{C}(D)$; $\partial_y f \in \mathcal{C}(D)$

$\Rightarrow$ Local existence and uniqueness holds.

$$y \equiv C \text{ is a sol.} \underset{y' \equiv 0}{\Leftrightarrow} 0 = \frac{\operatorname{tg} C}{1 + C^2}$$

$$\Leftrightarrow \operatorname{tg} C = 0 \Leftrightarrow C = k\pi \quad k \in \mathbb{Z}$$

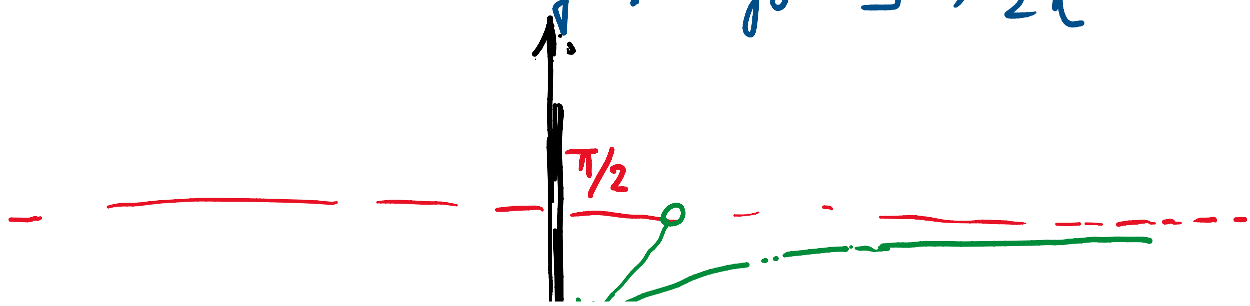
Now: $y \nearrow \Leftrightarrow 0 \leq y' \underset{\text{eqn}}{=} \dfrac{\operatorname{tg} y}{1+y^2} \Leftrightarrow \operatorname{tg} y \geq 0$

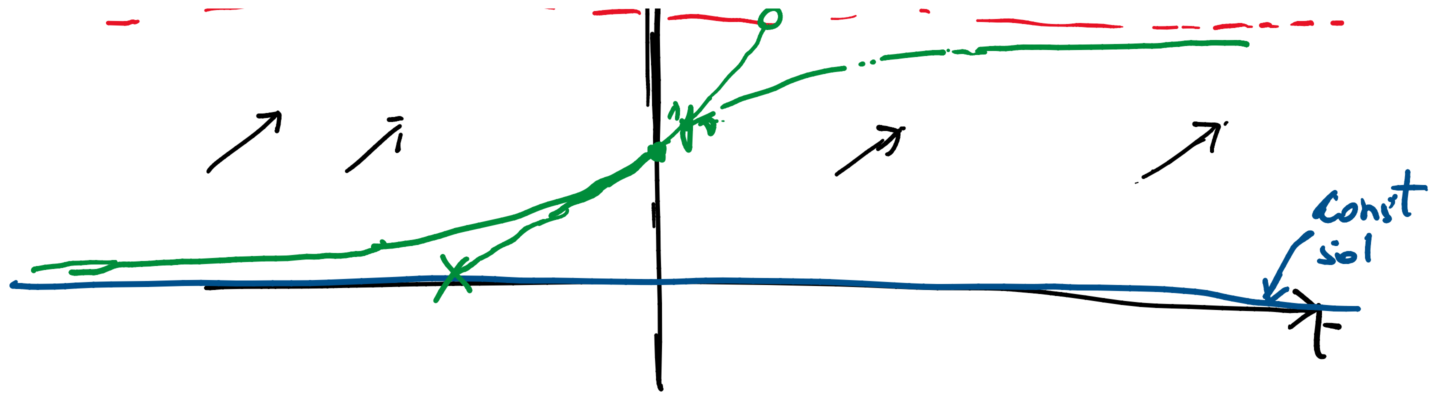


$$\Leftrightarrow k\pi \leq y < \frac{\pi}{2} + k\pi \quad k \in \mathbb{Z}$$

. . .

ii) Consider CP $y(0) = y_0 \in \left]0, \frac{\pi}{2}\right[$





(Show that y is monotone) and ($\alpha = -\infty$)

($y :]\alpha, \beta[\longrightarrow \mathbb{R}$)

<u>Monotonicity:</u> We prove that $y \nearrow$.

