Do 7.4.2/3/4

7.4.4 Consider the eqn

$$y' = \frac{1}{\log y + t}$$

i) Det domain where local $\exists!$ holds

ii) Are there st sols?

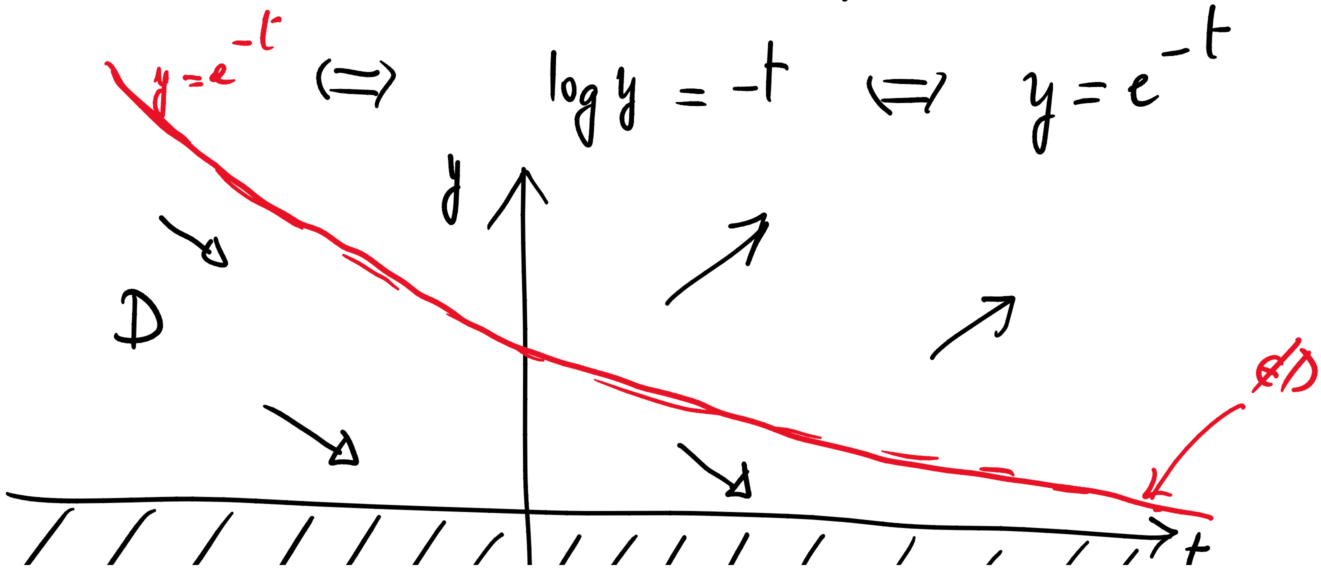
iii) Det regions where sols are incr/decreasing.

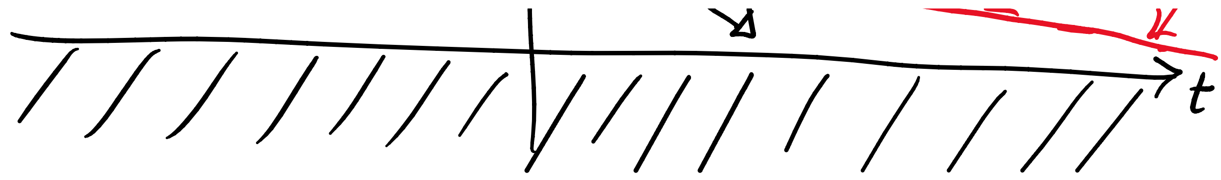
Sol: The eqn is $y' = f(t,y) := \dfrac{1}{\log y + t} = \dfrac{1}{t + \log y}$

$$D = \{(t,y) \in \mathbb{R}^2 : y > 0, \ t + \log y \neq 0\}$$

$$t + \log y = 0 \iff t = -\log y$$

$$\iff \log y = -t \iff y = e^{-t}$$





$f, \partial_y f \in \mathcal{C}(D) \Rightarrow$ loc. exist and uniq. holds true.

ii) $y \equiv C$ is sol $\iff 0 = \dfrac{1}{\log C + t}$ no values for C.

