Systems of Diff Eqns

Let's consider a system of diff eqns
$\downarrow$
2×2

$$\begin{cases} x'(t) = f(t, x(t), y(t)) \\ y'(t) = g(t, x(t), y(t)) \end{cases}$$

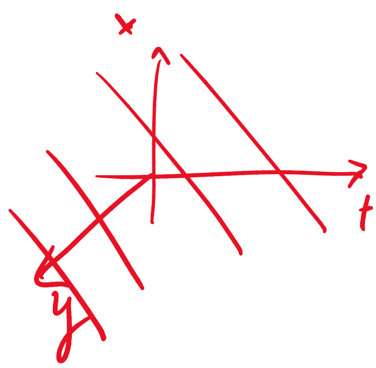
Here, for simplicity we will consider the case of autonomous systems

$$\boxed{\begin{cases} x'(t) = f(x(t), y(t)) \\ y'(t) = g(x(t), y(t)) \end{cases}}$$

The CP for such systems may be stated as follows:

$$\begin{cases} x'(t) = f(x(t), y(t)) \\ y'(t) = g(x(t), y(t)) \\ x(t_0) = x_0 \\ y(t_0) = y_0 \end{cases}$$

$\cancel{y(t_1) = y_0}$



Thm: ($\exists !$ local)

Assume that $f = f(x,y)$, $g = g(x,y)$, $f, g : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$. If

i) $f, g \in \mathcal{C}(D)$

ii) $\underset{\smile}{\partial_x}, \underset{\smile}{\partial_y}$ of $f, g \in \mathcal{C}(D)$

Then $\forall (t_0, x_0, y_0)$ $\exists$ sol $\begin{matrix} x = x(t) \\ y = y(t) \end{matrix} : \,]\alpha, \beta[\longrightarrow \mathbb{R}$

__Existence__

such $]\alpha, \beta[$ cannot be extended.

__Uniqueness:__ If $\begin{pmatrix} x \\ y \end{pmatrix}$ and $\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$ are sol of the system s.t.

$$\begin{pmatrix} x(t_0) \\ y(t_0) \end{pmatrix} = \begin{pmatrix} \tilde{x}(t_0) \\ \tilde{y}(t_0) \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} \equiv \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$$

__Prey - Predator system__

$x = x(t)$ # gazelles at time t

$y = y(t)$ # lions " " t

$x(t+h) - x(t)$ ... the rate of the x per unit of

$$\frac{x(t+h)-x(t)}{h\,x(t)} = \text{growth rate of pop } x \text{ per unit of time}$$

$$= \underbrace{\lambda}_{>0} - \underbrace{\alpha}_{>0} y(t)$$

assumption

$$\left(\frac{x(t+h)-x(t)}{h\,\cancel{x(t)}} = \lambda \Rightarrow \frac{x'}{\cancel{x}} = \lambda x \qquad x(t) = c\,e^{\lambda t} \right)$$

$$h \to 0 \quad \frac{x'(t)}{x(t)} \leftarrow \frac{x(t+h)-x(t)}{h}\,\frac{1}{x(t)} = \lambda - \alpha y(t)$$

$$\Rightarrow \boxed{x'(t) = (\lambda - \alpha y(t))\,x(t)}$$

$$\frac{y(t+h)-y(t)}{h\,y(t)} = -\underbrace{\mu}_{>0} + \underbrace{\beta}_{>0} x(t)$$

