We're considering a 2x2 autonomous system of diff eqns

$$(S) \quad \begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases}$$

<u>Goal</u>: to plot typical oriented orbits of this system.

<u>Orbit</u>: $\{(x(t), y(t)) : t \in]\alpha, \beta[\}$ $\quad (x,y)$ is a sol of (S)

For st/const sols $\begin{cases} x(t) \equiv x_0 \\ y(t) \equiv y_0 \end{cases} \Rightarrow$ orbits are $\{(x_0, y_0)\}$

For non st/const sol. we may find orbits in the form of graphs $y = y(x)$, where $y = y(x)$ solves

$$\boxed{\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)} \qquad \text{total eqn}}$$

<u>Rmk</u>: Mnemonic rule: $\begin{cases} \frac{dx}{dt} = f(x,y) \\ \frac{dy}{dt} = g(x,y) \end{cases} \Rightarrow \frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}$

If tot. eqn is a sep vars eqn

$$\frac{dy}{dx} = \frac{a(x)}{\ldots} \iff b(\ldots)\,dy = a(x)\,dx$$

$$\frac{dy}{dx} = \frac{a(x)}{b(y)} \iff b(y)\,dy = a(x)\,dx$$

$$\iff \boxed{\int b(y)\,dy = \int a(x)\,dx + C}$$

$$\iff \boxed{\int b(y)\,dy - \int a(x)\,dx = C}$$

Let

$$E(x,y) := \int b(y)\,dy - \int a(x)\,dx$$

This function has a remarkable property:

Prop: E is constant on sols of (S)

$$\Big(\quad E(\underbrace{x(t), y(t)}_{\text{sol of } S}) \equiv E(x(t_0), y(t_0)) \quad \Big)$$

E is called first integral / energy of the system

Proof: To show that $E(x(t), y(t))$ is const in t we show that

$$\frac{d}{dt}\Big[E(x(t), y(t))\Big] \equiv 0$$

Now

$$\frac{d}{dt} E\big(x(t), y(t)\big) = \partial_x E\big(x(t), y(t)\big) \cdot x'(t) + \partial_y E\big(x(t), y(t)\big) \cdot y'(t) \overset{(*)}{=}$$

$$\partial_x E = -a(x), \qquad \partial_y E = b(y)$$

$$E = \int b(y)\, dy - \int a(x)\, dx$$

$$\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases}$$

$$\overset{(*)}{=} -a(x) \cdot f(x,y) + b(y) \cdot g(x,y) \equiv 0$$

(this is because, once we assumed the tot eqn be a sep vars eqn $\frac{dy}{dx} = \boxed{\frac{g(x,y)}{f(x,y)} = \frac{a(x)}{b(y)}}$) ☒

In particular:

- orbits are contained in sets where $\underbrace{\{E = \text{Constant}\}}_{\text{level set of } E}$

Ex. 8.4.6 Consider

$$\begin{cases} x' = y(x-y) \\ y' = -x(x-y) \end{cases}$$

i) Find st sols.

ii) Det a non trivial first integral

iii) Plot the phase portrait

iii) Plot the phase portrait of the system.

Sol. i) $(x,y)=(a,b)$ is sol $\Leftrightarrow$ $\begin{cases} 0 = b(a-b) \\ 0 = -a(a-b) \end{cases}$ $\leftarrow$

