Ex 7.4.2

Consider

$$y' = \frac{\log y}{1+y}$$

i) Find domain of local $\exists$ and uniqueness

ii) Stationary sols

iii) Regions where sols are increasing/decreasing

Let now $y:]\alpha, \beta[\longrightarrow \mathbb{R}$ be the sol of CP $y(0) = y_0$ with $y_0 \in]0,1[$.

iv) Show that y is monotone and determine the concavity

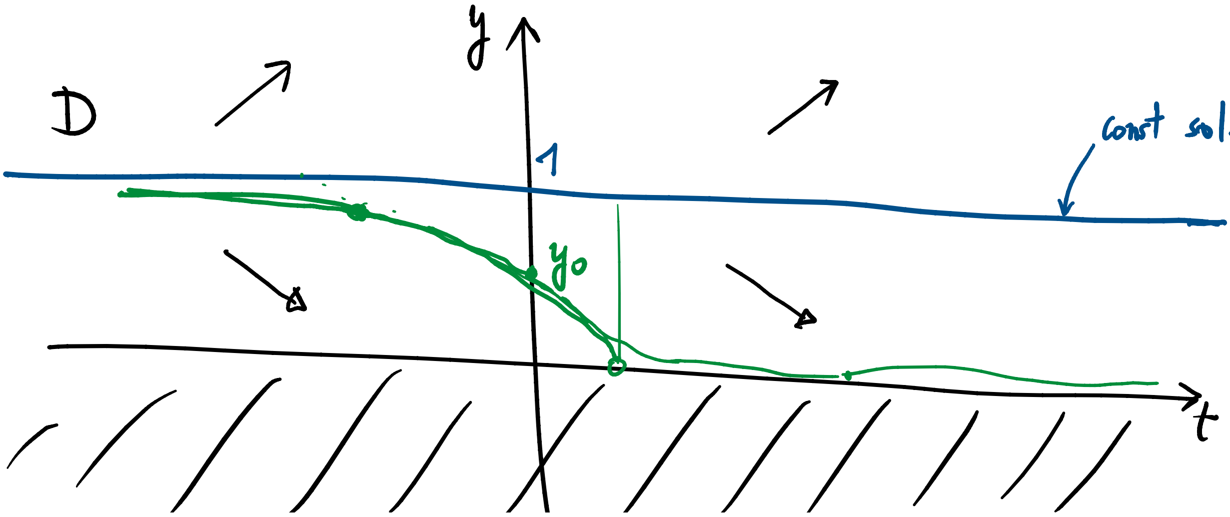
v) Show $\boxed{\alpha = -\infty}$ and compute $\lim_{t \to -\infty} y(t)$

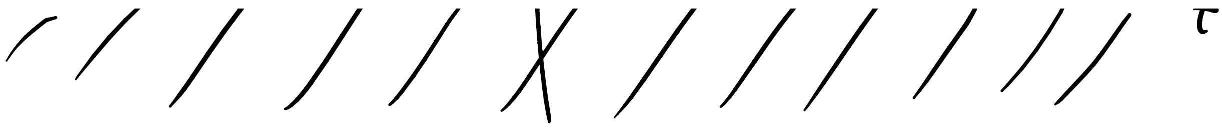
vi) Is $\beta < +\infty$? Compute $\lim_{t \to \beta^-} y(t)$, $\lim_{t \to \beta^-} y'(t)$

vii) Graph of y

Sol: i) The eqn is $y' = f(t,y) = \dfrac{\log y}{1+y}$

defined on $D = \{ (t,y) \in \mathbb{R} \times \mathbb{R} : y > 0, \cancel{1+y \neq 0} \}$





Clearly $f, \partial_y f \in \mathcal{C}(D) \Rightarrow$ local $\exists!$ holds on D

ii) $y \equiv C$ is sol $\iff 0 = \dfrac{\log C}{1+C} \iff \log C = 0$

$\iff C = 1$

iii) $y \nearrow \iff 0 \leq y' = \dfrac{\log y}{1+y} = f(t,y) \geq 0$

$\overset{y>0 \text{ on } D}{\iff} \log y \geq 0 \iff y \geq 1$

Now, let $y :]\alpha,\beta[\longrightarrow \mathbb{R}$ be the sol of CP $y(0) = y_0 \in]0,1[$

iv) We show $y \searrow$: we prove that

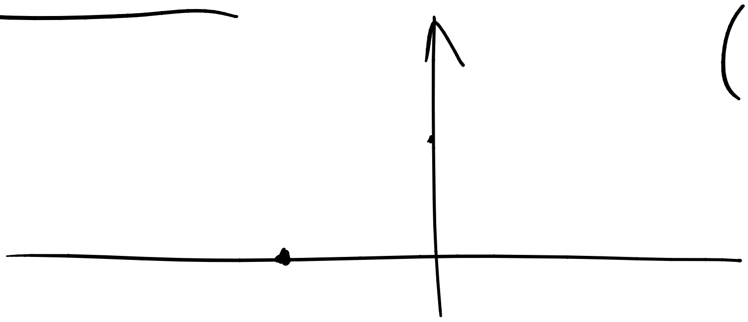
$$\underbrace{0 < y}(t) < 1 \qquad \forall t \in]\alpha,\beta[.$$