$$h \to 0 \downarrow$$

$$\frac{y'(t)}{y(t)}$$

$$\Rightarrow \boxed{y'(t) = (-\mu + \beta x(t))\,y(t)}$$

Putting together these eqns we arrive to the system

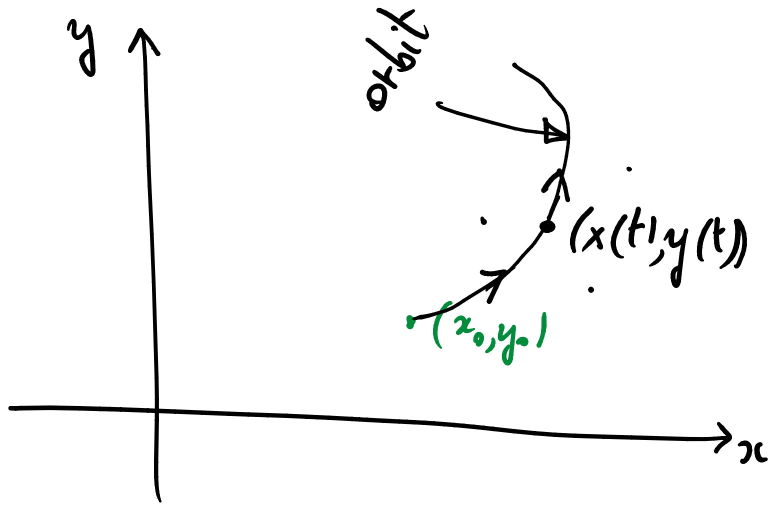
system

$$\begin{cases} x' = (\lambda - \alpha y)x \\ y' = (-\mu + \beta x)y \\ x(t_0) = x_0 \\ y(t_0) = y_0 \end{cases}$$

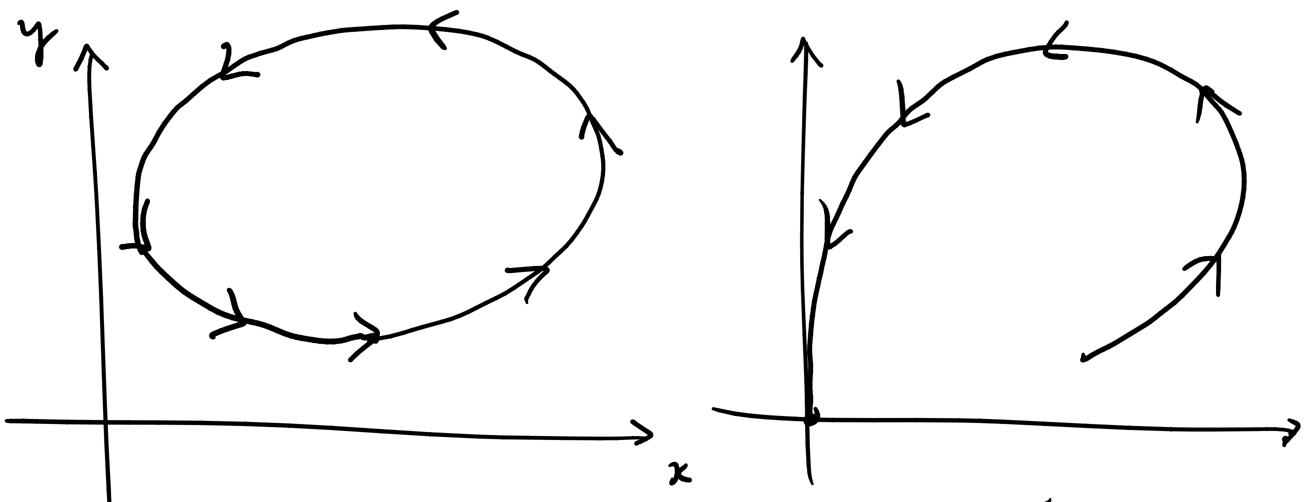
$\boxed{\lambda, \mu, \alpha, \beta > 0}$

(known)

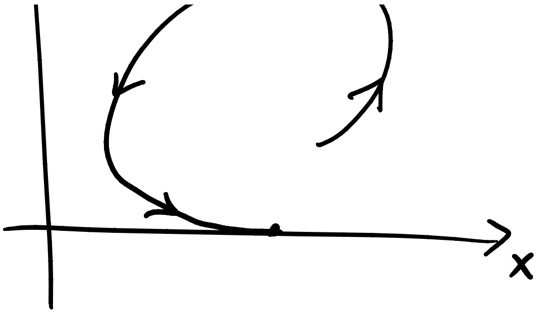
How do we represent the behavior of sols x, y?