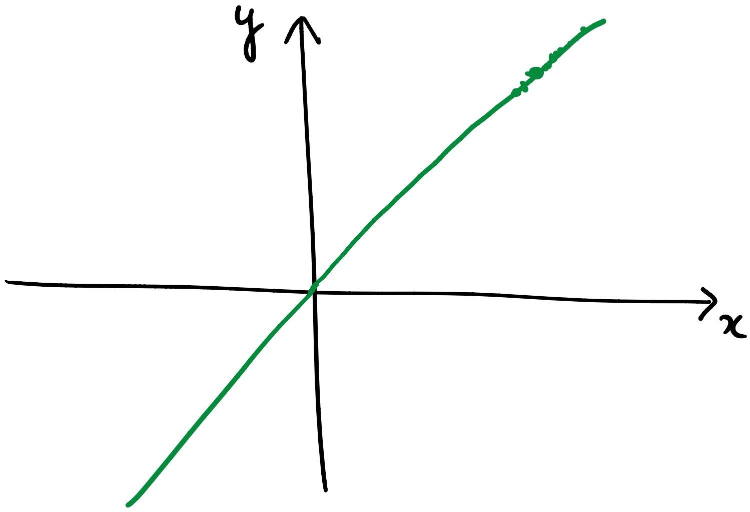
$(x(t),y(t))$

$\Leftrightarrow$ $\begin{cases} b=0 \\ 0=a^2 \end{cases}$ $\vee$ $\begin{cases} a-b=0 \\ 0=0 \end{cases}$

$\Downarrow$

$(a,b)=(0,0)$ $\qquad$ $(a,b)=(a,a)$ $\quad a\in\mathbb{R}$

Conclusion: st points are (a,a) $a\in\mathbb{R}$.



ii) Let's write the tot. eqn:

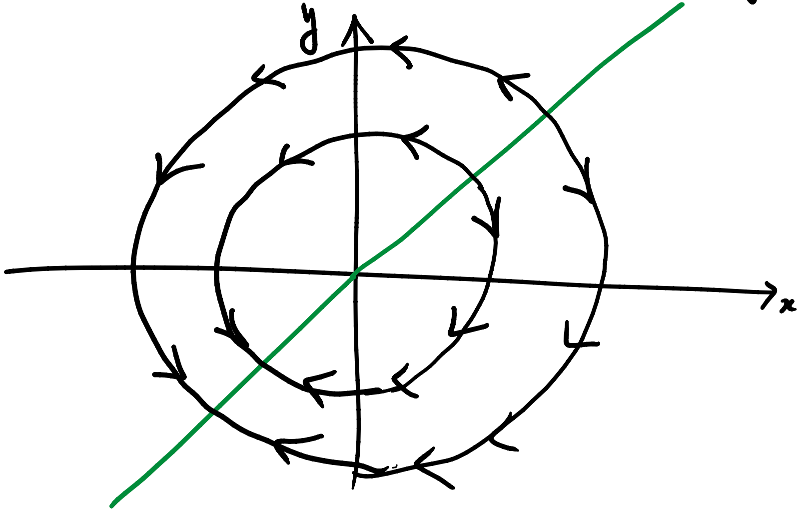
$$\frac{dy}{dx} = \frac{-x\cancel{(x-y)}}{y\cancel{(x-y)}} = -\frac{x}{y} \qquad \text{sep vars eqn}$$

$$\Leftrightarrow \quad y\,dy = -x\,dx \quad \Rightarrow \quad \frac{y^2}{\cancel{2}} = -\frac{x^2}{2} + C$$

$\Leftrightarrow \quad y\,dy = -x\,dx \quad \Rightarrow \quad \frac{y^2}{2} = -\frac{x^2}{2} + C$

$\Leftrightarrow \quad \underbrace{y^2 + x^2}_{E(x,y)} = C \qquad\qquad \frac{y^2}{2} - \frac{x^2}{2}$

iii) Orbits are contained in lines $y^2 + x^2 = C$

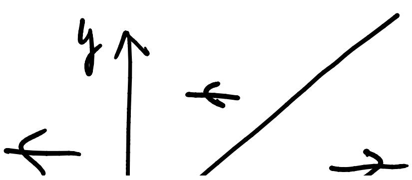


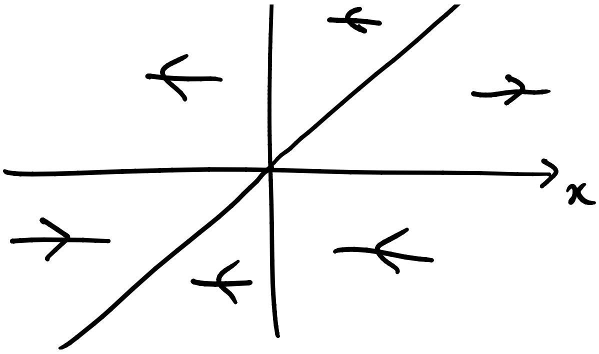
Orientation: $x \nearrow \Leftrightarrow 0 \leq x' = y(x-y) \Leftrightarrow y(x-y) \geq 0$