First: $\underline{y(t) > 0}$ $\forall t$. If false $\exists \hat{t} : y(\hat{t}) \leq 0$

$\big((t, y(t)) \in D \ \forall t$

$\Downarrow$

$y(t) > 0 \ \forall t$



$\Rightarrow (\hat{t}, y(\hat{t})) \notin D \qquad \big(D = \{(t,y) : y > 0\} \big)$

impossible for a sol.

Second: $y(t) < 1 \ \forall t$. If false $\exists \hat{t} : y(\hat{t}) \geq 1$

Second: $y(t)<1 \ \forall t$. If false $\exists\, \hat{t} : y(\hat{t}) \geq 1$

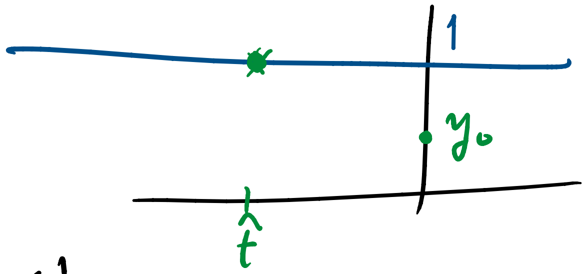
If $y(\hat{t}) = 1$

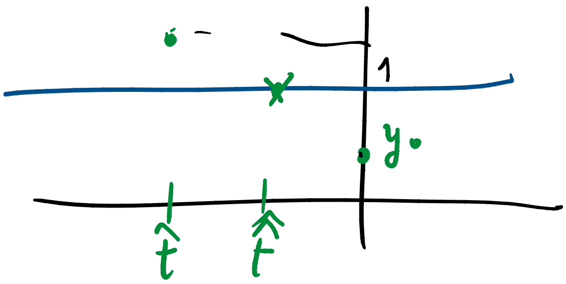
$\Downarrow$ (by uniqueness)

$y \equiv 1$ but $y(0) = y_0 < 1$

$\Downarrow$

imposs.





If $y(\hat{t}) > 1$, by cont

$\exists\, \hat{\hat{t}} : y(\hat{\hat{t}}) = 1$ (prev. case) $\Rightarrow$ imposs

$\Rightarrow y \searrow$

Concavity:

$$y'' = \left(\frac{\log y}{1+y} \right)' = \frac{\frac{1}{y} y'(1+y) - (\log y)\, y'}{(1+y)^2}$$

$$= y' \, \frac{1 + 1/y - \log y}{(1+y)^2}$$

$$y'' \geq 0 \iff 1 + \underbrace{1/y}_{>1} - \underbrace{\log y}_{<0} \leq 0 . \quad \text{never}$$

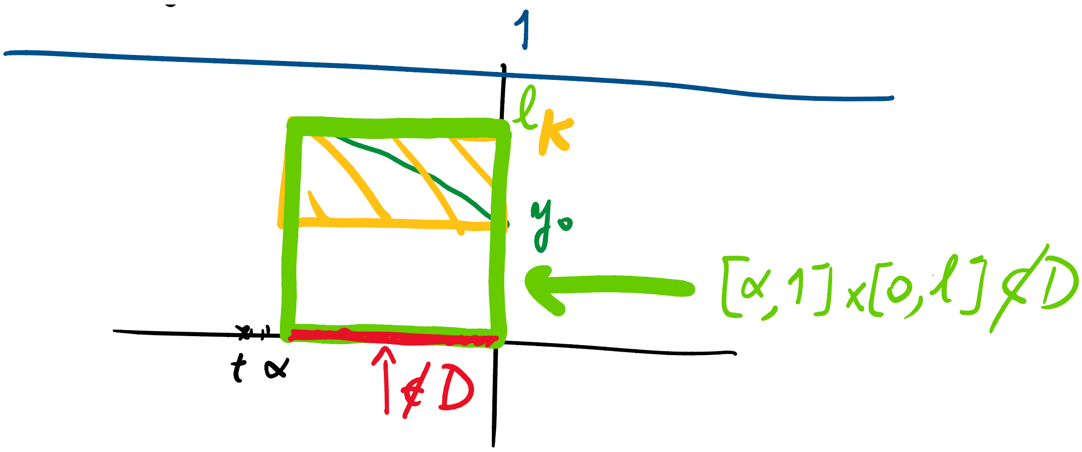
$\boxed{0 < y < 1}$

$\Rightarrow y'' < 0$ always $\Rightarrow y$ is concave.