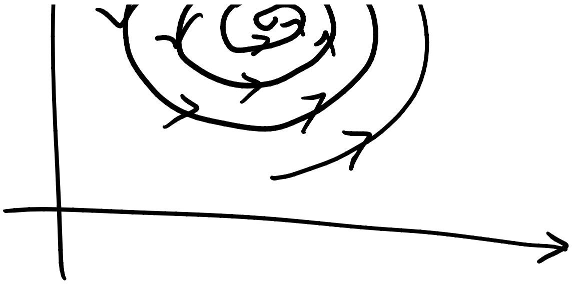
State of the system : $(x(t), y(t))$



lions



gaz



Def: An orbit is a set of points

$$\{ (x(t), y(t)) \;:\; t \in]\alpha, \beta[\}$$

where $(x, y) :]\alpha, \beta[$ is a sol. of the system.

The orientation of an orbit is the direction $(x(t), y(t))$ runs on the orbit as $t \nearrow$

Let's focus a bit on constant stationary solutions

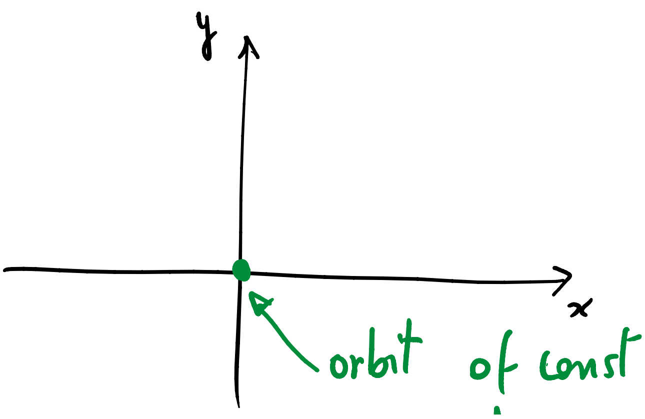
$$\begin{pmatrix} x \\ y \end{pmatrix} \equiv \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \text{ is a sol} \iff \begin{cases} 0 = f(x_0, y_0) \\ 0 = g(x_0, y_0) \end{cases}$$

Orbits for const/st. sols are made by singletons

$$\{(x_0, y_0)\}$$

Ex 1

$$\begin{cases} x' = y = f(x,y) \\ y' = x = g(x,y) \end{cases}$$



St. sols are $(x, y) \equiv (a, b)$

St. sols are $(x,y)\equiv(a,b)$ | orbit of const sol.

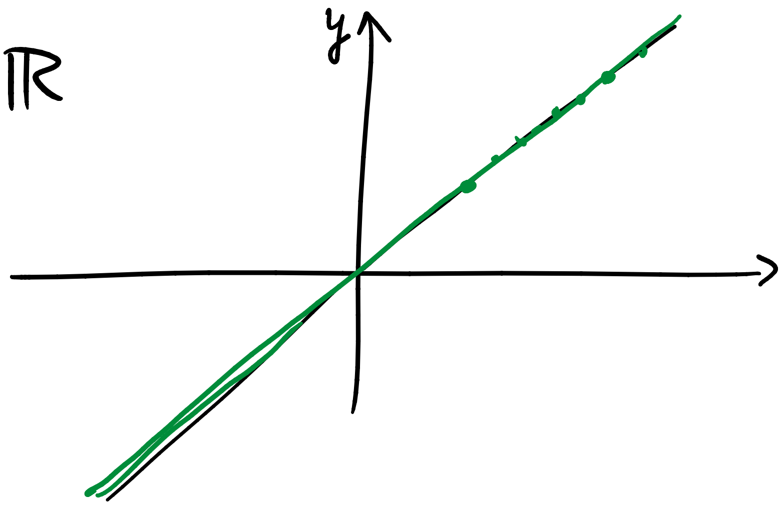
$$\iff \begin{cases} 0=b \\ 0=a \end{cases} \iff (a,b)=(0,0)$$