$$\Leftrightarrow \begin{cases} y \geq 0 \\ x - y \geq 0 \end{cases} \vee \begin{cases} y \leq 0 \\ x - y \leq 0 \end{cases}$$

$$\Downarrow \qquad\qquad \Updownarrow$$

$$\begin{cases} y \geq 0 \\ y \leq x \end{cases} \vee \begin{cases} y \leq 0 \\ y \geq x \end{cases}$$





An important consequence of key property fulfilled by the first int. is that we can compute explici sols to the system:

$$\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases} \quad + \quad E(x,y) \equiv C$$

$$\Updownarrow$$

$$y = y(x)$$

$$x' = f(x, y(x)) = F(x) \xrightarrow{\text{SOLVE}} x = x(t)$$

<u>Ex. 8.4.8</u> Consider the system

$$\begin{cases} x' = xy \\ y' = -x^2 + 2x^4 \end{cases}$$

i) St. sols.

ii) First Int.

iii) Phase portrait : are there periodic sols ?
are there global sols ?
($]\alpha, \beta[\; = \;]-\infty, +\infty[$)

$( ]\alpha,\beta[\; = \;]-\infty,+\infty[\,)$

iii) Find sol of CP $x(0)=2$, $y(0)=2\sqrt{3}$.

Sol: i) $(x,y)=(a,b)$ is sol $\iff \begin{cases} 0 = ab \\ 0 = -a^2+2a^4 \end{cases}$

$$\iff \begin{cases} a=0 \\ 0=0 \end{cases} \quad \vee \quad \begin{cases} b=0 \\ 0=-a^2+2a^4 \end{cases}$$

$$\Updownarrow$$

$$\cancel{(0,0)}$$

$$\boxed{(0,b) \quad \forall b \in \mathbb{R}} \quad \vee \quad \begin{cases} b=0 \\ a^2(-1+2a^2)=0 \end{cases}$$

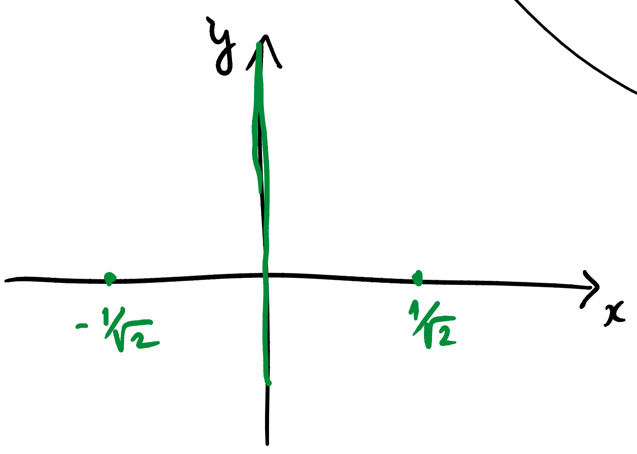
~~no sols~~

$$\Updownarrow$$

$$\begin{cases} b=0 \\ a^2=0 \end{cases} \quad \vee \quad \begin{cases} b=0 \\ 2a^2=1 \end{cases}$$

$$\Updownarrow$$

$$(0,0) \quad \vee \quad \left(\pm\frac{1}{\sqrt{2}},0\right)$$



ii) First int: the total eqn is

$$\frac{dy}{dx} = \frac{-x^2+2x^4}{xy} = \frac{-x+2x^3}{y} \quad \text{sep vars}$$