v) $\alpha = -\infty$

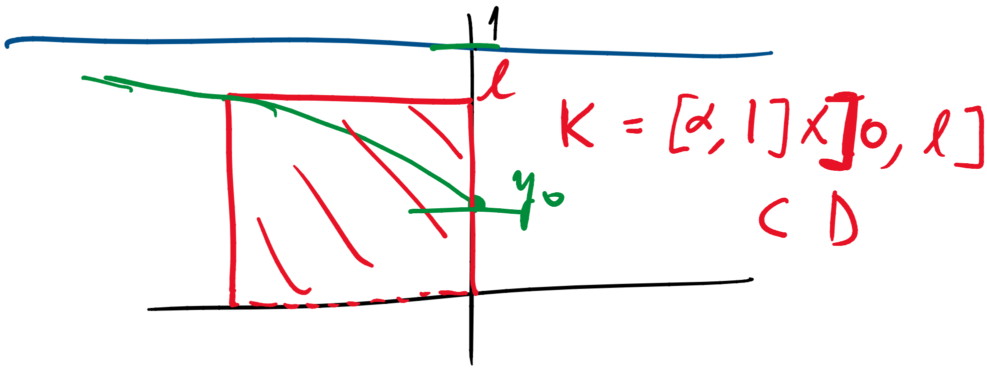
$0 < y < 1$

$y \searrow$



If $\alpha > -\infty$, setting $K = [\alpha,1]\times[y_0,1] \subset D$ and compact and sol does not exit K in the past $\Rightarrow$ imposs.

$\Rightarrow \alpha = -\infty$.



$\lim\limits_{t\to-\infty} y(t) =: \ell$. First, the lim <u>exists</u> because y is decreasing.

We know $y \searrow$, $0 < y < 1$

$\Downarrow$

$$\boxed{y_0 \le \ell \le 1}$$

Moreover

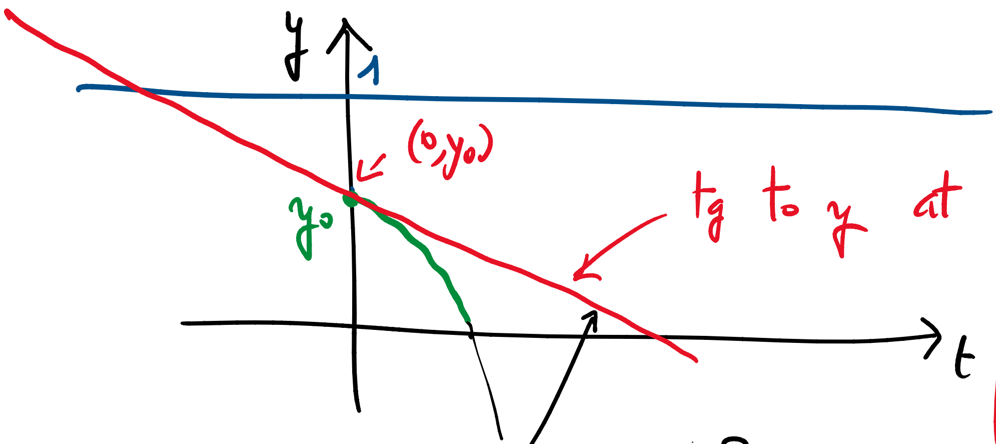
$$y'(t) = \frac{\log y(t)}{1+\dots} \longrightarrow \frac{\log \ell}{\dots}$$

($y(t) \to \ell$, $\log y(t) \to \log \ell$)

$$y'(t) = \frac{\dots}{1+y^{A\dots}} \longrightarrow \frac{\dots}{1+\ell} \longrightarrow \frac{\log \ell}{1+\ell}$$

$$\Longrightarrow \quad \frac{\log \ell}{1+\ell} = 0 \quad \Rightarrow \quad \log \ell = 0 \quad \Rightarrow \quad \ell = 1 .$$

vi) Guess: $\beta < +\infty$



tg to y at $(0, y_0)$:

$$\boxed{y = y(t_0) + y'(t_0)(t - t_0)}$$

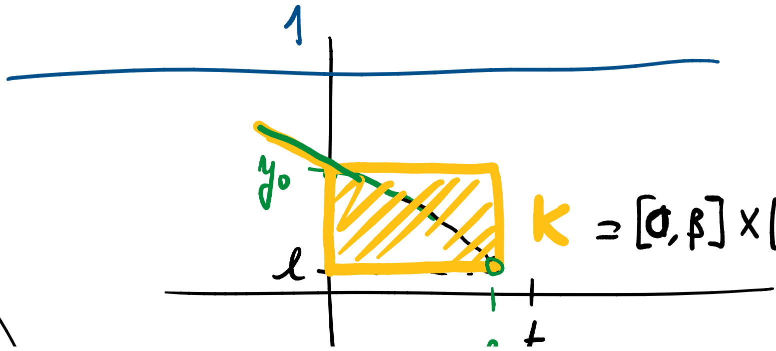
For us $y = y_0 + \underbrace{y'(0)}_{\substack{\wedge \\ 0}} t$, $\quad y'(0) = \dfrac{\log y_0}{1+y_0}$ and because y is concave

$$y(t) < y_0 + y'(0)\, t \quad \forall\, t \in \,]\alpha, \beta[$$

If $\beta = +\infty$ $\quad 0 \leq y(t) < y_0 + y'(0)\, t \quad t \to +\infty$

$y(t) \downarrow$, $\quad y_0 + y'(0)\,t \downarrow -\infty$ $\quad$ imposs.

$\Longrightarrow \beta < +\infty .$

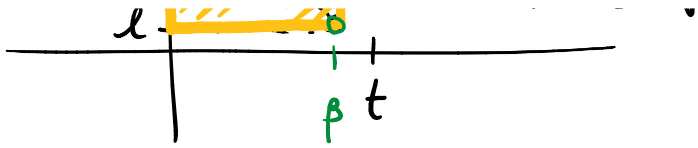


$K = [0, \beta] \times [\ell, y_0] \subset D$

$$\lim_{t \to \beta -} y(t)$$

$t \to \beta$

$\lim_{t \to \beta^-} y'(t)$



Because $y \downarrow$ $\exists$; because $0 < y \le y_0$

$\Downarrow$

$0 \le \ell \le y_0$

If $\ell > 0$, $K = [0, \beta] \times [\ell, y_0]$ is compact $\subset D$

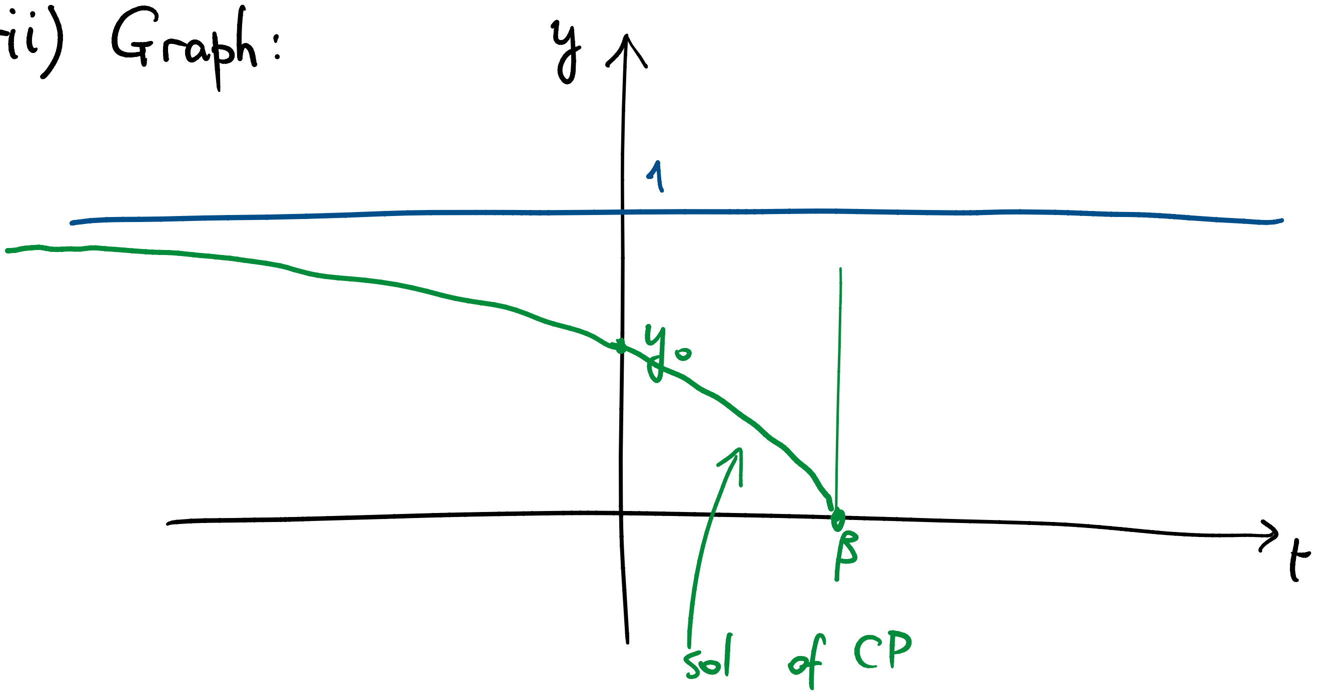
and sol does not get out of D in the future