To prove this we prove that

$$\underbrace{0 < \overbrace{y < \pi/2}} \quad \forall\, t \in]\alpha, \beta[$$

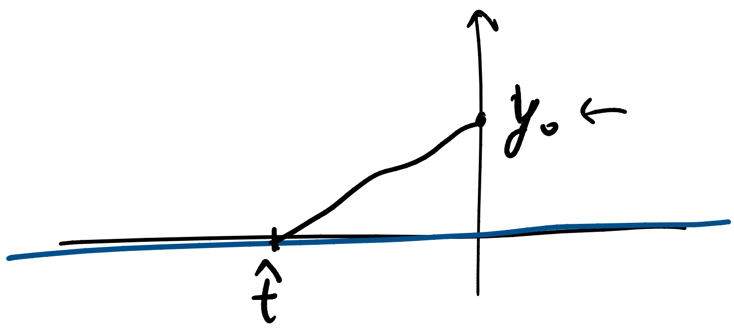
1. $y > 0 \quad \forall t$: if false $\exists\, \hat{t}$: $y(\hat{t}) \leq 0$

This is imposs:

- if $y(\hat{t}) = 0$

$\Downarrow$

y is equal to a (constant) sol

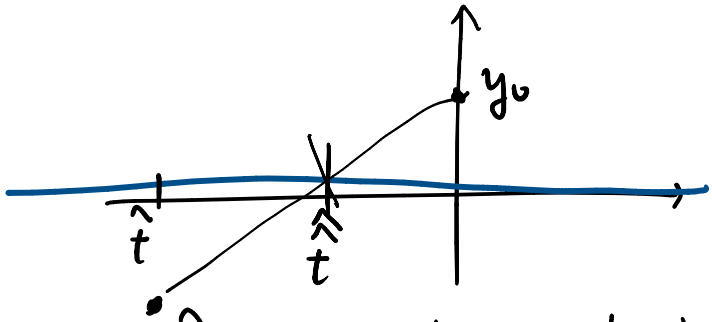


$\Downarrow$

$y \equiv 0$ but $y(0) = y_0 > 0$: impossible!

. if $y(\hat{t}) < 0$

$\Downarrow$



y cont $\exists\, \hat{\hat{t}}$: $y(\hat{\hat{t}}) = 0$ (zeroes thm)

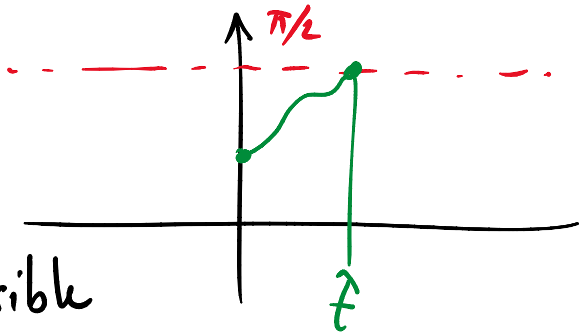
$\Downarrow$

impossible!

$\Rightarrow \quad y(t) > 0 \quad \forall t.$

2. $y < \pi/2 \quad \forall t$. If false $\exists\, \hat{t}$: $y(\hat{t}) \geq \pi/2$.

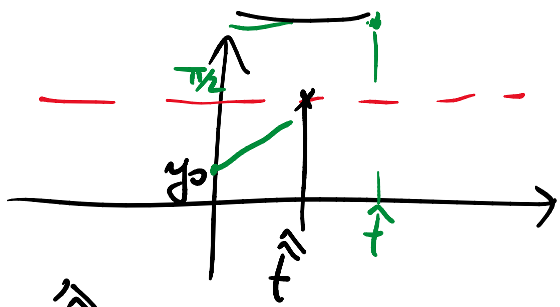
if $y(\hat{t}) = \pi/2$



$\Downarrow$

$(\hat{t}, y(\hat{t})) \notin D$ impossible

if $y(\hat{t}) > \pi/2$



$\Downarrow$

by continuity $\exists\, \hat{\hat{t}}$: $y(\hat{\hat{t}}) = \pi/2 \Rightarrow$ imposs.