$\Rightarrow$ no st. sols.

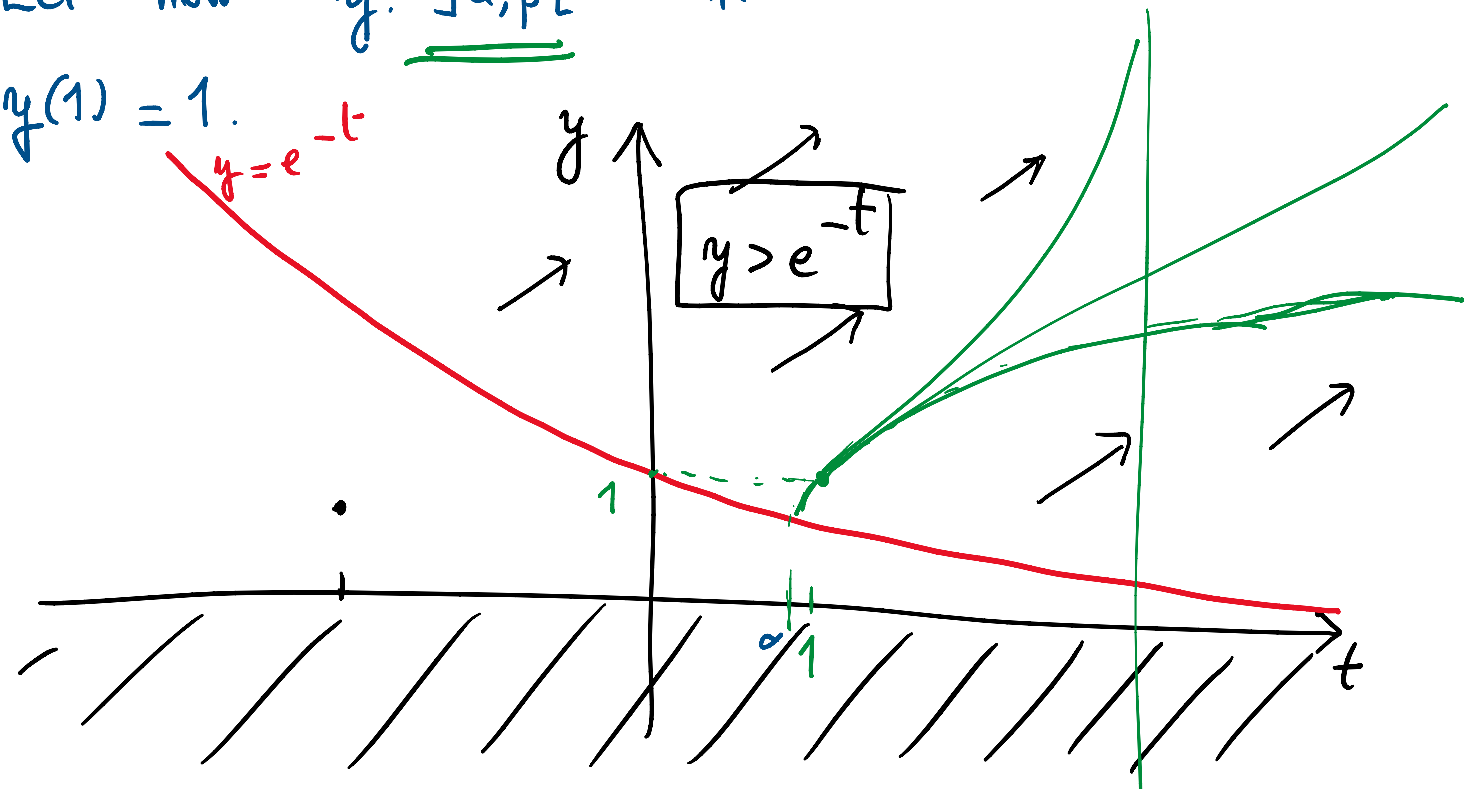
iii) $y \nearrow \iff 0 < y' = \dfrac{1}{\log y + t} \iff$