__Ex 2__

$$\begin{cases} x' = x-y = f(x,y) \\ y' = y-x = g(x,y) \end{cases}$$

$$(x,y)\equiv(a,b) \text{ is sol} \iff \begin{cases} 0 = a-b \\ \cancel{0 = b-a} \end{cases} \iff b=a$$

$$\Rightarrow (a,a) \quad a\in\mathbb{R}$$



__Ex 3:__

$$\begin{cases} x' = x(x-y) \\ y' = y+x \end{cases}$$

Here $(x,y)\equiv(a,b)$ is st. sol $\iff$

$$\begin{cases} 0 = a(a-b) \leftarrow \\ 0 = a+b \end{cases}$$

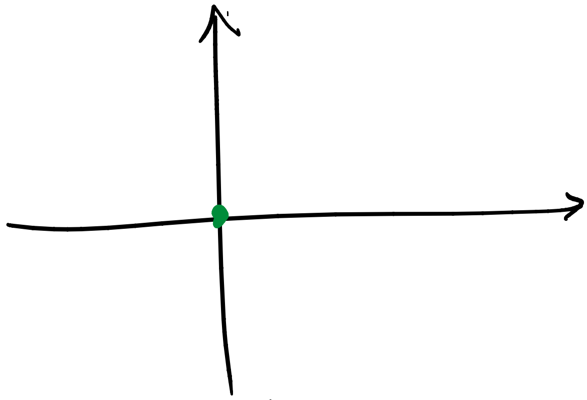
$$\iff \begin{cases} a=0 \\ \end{cases} \vee \begin{cases} a-b=0 \\ \end{cases}$$

$$\Leftrightarrow \begin{cases} b = 0 \end{cases} \vee \begin{cases} a+b = 0 \end{cases}$$

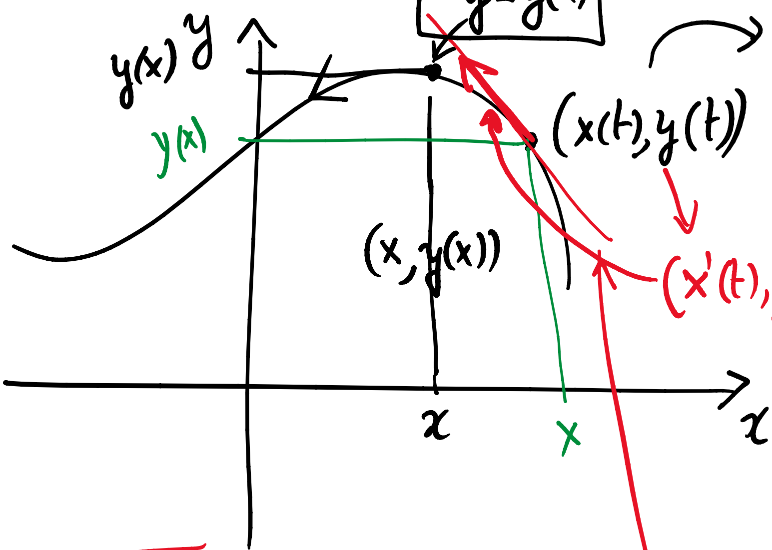
$\Updownarrow$ $(0,0)$

$$\Updownarrow \quad \begin{cases} 2a = 0 \\ b = a \end{cases} \Leftrightarrow (0,0)$$

St pts: $(0,0)$.



How can we plot an orbit $\boxed{y=y(x)}$ for a non-const sol? $\rightarrow$ sols of $\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases}$



$(x'(t), y'(t))$

slope $= \dfrac{y'(t)}{x'(t)}$

$\|$

angular coeff $= \dfrac{dy}{dx}$

$\boxed{f = f(x)}$ $\quad \dfrac{df}{dx}$

$$\Rightarrow \quad \boxed{\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}} \qquad F(x, y(x))$$

(total eqn)