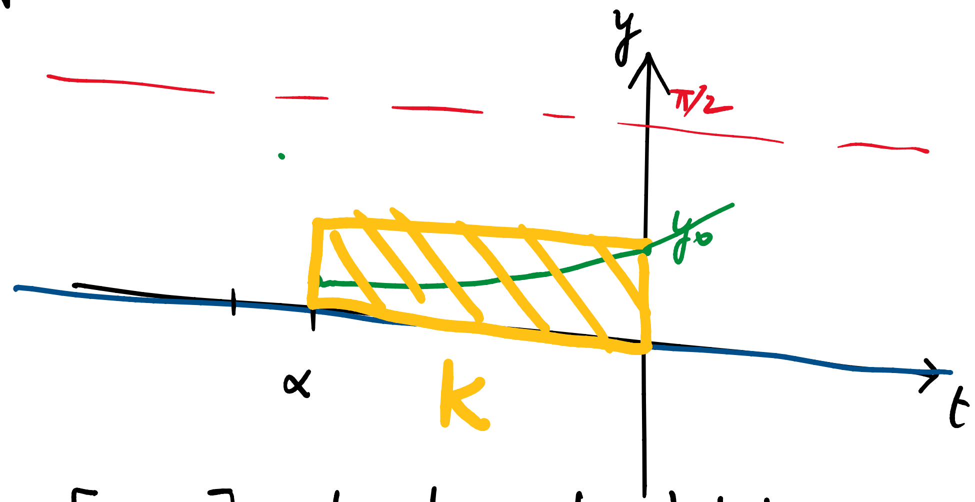
$\Rightarrow \quad y < \pi/2 \quad \forall t$

$\Rightarrow \quad 0 < y < \pi/2 \quad \forall t \quad \overset{}{\underset{i)}{\Longrightarrow}} \quad y \nearrow.$

$\Rightarrow \quad 0 < y < \pi/2 \quad$ or $\quad \overset{\Longrightarrow}{i)} \quad y \nearrow.$

$\boxed{\alpha = -\infty}$

If $\alpha > -\infty$



$K = [\alpha, 0] \times [0, y_0]$ closed and bded $\Rightarrow$ compact

Here $\nexists\ \sigma : \quad (t, y(t)) \notin K \quad t < \sigma$

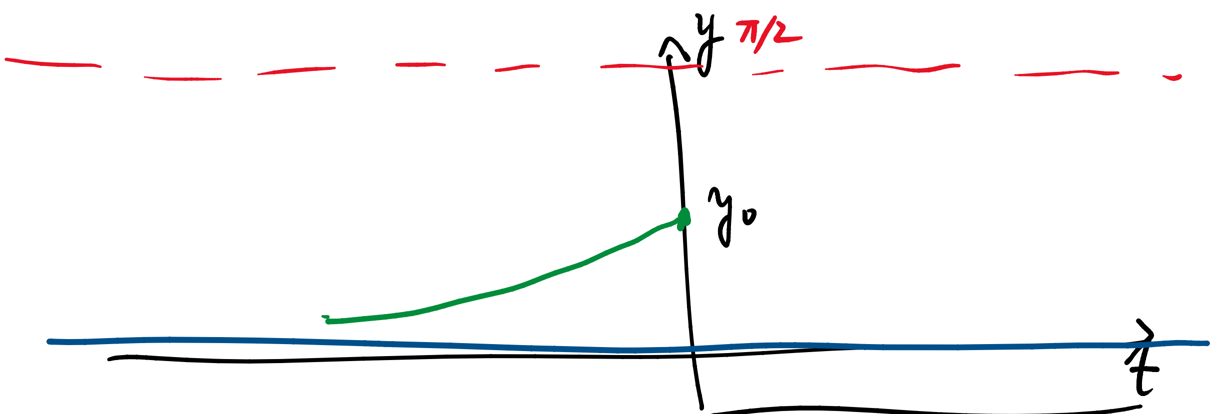
$\Rightarrow$ impossible according to the argument of comp. sets. $\quad \Rightarrow \alpha = -\infty$

Show that $\exists \lim_{t \to -\infty} y(t)$ and det this limit.