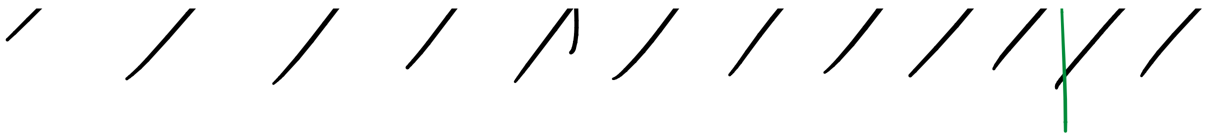
$$\log y + t > 0$$

$$\iff \log y > -t \iff y > e^{-t}$$

Let now $y:]\alpha,\beta[\to \mathbb{R}$ be the sol of CP

$y(1) = 1$.





iv) Show that y is increasing

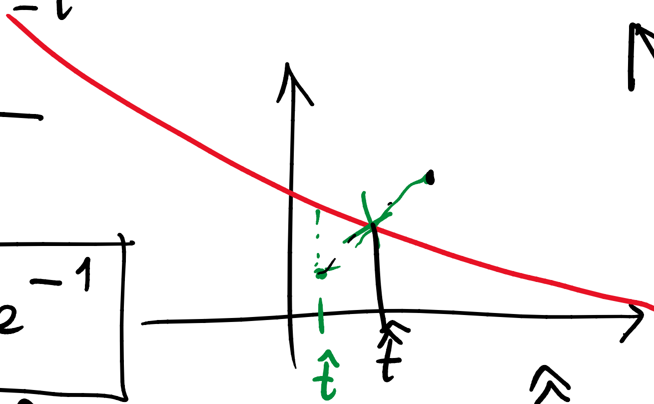
To prove $y \nearrow$ let's prove that

$$\boxed{y(t) > e^{-t} \quad \forall t \in \,]\alpha,\beta[}$$

If false $\exists\, \hat{t} : y(\hat{t}) \leq e^{-\hat{t}}$. But

if $\quad y(\hat{t}) = e^{-\hat{t}} \implies (\hat{t}, y(\hat{t})) \notin D$

if $\quad \underline{y(\hat{t}) < e^{-\hat{t}}}$



being also $\boxed{y(1) = 1 > e^{-1}}$

$\implies$ by cont $\exists\, \hat{\hat{t}} : y(\hat{\hat{t}}) = e^{-\hat{\hat{t}}}$

$\implies y > e^{-t} \quad \forall t \implies y \nearrow.$

v) Concavity

$$y' = \frac{1}{t + \log y} \implies y'' = \left(\frac{1}{t + \log y}\right)'$$

$$\left(\frac{1}{f}\right)' = -\frac{f'}{f^2} = \ominus\,\frac{1+\frac{y'}{y}}{\underset{\oplus}{(t+\log y)^2}}$$

so $y'' \geq 0 \iff 1 + \overset{+}{\frac{y'}{y}} \leq 0$

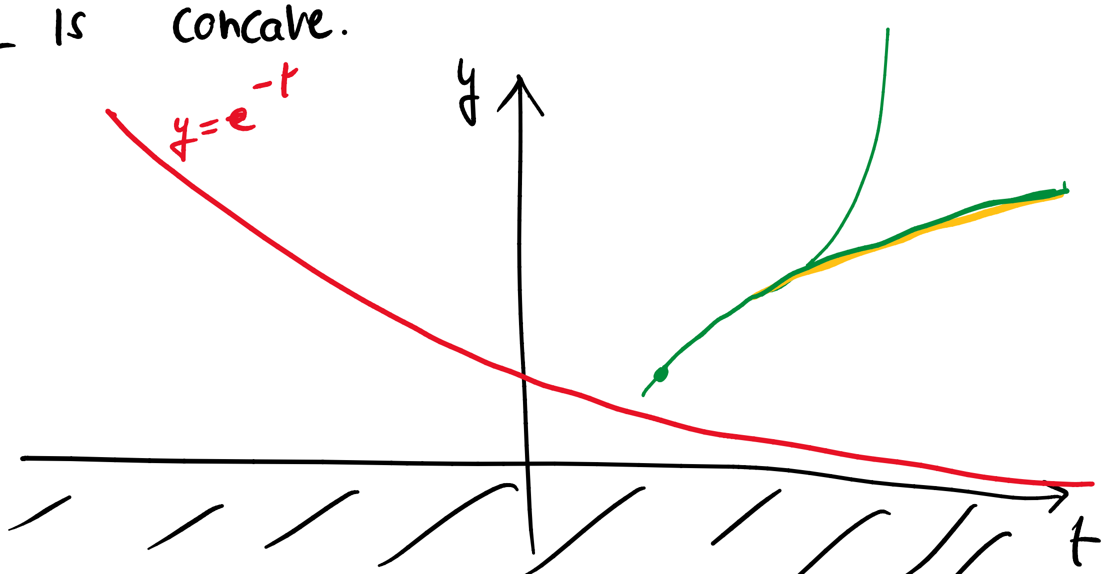
We know $\underset{(>)}{y' \geq 0}$, $y > 0$