$$\iff \quad y\,dy = (-x+2x^3)\,dx$$

$$\Leftrightarrow \quad y\,dy = (-x + 2x^3)\,dx$$

$$\Leftrightarrow \quad \int y\,dy = \int (-x + 2x^3)\,dx + C$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + \cancel{2}\frac{x^4}{\cancel{4}2} + C$$

$$\Leftrightarrow \quad y^2 = x^4 - x^2 + C$$

$$E(x,y) = y^2 - (x^4 - x^2).$$

iii) <u>Phase portrait:</u> Orbits are contained in lines

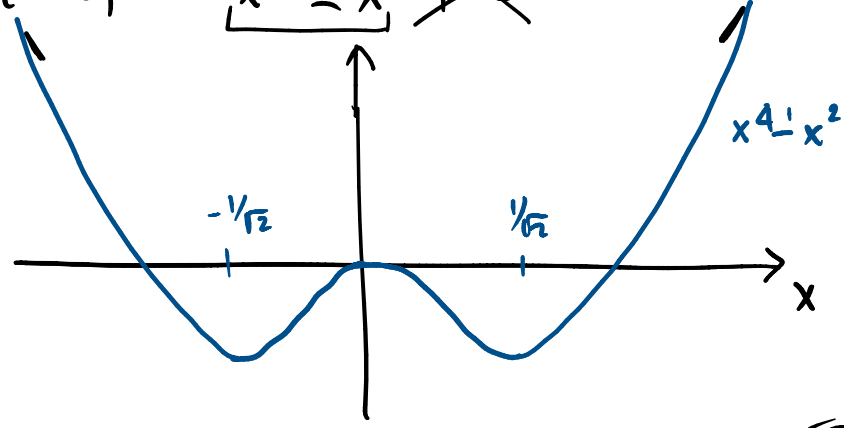
$$y^2 = x^4 - x^2 + C \qquad \cancel{\boxed{x^2+y^2 = x^4 + C}}$$

~~circle radius~~ $\cancel{\sqrt{x^4+C}}$ ...