$\Rightarrow$ imposs $\Rightarrow$ $\boxed{\ell = 0}$

$$\Rightarrow \quad \lim_{t \to \beta^-} y(t) = 0, \quad \lim_{t \to \beta^-} y'(t) = \lim_{t \to \beta^-} \frac{\overbrace{\log \overbrace{y(t)}^{0+}}^{-\infty}}{\underbrace{1 + \underbrace{y(t)}_{0}}_{1}}$$

$$= -\infty$$

vii) Graph:



Ex 7.4.3.

$$\begin{cases} y' = t(y^2-1)\,\mathrm{arctg}\, y \\ \boxed{y(0) = y_0} \end{cases}$$

i) Local $\exists !$

ii) Stationary sols

iii) Regions where sols are $\nearrow, \searrow$

$y :]\alpha, \beta[\to \mathbb{R}$

Focus on $y_0 \in]0,1[$

iv) Show y is even

v) Discuss monotonicity and nature of $t=0$

vi) α? β? limits $t \to \alpha+$, $t \to \beta-$

vii) Graph

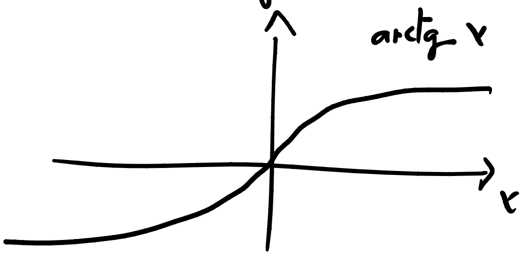
Sol: i) Here $y' = f(t,y) = t(y^2-1)\,\mathrm{arctg}\, y$

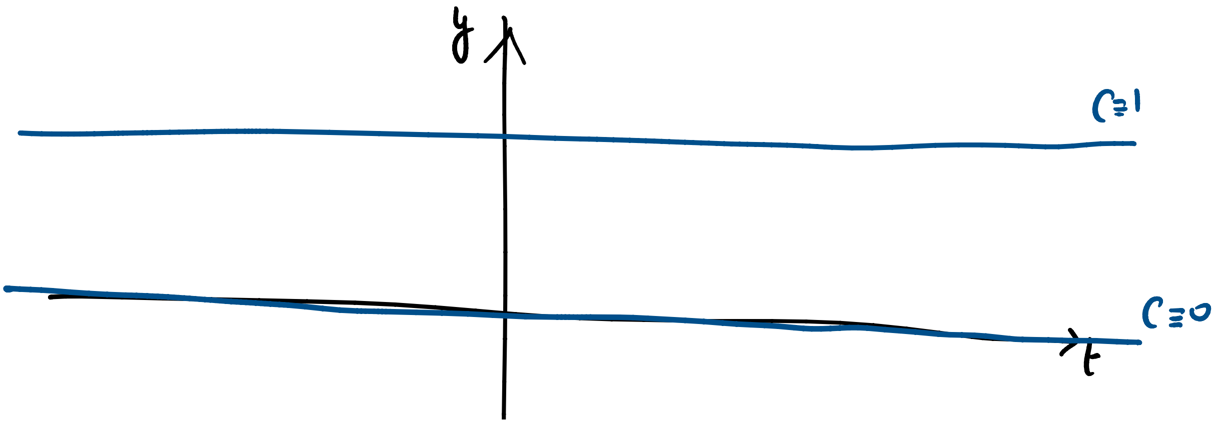
$D = \{(t,y) \in \mathbb{R} \times \mathbb{R}\} = \mathbb{R}^2$ and clearly $f, \partial_y f \in \mathcal{C}$