(yes because $y > e^{-t}$ in the case of our CP

or $y > 0$ because sol. y lives in D.)

$$\Rightarrow \quad 1 + \frac{y'}{y} > 0 \Rightarrow y'' < 0$$

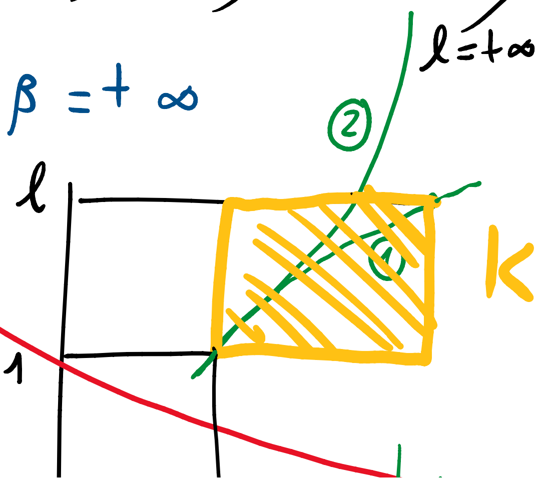
$\Rightarrow$ y is concave.



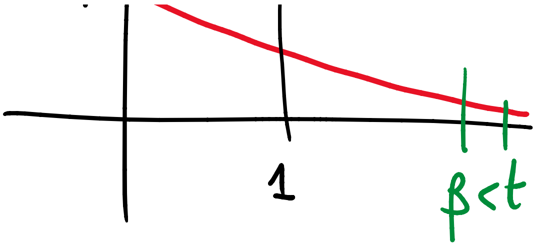
vi) Prove that $\beta = +\infty$

If $\boxed{\beta < +\infty}$

$\|$



$K = [1,\beta] \times [1,\ell]$

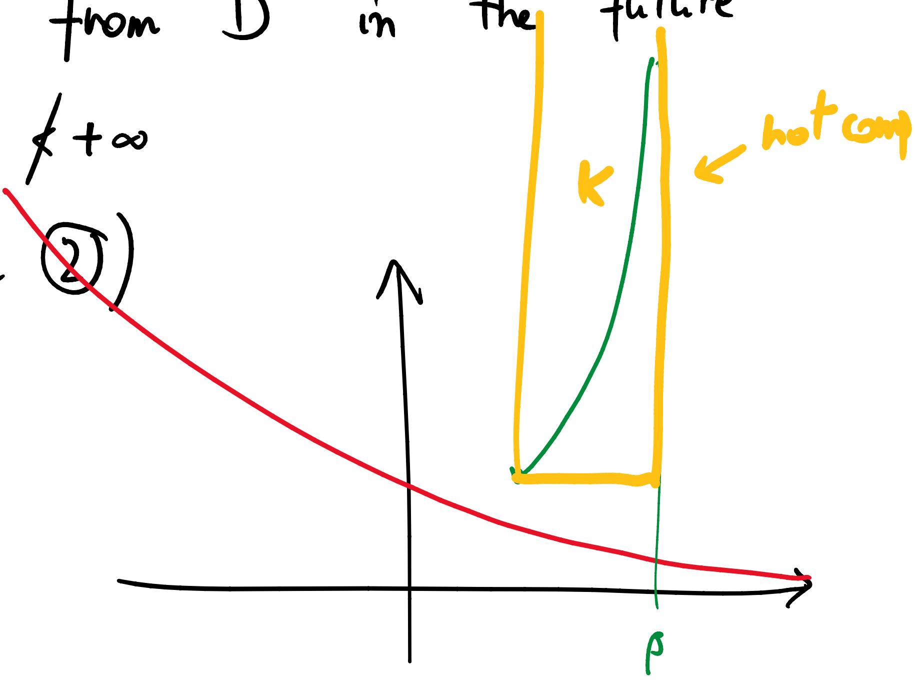


$$\boxed{\exists \lim_{t\to\beta} y(t) =: \ell}$$

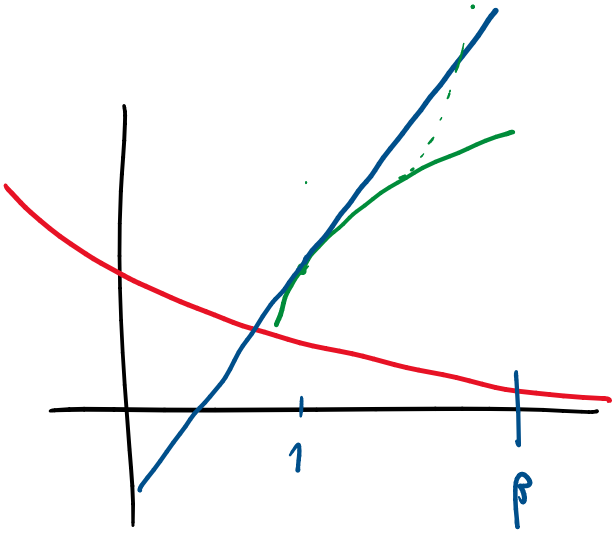
If $\ell < +\infty$ (case ①), taking $K = [1,\beta]\times[1,\ell]$

K is closed & bdd ($\Rightarrow$ compact) $\subset D$ and y does not escape from D in the future

$\Rightarrow$ imposs $\Rightarrow$ $\ell \not< +\infty$



If $\boxed{\ell = +\infty}$ (case ②)



tg to y at $(1,1) = (1, y(1))$