$$\Updownarrow$$

$$y = \pm\sqrt{\underbrace{x^4 - x^2 + C}}$$

Plot of $\underbrace{x^4 - x^2} \cancel{+C}$



$\varphi(x) = x^4 - x^2$

φ even

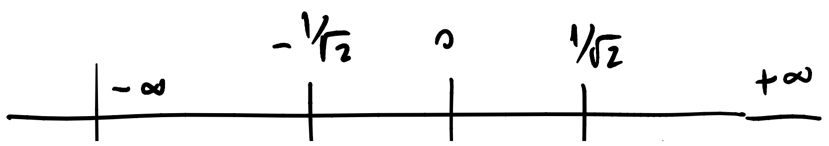
$\varphi(\pm\infty) = +\infty$

$\varphi(0) = 0$

$\varphi' = 4x^3 - 2x$

$\geqslant 0$

$\Leftrightarrow \quad x(4x^2 - 2) \geqslant 0$

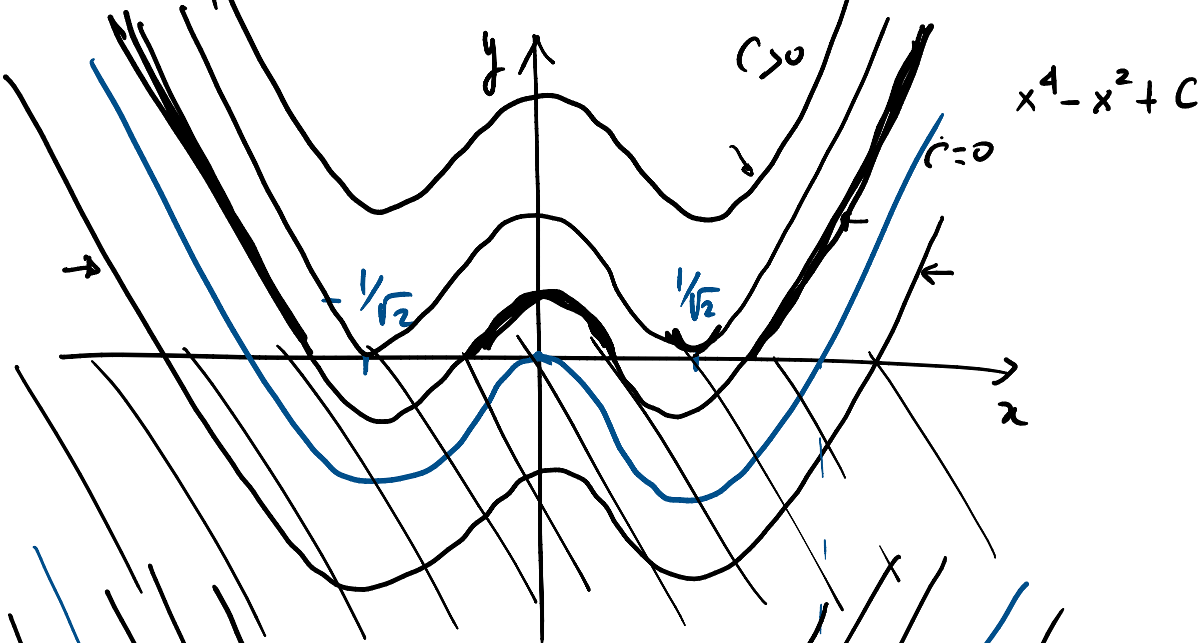


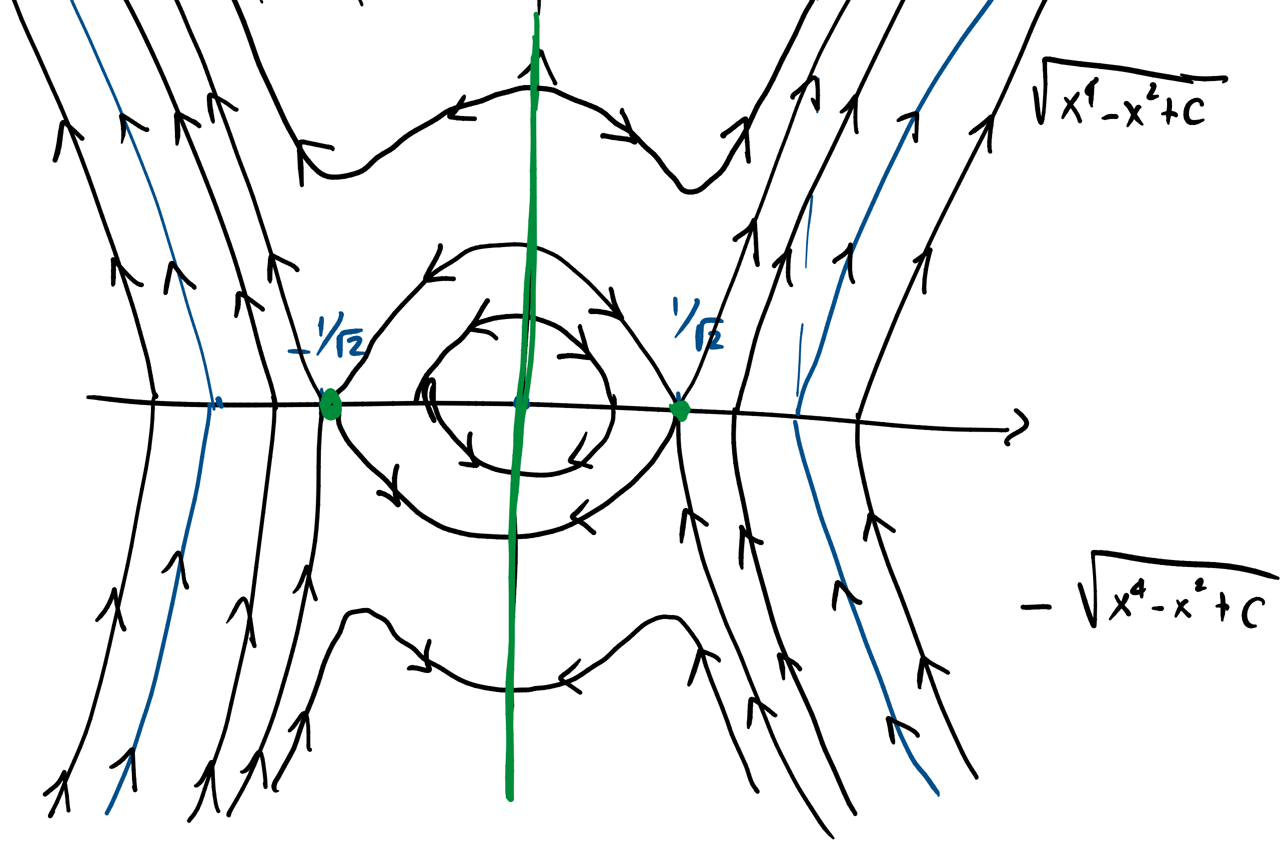
$4x^2 - 2 \geqslant 0 \Leftrightarrow x^2 \geqslant 1/2$