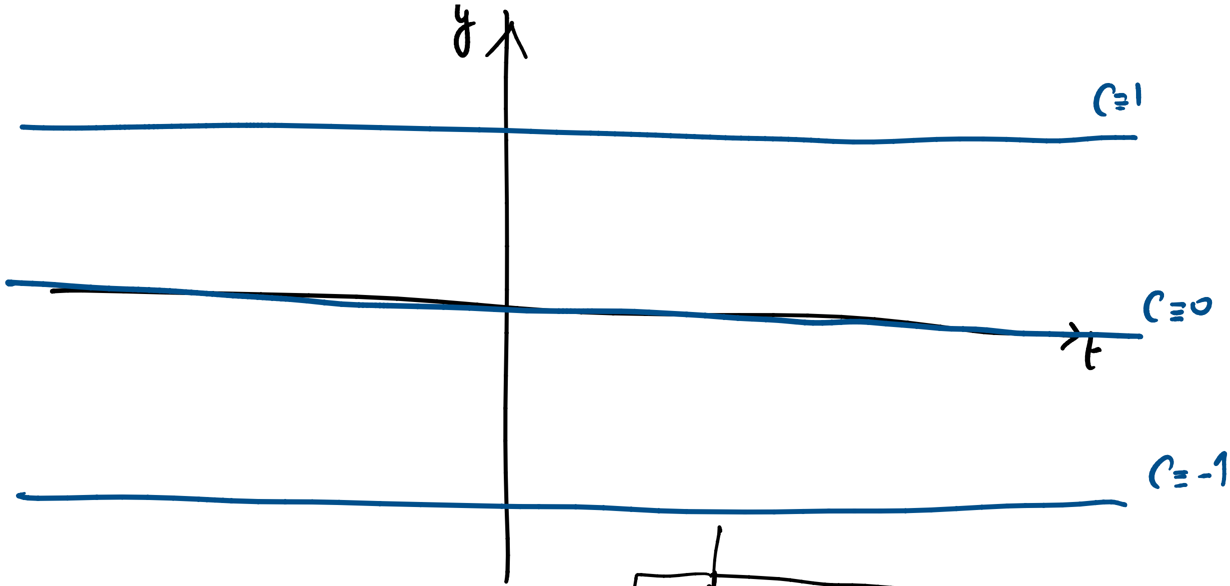
ii) $y \equiv C$ is sol $\iff 0 = t(C^2-1)\,\mathrm{arctg}\, C \quad \forall t$

$\iff (C^2-1)\,\mathrm{arctg}\, C = 0$

$\iff C = \pm 1, \quad C = 0$







iii) $y \nearrow \iff y' \geq 0 \iff \boxed{t\,\underbrace{(y^2-1)\,\operatorname{arctg} y}} \geq 0$