The limit exists because y is monotone

Let

$$\ell := \lim_{t \to -\infty} y(t)$$



Because $\quad 0 < y < \pi/2 \quad \Rightarrow \boxed{0 \le \ell \le y_0}$

Because $\quad 0 < y < \pi/2 \implies \boxed{0 \leq \ell \leq y_0.}$

and

$$y' = \frac{\overset{\nearrow \tan \ell}{\tan y}}{1+y^2 \searrow 1+\ell} \longrightarrow \frac{\tan \ell}{1+\ell^2}$$

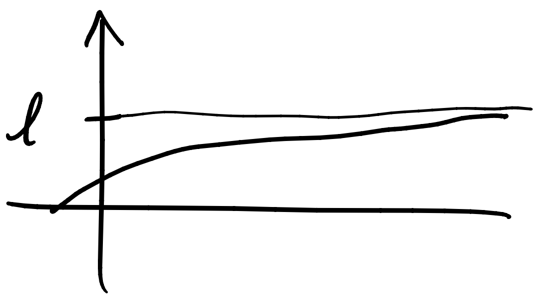
$$\implies \quad y' \longrightarrow m$$

Gen fact: $\quad y \to \ell \in \mathbb{R}$, $\quad t \to \pm\infty.$

$y' \to m$

$$\implies \boxed{m = 0}$$

Proof: $0 = \lim_{t \to \pm\infty} \dfrac{y(t)}{t} \underset{H}{=} \lim_{t \to \pm\infty} y'(t) = m$ $\square$



Rmk: In general to know that $y \to \ell$ $\quad t \to \pm\infty$

$$\not\Downarrow$$

$$y' \to 0$$

(false: $\quad y(t) = \dfrac{1}{t} \sin(t^3) \underset{t \to \pm\infty}{\longrightarrow} 0$

$$y'(t) = -\frac{1}{t^2} \sin t^3 + \frac{1}{t} \cos(t^3) (3t)^2$$

$$= -\underbrace{\frac{1}{t^2}}_{\downarrow\, 0}\sin t^3 + \underbrace{3t\cos t^3}_{\text{unbeded}} \not\to 0$$

but if $y \to l \in \mathbb{R}$ AND $y' \to m$ $\Longrightarrow$ $m = 0$.

Therefore

$$\frac{\operatorname{tg} l}{1+l^2} = 0 \iff \operatorname{tg} l = 0$$

$$\iff l = k\pi$$

and because $0 \leq l \leq y_0 < \pi/2 \Longrightarrow \boxed{l = 0}$

iii) Discuss concavity of y

y is convex $\iff y'' \geq 0$.

Where do we take y'? Because

$$y' = \frac{\operatorname{tg} y}{1+y^2}$$

$$\downarrow$$

$$y'' = \left(\frac{\operatorname{tg} y}{1+y^2}\right)' \qquad y = y(t)$$

$$= \frac{(\operatorname{tg} y)' \cdot (1+y^2) - (\operatorname{tg} y)(1+y^2)'}{(1+y^2)^2}$$

$$(\operatorname{tg} y)' = (1 + (\operatorname{tg} y)^2)\, y'$$

$$(1+y^2)' = 2yy'$$

$$= \frac{y'(1+(\operatorname{tg} y)^2)(1+y^2) - (\operatorname{tg} y)\, 2y y'}{(1+y^2)^2}$$

$$y'' = \underset{+}{y'} \frac{\boxed{(1+(\operatorname{tg} y)^2)(1+y^2) - 2y \operatorname{tg} y}}{\underbrace{(1+y^2)^2}_{+}}$$

$$y'' \geq 0 \iff (1+(\operatorname{tg} y)^2)(1+y^2) - 2y \operatorname{tg} y \geq 0$$