	$-\infty$	$-1/\sqrt{2}$	0	$1/\sqrt{2}$	$+\infty$
sgn x	$-$	$-$	$+$	$+$	
sgn$(4x^2-2)$	$+$	$-$	$-$	$+$	
sgnφ'	$-$	$+$	$-$	$+$	
φ	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$	

$$4x^2 - 2 \geq 0 \iff x^2 \geq 1/2$$
$$\iff x \leq -1/\sqrt{2} \ \lor$$
$$x \geq 1/\sqrt{2}$$

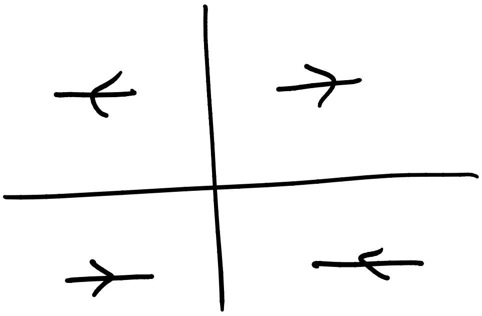
$$\varphi\left(\pm \frac{1}{\sqrt{2}}\right) = \frac{1}{4} - \frac{1}{2} = -1/4$$





$$x \nearrow \iff 0 \leq x' = xy \iff \{x \geq 0 \quad \lor \quad \{x \leq 0$$

$x \nearrow \iff 0 \le x' = xy \iff \begin{cases} x \geq 0 \\ y \geq 0 \end{cases} \vee \begin{cases} x \leq 0 \\ y \leq 0 \end{cases}$



iv) To find the sol of CP $\begin{cases} x(0) = 2 \\ y(0) = 2\sqrt{3} \end{cases}$