Suppose the total eqn be a separable vars eqn

$$\boxed{\frac{dy}{dx} = \frac{a(x)}{b(y)}} \qquad \left(A(x)\, B(y) \right)$$

Then, by sep of vars

$$b(y)\,dy = a(x)\,dx \quad \Rightarrow$$

$$\int b(y)\,dy = \int a(x)\,dx + C$$

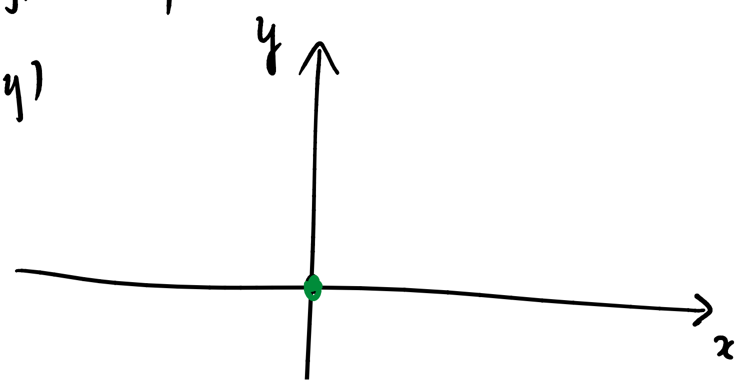
$$\Updownarrow$$

$$\boxed{\int b(y)\,dy - \int a(x)\,dx \equiv C}$$

(eqn in x, y representing the orbit itself)

Ex 1:

$$\begin{cases} x' = y = f(x,y) \\ y' = x = g(x,y) \end{cases} \qquad \text{St pts } (0,0)$$



Total eqn:

$$\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)} = \frac{x}{y}$$

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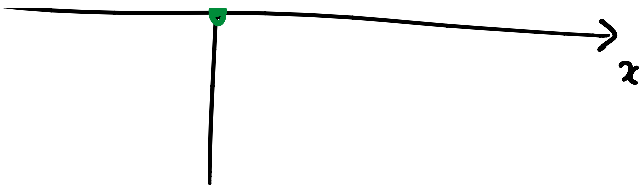


sep vars eqn $\iff$

$$y\,dy = x\,dx$$

$$\Downarrow$$

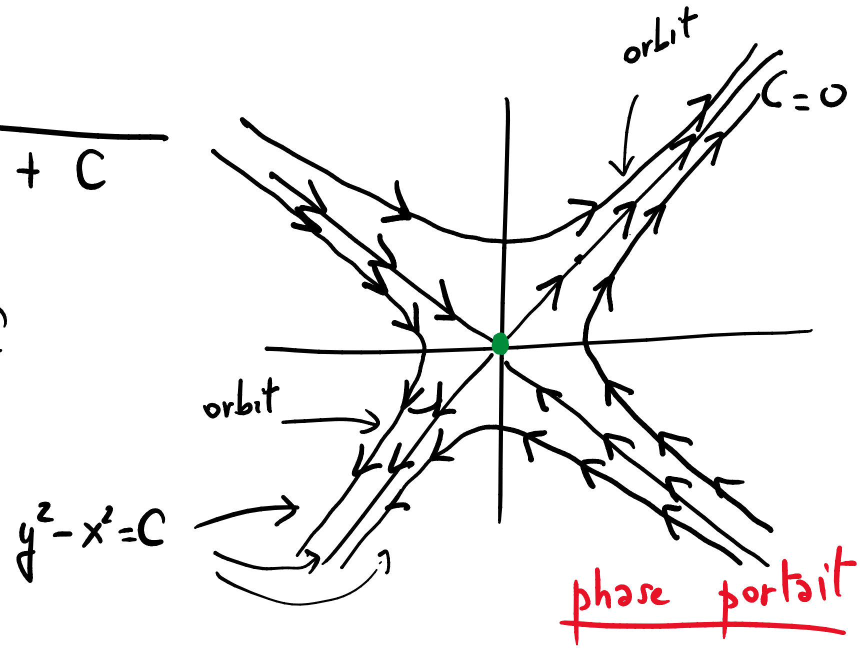
$$\frac{y^2}{2} = \int y\,dy = \int x\,dx + C = \frac{x^2}{2} + C$$