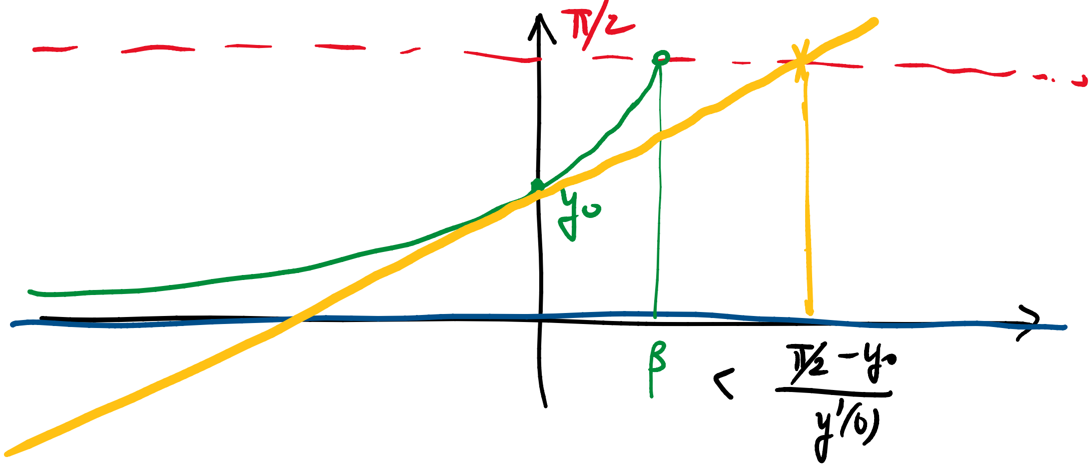
$$1 + (\operatorname{tg} y)^2 + y^2 + y^2 (\operatorname{tg} y)^2 - 2y \operatorname{tg} y \geq 0$$

$$\iff 1 + (\operatorname{tg} y - y)^2 + y^2 (\operatorname{tg} y)^2 \geq 0$$

yes!

<u>Conclusion</u>: y is convex!

iv) Use iii) to deduce if $\beta < +\infty$ or less.



Because y is convex, y is above every of its tgs, in particular

$$y(t) \geqslant \overset{y_0}{y(0)} + \overset{>0}{y'(0)}\,t \qquad \forall\, t \in \,]\alpha, \beta[$$

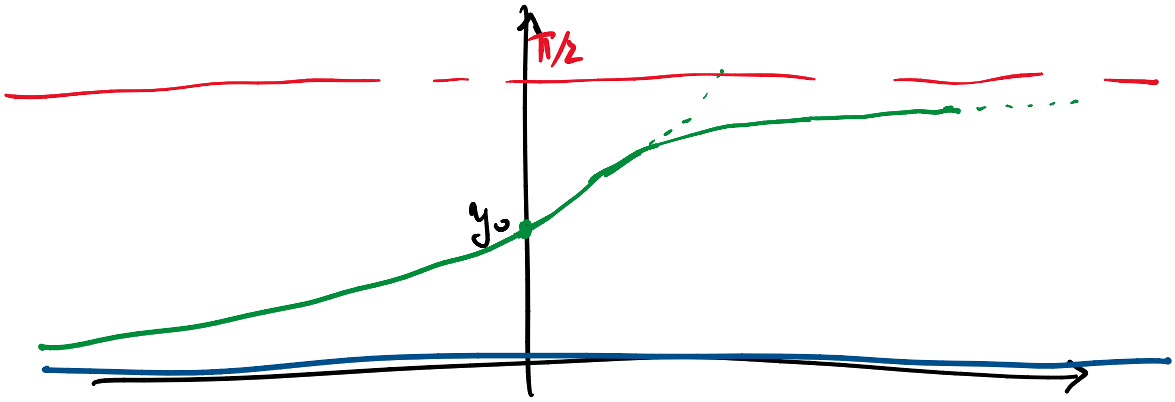
$\ulcorner$ tg at point t_0: $\quad y = y(t_0) + y'(t_0)(t - t_0)$ $\lrcorner$

and because $\frac{\pi}{2} \geqslant y(t) > y_0 + y'(0)\,t$

$$\Rightarrow \quad t < \frac{\pi/2 - y_0}{y'(0)}$$

so $\beta < +\infty$.

Alternative way to prove $\beta < +\infty$.



We know $y \nearrow$, $0 < y < \pi/2$. We want to discuss if $\beta < +\infty$ or less.

β can be $\beta < +\infty$ or $\beta = +\infty$.