we notice that, because $E(x(t), y(t)) \equiv E(x(0), y(0))$

$= E(2, 2\sqrt{3})$

where $E(x,y) = y^2 - (x^4 - x^2)$

$\Rightarrow$

$12 - 16 + 4$

$$y^2 - (x^4 - x^2) = E(2, 2\sqrt{3}) = 4 \cdot 3 - (16 - 4)$$

$$= 0$$

$\Rightarrow \boxed{y^2 = x^4 - x^2}$ $\qquad y = \pm\sqrt{x^4 - x^2}$

$\left(t=0 : \; 2\sqrt{3} = \pm\sqrt{16 - 4}\right)$

$\Downarrow \qquad \Downarrow$

$\oplus$

$y = \sqrt{x^4 - x^2} = |x|\sqrt{x^2 - 1}$

$t = 0 \qquad y = \pm x\sqrt{x^2 - 1}$

$2\sqrt{3} = \pm 2\sqrt{3}$

$\Downarrow$

$\oplus$

$y = x\sqrt{x^2 - 1}$

$\Rightarrow x' = xy$

$= x \cdot x\sqrt{x^2 - 1}$

$= x \cdot x \sqrt{x^2 - 1}$ $\Leftrightarrow$ $y = x\sqrt{x^2-1}$

$$\Rightarrow \quad \boxed{x' = x^2\sqrt{x^2-1}}$$

This is a sep vars eqn $\Leftrightarrow$

$$\frac{dx}{x^2\sqrt{x^2-1}} = dt$$

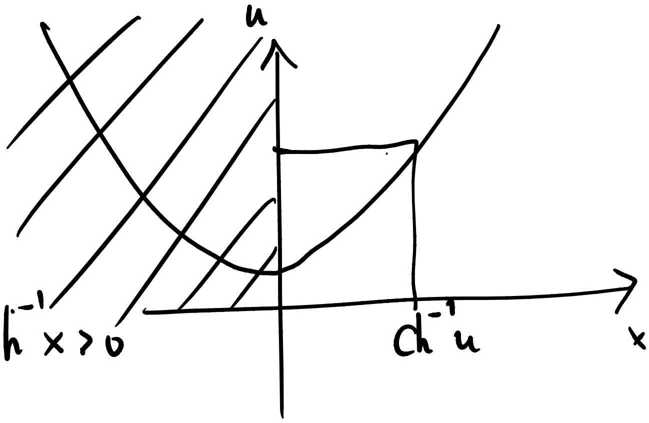
$$\Rightarrow \int \frac{dx}{x^2\sqrt{x^2-1}} = \int dt + C = t + C$$

$Ch^2 - Sh^2 = 1$

$Sh^2 = Ch^2 - 1$

$x = Ch\,u \qquad u = Ch^{-1} x > 0$

$dx = Sh\,u\,du$



$$= \int \frac{Sh\,u\,du}{(Ch\,u)^2\sqrt{(Sh\,u)^2}} = \int \frac{Sh\,u}{(Ch\,u)^2\,|\underset{>0}{Sh\,u}|}\,du$$

$$= \int \frac{\cancel{Sh\,u}}{(Ch\,u)^2\,\cancel{Sh\,u}}\,du = \int \frac{1}{(Ch\,u)^2}\,du$$

$$Ch\,u = \frac{e^u + e^{-u}}{2} \qquad = \int \frac{4}{(e^u + \frac{1}{e^u})^2}\,du$$

$$= 4\int \frac{e^{2u}}{(e^{2u}+1)^2}\,du$$

$$= 4 \int \frac{-}{(e^{2u}+1)^2}\, du$$

$$= \cancel{4}2 \int (e^{2u}+1)^{-2}\, 2e^{2u}\, du = 2\, \frac{(e^{2u}+1)^{-1}}{-1}$$

$$2e^{2u} = (e^{2u}+1)'$$

$$= -\frac{2}{e^{2u}+1} = -\frac{2}{e^{2Ch^{-1}x}+1}$$

$$\Rightarrow \quad -\frac{2}{e^{2Ch^{-1}x}+1} = t + C \qquad \left(\begin{matrix} t=0 \\ C = -\frac{2}{e^{2Ch^{-1}2}+1} \end{matrix} \right)$$

$$\Rightarrow \quad \frac{\cancel{2}\,1}{e^{2Ch^{-1}x}+1} = \frac{-t-C}{2}$$

$$\Rightarrow \quad e^{2Ch^{-1}x}\cancel{+1} = -\frac{2}{t+C} - 1$$

$$\Rightarrow \quad \cancel{2}Ch^{-1}x = \frac{1}{2}\log\left(-\frac{2}{t+C} - 1\right)$$

$$\Rightarrow \quad x(t) = Ch\left[\frac{1}{2}\log\left(-\frac{2}{t+C} - 1\right)\right]$$

$$\boxed{\text{Do Ex 8.7.2 \#1 -- 4}}$$