Because y is concave,

$$y(t) \leq \underbrace{y(1) + y'(1)(t-1)}_{}$$

$$= 1 + (t-1)$$

$$y'(1) = \frac{1}{\underset{\underset{0}{\shortparallel}}{\log}(\underset{\underset{1}{\shortparallel}}{y(1)}) + 1} = 1$$

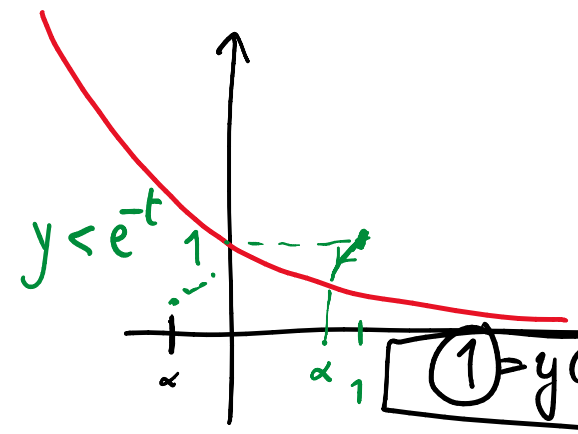
$$\boxed{y(t) \leq t} \quad \forall t \in]\alpha, \beta[$$

If $t \to \beta$ and $y \to +\infty$

$+\infty \le \beta < +\infty$ imposs.

$\Rightarrow$ in any case we got a contradiction $\Rightarrow \beta = +\infty$.

vii) Show that $\underline{\alpha > -\infty}$ (actually $\alpha > 0$). Plot a graph of y.



If $\boxed{\alpha \le 0}$, being $y > e^{-t}$ $\forall t \in]\alpha, \beta[$

$t \to \alpha$

$$\boxed{1 = y(1) > \lim_{t \to \alpha} y(t) \ge e^{-\alpha} \ge 1}$$

$\Downarrow$

impossible.

$\Rightarrow \alpha > 0$.

Do 7.4.5 (similar to 7.4.4)

Do 7.4.2/3/1