If $\boxed{\beta = +\infty}$, because $y \nearrow$

$$\exists \lim_{t \to +\infty} y(t) = \ell$$

$$\boxed{0 < y_0 < \ell \leq \pi/2}$$

$$\Rightarrow \quad y' = \frac{tg\, y}{1+y^2} \longrightarrow \begin{cases} \ell = \pi/2 & \dfrac{+\infty}{1+\left(\frac{\pi}{2}\right)^2} = +\infty \\ \ell < \pi/2 & \dfrac{tg\, \ell}{1+\ell^2} \in \mathbb{R} \end{cases}$$

$= 0$

$\Uparrow$

but by asympt thm

$\dfrac{tg\,\ell}{1+\ell^2} \in \mathbb{R}$ $\Downarrow$ $= 0$

$tg\,\ell = 0$

$\Downarrow$

$\ell = k\pi$ IMPOSS

$\beta = +\infty \quad \Leftarrow$ leads to contradictions

$\Downarrow$

$$\Downarrow$$
$$\beta < +\infty.$$

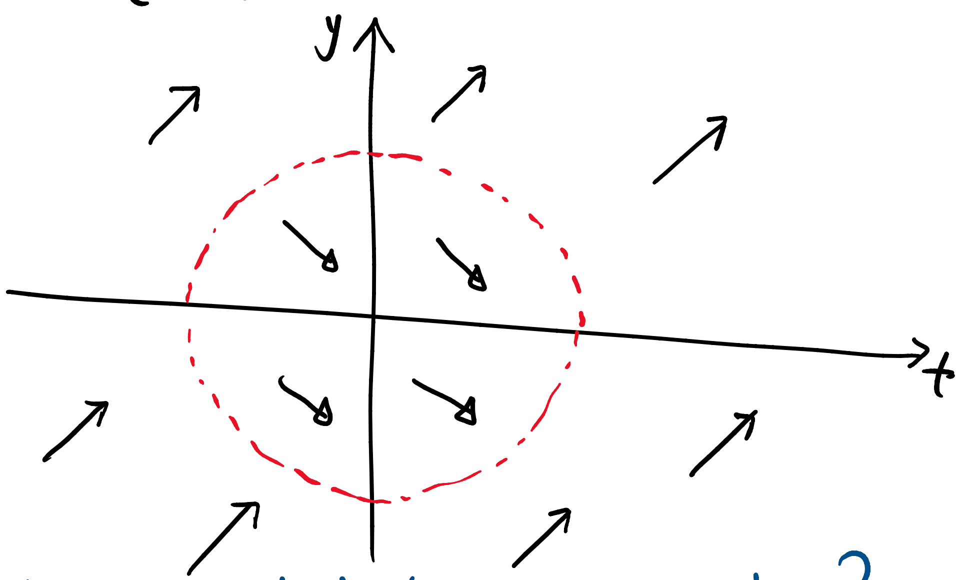
Ex 7.3.3 Consider the eqn

$$y' = \frac{1}{t^2+y^2-1} = f(t,y)$$

i) Det D of local $\exists$ and uniq.

Here f is well defined on

$$D = \{(t,y) \in \mathbb{R}^2 : t^2+y^2 \neq 1\}$$



ii) Are there constant/stationary sols ?