$$\operatorname{sgn} t = \begin{cases} +1 & t > 0 \\ -1 & t < 0 \end{cases}$$

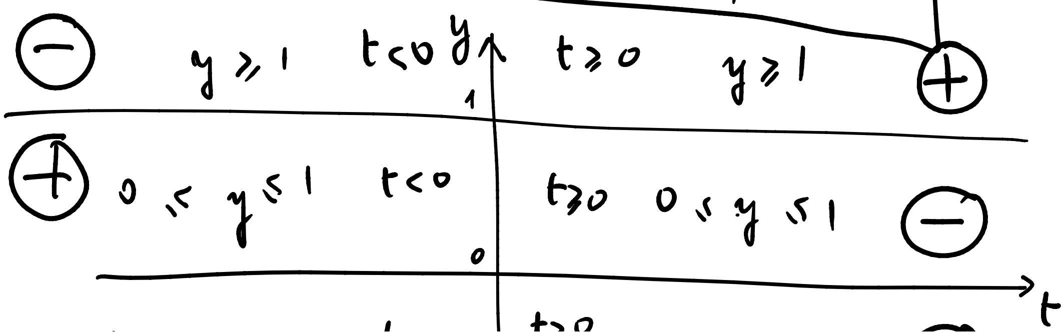
$y - t > 0$

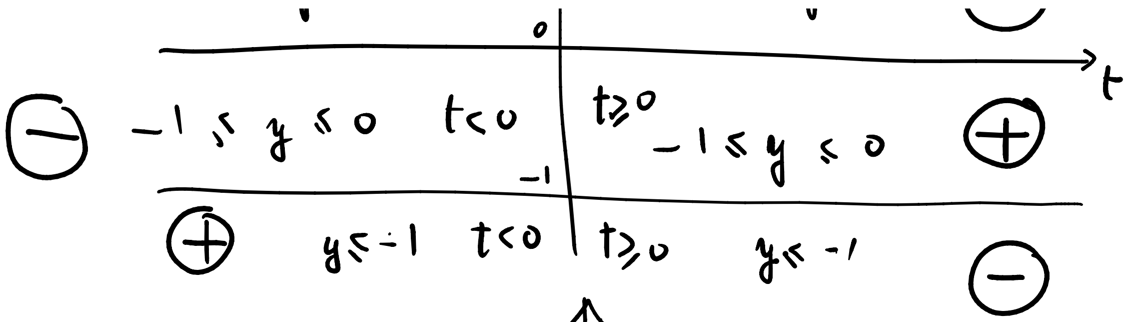
$(y^2-1)\operatorname{arctg} y \geq 0$

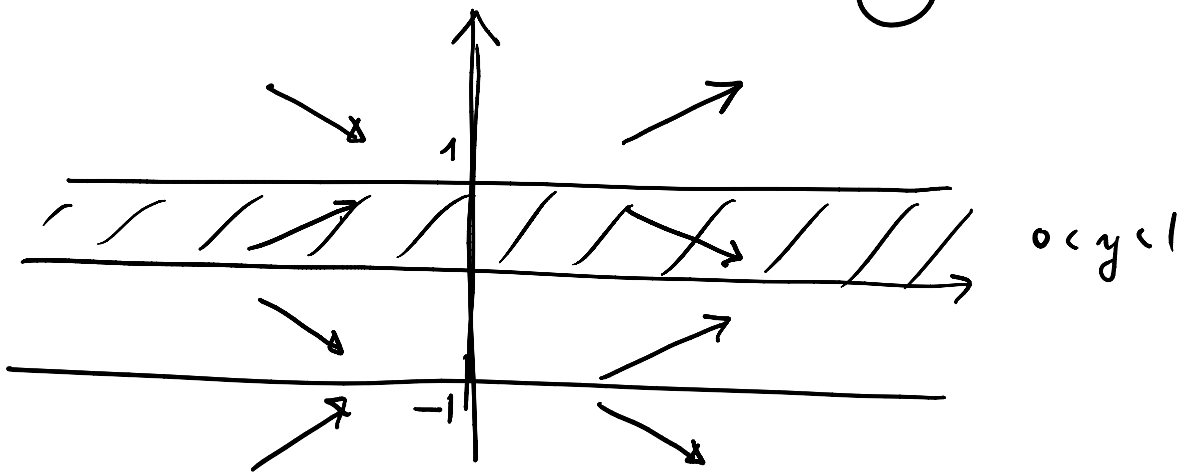
$y^2 - 1 \geq 0 \iff y^2 \geq 1 \iff \boxed{y \leq -1 \ \vee \ y \geq 1}$

$\operatorname{arctg} y \geq 0 \iff y \geq 0$

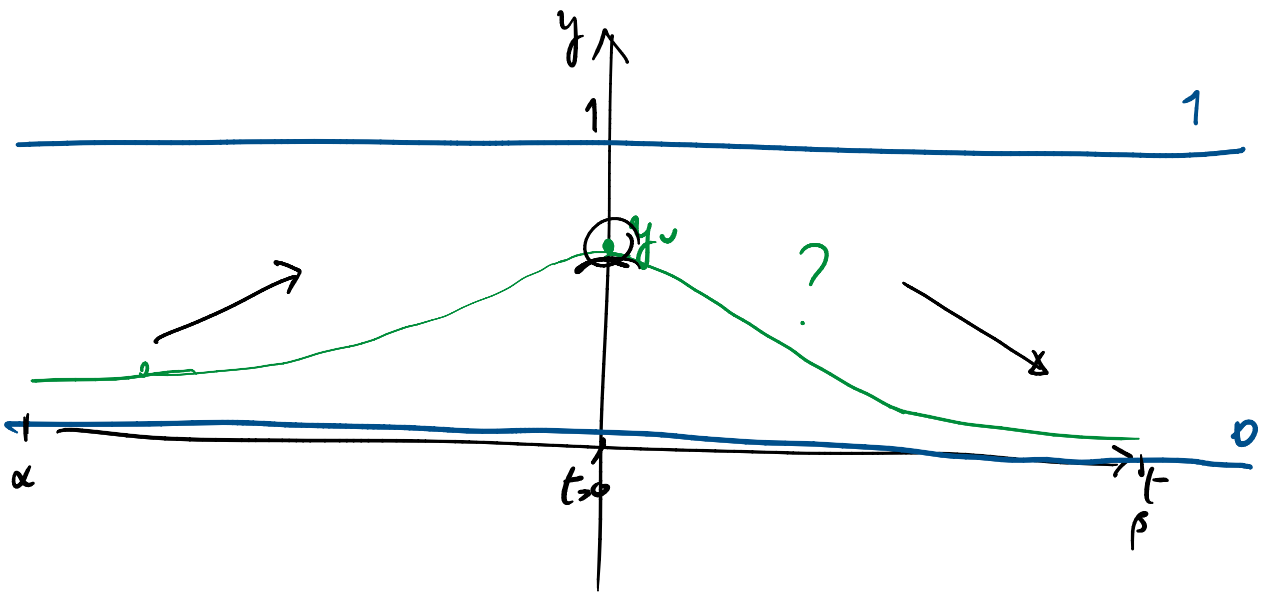
	$-\infty$ … -1	-1 … 0	0 … $+1$	$+1$ … $+\infty$
y^2-1	$+$	$-$	$-$	$+$
$\operatorname{arctg} y$	$-$	$-$	$+$	$+$
$(y^2-1)\operatorname{arctg} y$	$-$	$+$	$-$	$+$