$$\Rightarrow \quad \boxed{y^2 = x^2 + C \qquad C \in \mathbb{R}} \leftarrow$$

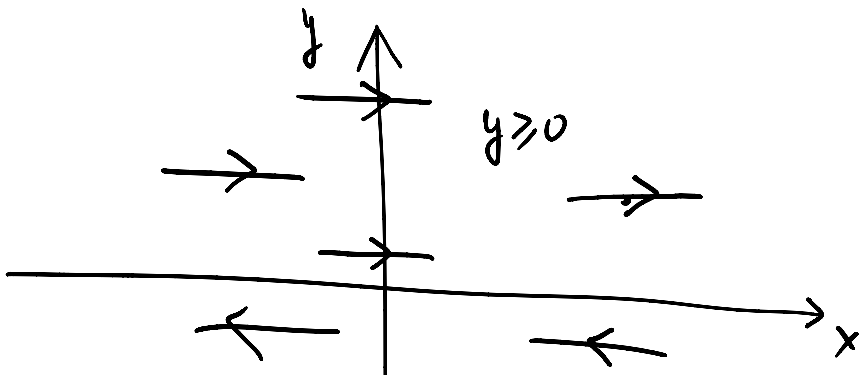
$$\Downarrow$$

$$y = \pm\sqrt{x^2 + C}$$

$$y^2 - x^2 = C$$

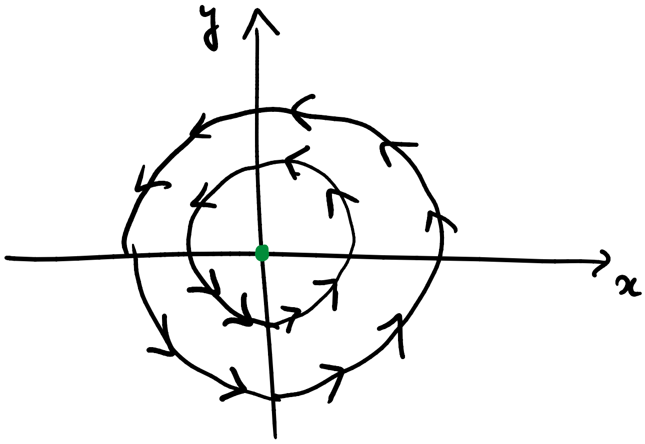


Orientation: $x \nearrow \iff 0 \leq x' = y \iff y \geq 0$



Ex 2:
$$\begin{cases} x' = -y \\ y' = x \end{cases}$$

St. sols: $(x,y) \equiv (a,b)$ is a sol $\iff \begin{cases} 0 = -b \\ 0 = a \end{cases} \iff (a,b) = (0,0)$



Let's write the total eqn:

$$\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)} = \frac{x}{-y}$$

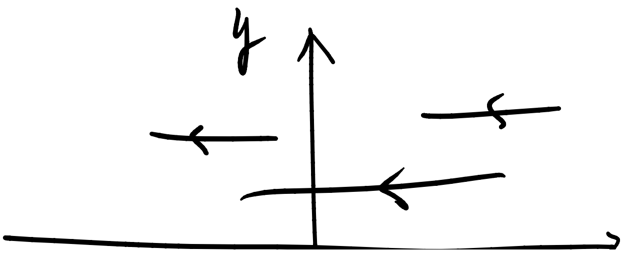
it is a sep vars eqn

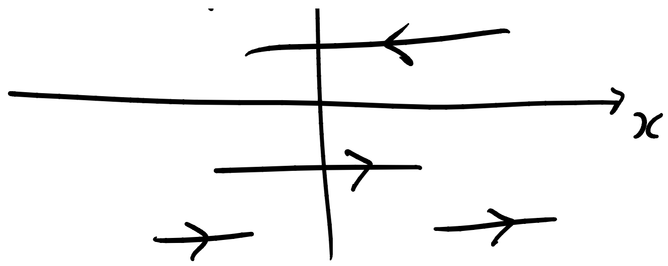
$$\iff -y\,dy = x\,dx \iff -\frac{y^2}{2} = \frac{x^2}{2} + C \quad C \in \mathbb{R}$$

$$\iff y^2 = -x^2 + C$$

$$\iff x^2 + y^2 = C$$

Orientation: $x \nearrow \iff 0 \leq x' = -y \iff y \leq 0$

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