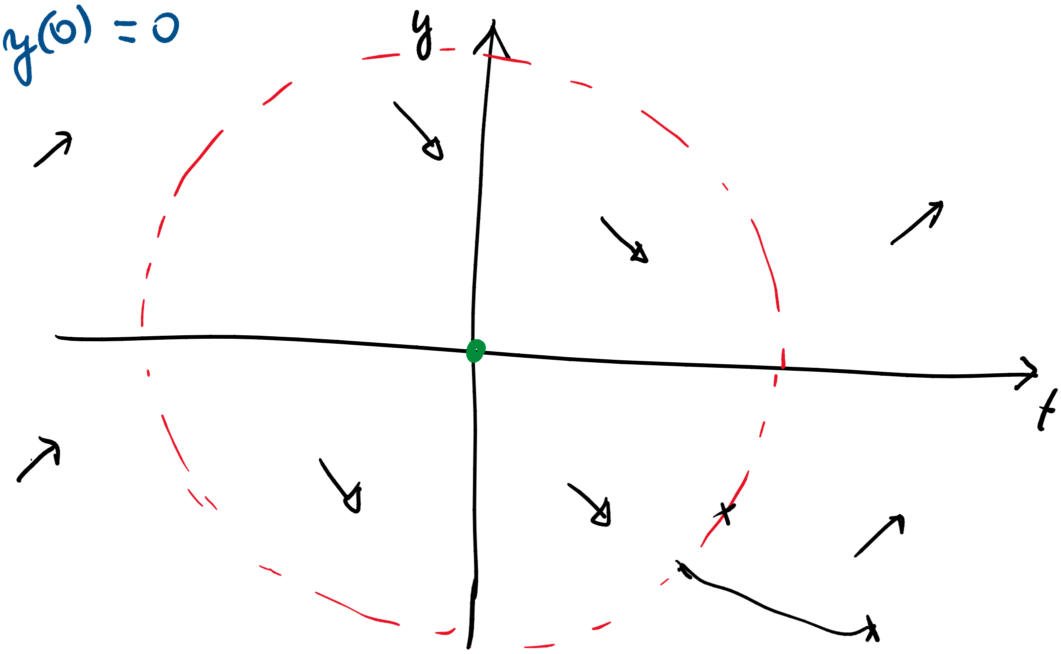
$y \equiv C$ is a sol $\iff$ $0 = \dfrac{1}{t^2+C^2-1}$ never

there're not const. sols.

iii) Det regions of D where sols are increasing/decreasing

$$y \nearrow \iff y' \geq 0 \iff \frac{1}{t^2+y^2-1} \geq 0$$
$$\iff t^2+y^2-1 > 0$$
$$\iff t^2+y^2 > 1$$

Now we consider the sol $y:]\alpha,\beta[\longrightarrow \mathbb{R}$ of CP $y(0)=0$



iv) Show that y is decreasing.

We claim that $t^2 + y(t)^2 < 1 \quad \forall t$

If false $\Rightarrow \exists \hat{t} : \quad \hat{t}^2 + y(\hat{t})^2 \geq 1$

but if $\quad \hat{t}^2 + y(\hat{t})^2 = 1 \Rightarrow (\hat{t}, y(\hat{t})) \notin D$

impossible

but if $\quad \hat{t}^2 + y(\hat{t})^2 > 1 \Rightarrow$ and because and if $\hat{t}^2 + y(\hat{t})^2 > 1 \Rightarrow$ and because

$$0^2 + y(0)^2 = 0 < 1$$

$\Downarrow$ by cont

$$\exists\ \hat{\hat{t}} : \hat{\hat{t}}^2 + y(\hat{\hat{t}})^2 = 1 \Rightarrow \text{imposs.}$$

$\Rightarrow y \searrow$.

v) Show that y is odd.

We have to prove $y(-t) = -y(t)\ \forall t$

$$\Updownarrow$$

$$y(t) = \underbrace{-y(-t)}_{v(t)}\ \forall t$$

Strategy: we prove v is

- sol of diff eqn
- $v(0) = y(0)$

$\overset{\text{uniqueness}}{\Longrightarrow} y \equiv v$

$\Downarrow$

$y(t) = -y(-t)\ \forall t$

$$v(0) = -y(-0) = -\underset{0}{\underset{\|}{y(0)}} = -0 = 0 = y(0)$$

$\Leftrightarrow$ $\boxed{v'(t) = \dfrac{1}{t^2 + v(t)^2 - 1}}$ $\quad y'(t) = \dfrac{1}{t^2 + y(t)^2 - 1}$

$$\Leftrightarrow \quad \boxed{v'(t) = \frac{1}{t^2+v(t)^2-1}} \qquad y'(t) = \frac{1}{t^2+y(t)^2-1}$$

$$v(t) = -y(-t)$$

$$v'(t) = -y'(-t)(-1) = y'(-t)$$

$$= \frac{1}{(-t)^2 + y(-t)^2 - 1} = \frac{1}{t^2 + (-y(-t))^2 - 1}$$

$$= \frac{1}{t^2 + v(t)^2 - 1}.$$

vi) Concavity:

Study of y'':