iv)



y even $\Longleftarrow$ $\underbrace{y(-t)}_{v(t)} = y(t) \quad \forall t$

We prove that
- v is a sol of diff eqn
- $v(0) = y(0)$

$\Downarrow$ uniqueness

$v \equiv y$

• $v(0) = y(0)$

$\Downarrow$ uniqueness

$v \equiv y$

$v(0) = y(-0) = y(0)$

$$v'(t) \overset{?}{=} t\,(v(t)^2 - 1)\,\mathrm{arctg}\, v(t)$$

$$\boxed{y'(t) = t(y(t)^2 - 1) \cdot \mathrm{arctg}\, y(t)}$$

$$\| $$

$$(y(-t))' = -y'(-t)$$

$$= -\left[-t\,(y(-t)^2 - 1)\,\mathrm{arctg}\, y(-t)\right]$$

$$v'(t) = +\, t\,(v(t)^2 - 1)\,\mathrm{arctg}\, v(t)$$

$\Longrightarrow$ v solves diff eqn.

v) Monot: we claim $y \nearrow$ for $t < 0$
$y \searrow$ for $t > 0$

This follows once we prove $0 < y < 1$

This is true because if false,

$\exists\, \hat{t}:\ y(\hat{t}) \leq 0 \Longrightarrow$ if $y(\hat{t}) = 0 \overset{\text{uniq}}{\Longrightarrow} y \equiv 0$
(imposs $y(0) = 1/2$)

if $y(\hat{t}) < 0 \overset{\text{by cont}}{\Longrightarrow} \exists\, \hat{\hat{t}}:$
$y(\hat{\hat{t}}) = 0$

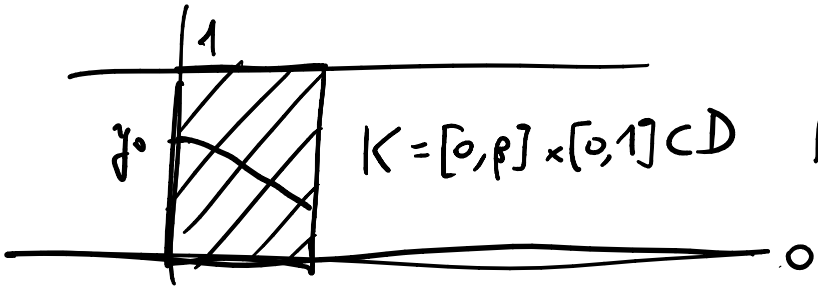
similarly $y < 1$ $\forall t$

similarly $y < 1 \quad \forall t$

Now because $\left.\begin{array}{l} y \nearrow \text{ for } t<0 \\ y \searrow \text{ for } t>0 \\ y \in \mathcal{C} \end{array}\right\} \Rightarrow t=0$ is a max for y

vi) α, β? Because y is even $\alpha = -\beta$.

$\beta = ?$



$K = [0,\beta] \times [0,1] \subset D \qquad \beta = +\infty$

β

If $\beta < +\infty \Rightarrow$ sol never leaves K in the future

impos.

$\Rightarrow \beta = +\infty$.

$\underset{t \to +\infty}{\lim} y(t) : \exists$ because y is $\searrow$ on $[0,+\infty[$

$\overset{\shortparallel}{\ell}$

$\boxed{0 \leq \ell \leq y_0} < 1$

$$y' = t\,(y^2-1)\,\text{arctg}\, y = \pm\infty$$

$t \to +\infty$, $(y^2-1) \to (\ell^2-1)$, $y \to \ell$

$(\ell^2-1)\cdot \text{arctg}\,\ell \leftarrow$ if $\neq 0$

(impos because: $y \to \ell \in \mathbb{R}$, $y' \to m \Rightarrow m = 0$)

$$\Downarrow$$

$$(l^2-1)\operatorname{arctg} l = 0 \Rightarrow l = \pm 1,\ \boxed{l = 0}$$

$$\Rightarrow l = 0. \quad \blacksquare$$