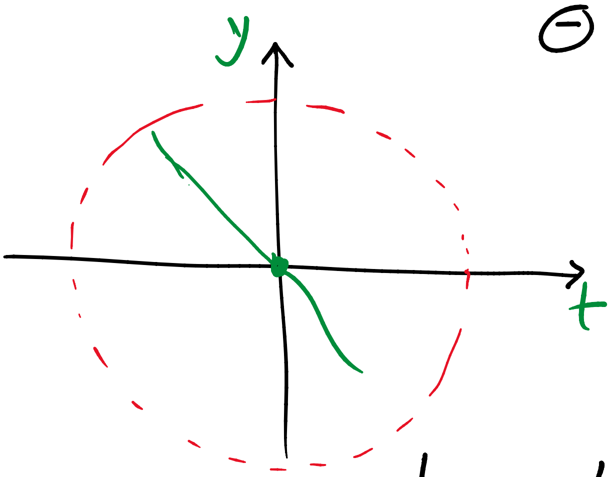
$$y'' = \left(\frac{1}{t^2+y^2-1}\right)' \qquad \left(\frac{1}{f}\right)' = -\frac{f'}{f^2}$$

$$= -\frac{(t^2+y^2-1)'}{(t^2+y^2-1)^2}$$

$$= - \frac{2t + 2yy'}{\underbrace{(t^2+y^2-1)^2}_{+}}$$

$$\boxed{y'' \geq 0 \Leftrightarrow \cancel{2}t + \cancel{2}yy' < 0}$$

$$\boxed{y'' \geq 0 \iff \cancel{2}t + \cancel{2}yy' < 0}$$



$\ominus$ because $y \searrow$

$$\Downarrow$$

$$y > 0 \iff t < 0$$

$t + yy'$

when $t > 0$

$$t + \overset{+}{yy'} > 0$$

$\oplus \quad \ominus \ominus$

when $t < 0$

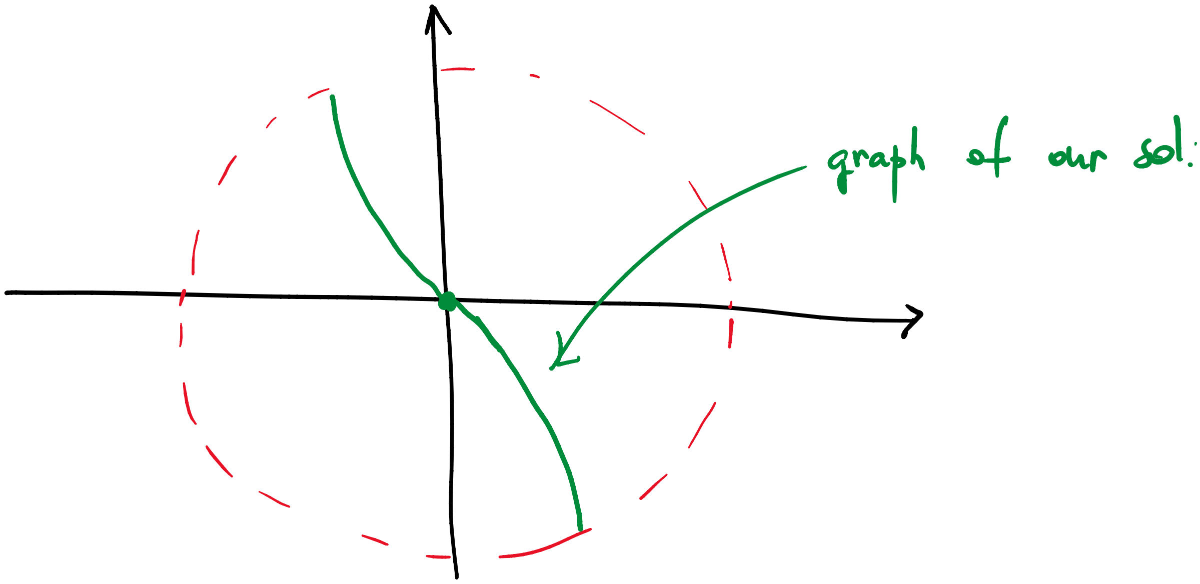
$$t + \overset{\ominus}{yy'} < 0$$

$\ominus \quad \oplus \ominus$

So $y'' \geq 0 \iff t + yy' < 0 \iff t < 0$

Thus

	$t<0$	$t>0$
y''	$\oplus$	$\ominus$
y	convex	concave


graph of our sol.