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Ex 8.7.2

#1 Consider
$$\begin{cases} x' = 2y(y-2x) \\ y' = (1-x)(y-2x) \end{cases}$$

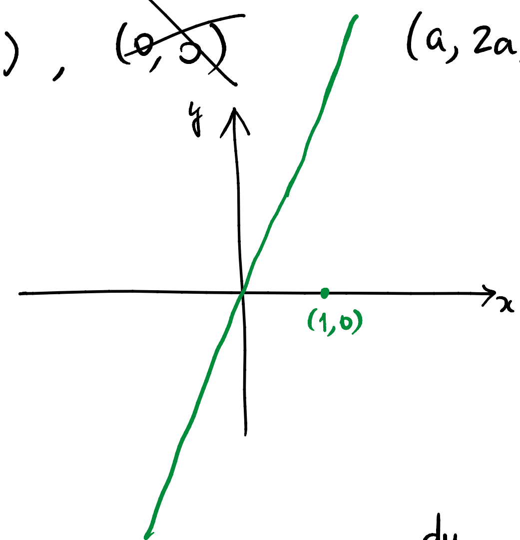
i) St sols
ii) First Int.
iii) Phase portr.

Sol: i) $(x,y) \equiv (a,b)$ is a sol $\iff \begin{cases} 0 = \cancel{2}b(b-2a) \\ 0 = (1-a)(b-2a) \end{cases}$

$$\iff \begin{cases} b = 0 \\ 0 = (1-a)(-2a) \end{cases} \vee \begin{cases} b - 2a = 0 \\ 0 = 0 \end{cases}$$

$$\iff \begin{cases} b = 0 \\ a = 1 \end{cases} \vee \begin{cases} b = 0 \\ a = 0 \end{cases} \vee \begin{cases} b = 2a \end{cases}$$

$(1,0)$, $\cancel{(0,0)}$ $\qquad (a, 2a) \quad a \in \mathbb{R}$



ii) The total eqn is $\displaystyle \frac{dy}{dx} = \frac{(1-x)\cancel{(y-2x)}}{2y\,\cancel{(y-2x)}} = \frac{1-x}{2y}$ sep vars

$$\iff 2y\,dy = (1-x)\,dx$$

$$\iff y^2 = x - x^2/2 + C$$

First int $E(x,y) = y^2 - \left(x - \frac{x^2}{2}\right)$

iii) We have to plot lines $y^2 = x - \frac{x^2}{2} + C$

$$\boxed{y^2 + \frac{x^2}{2} - x = C}$$

$$y^2 + \frac{1}{2}(x-1)^2 = c$$

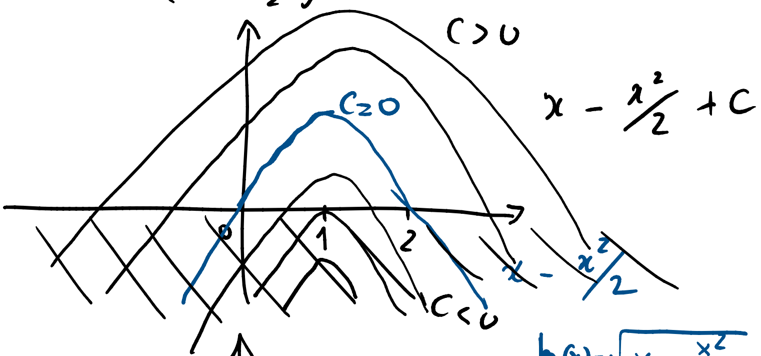
$$\frac{1}{2}(x^2 - 2x + 1) = c$$

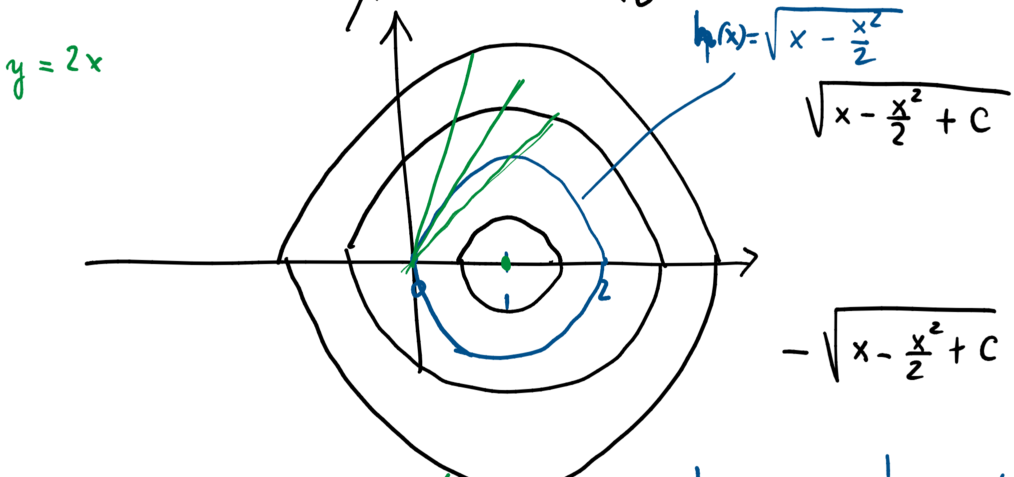
$$\Updownarrow$$

$$y = \pm\sqrt{x - \frac{x^2}{2} + C}.$$

Let's start by $g(x) = x - \frac{x^2}{2}$

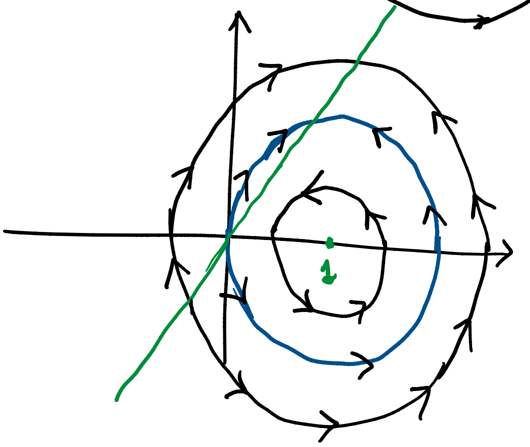
$g = 0 \iff x\left(1 - \frac{x}{2}\right) = 0 \iff x = 0 \vee 2$





$$h'(x) = \frac{1}{2\sqrt{x - \frac{x^2}{2}}} \cdot (1 - x)$$

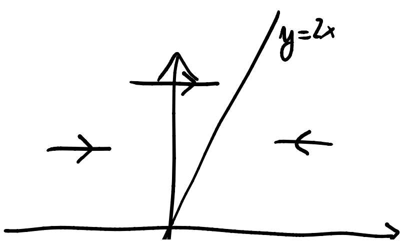
$$h'(0) = \frac{1}{0^+} \, 1 = +\infty$$



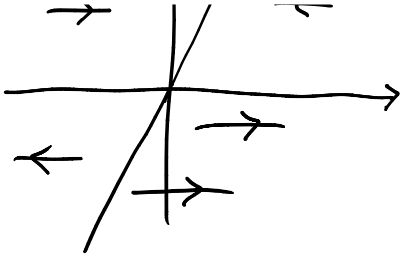
Orientation: $x \nearrow \iff 0 \leq x' = 2y(y - 2x) \iff$

$$2y(y - 2x) \geq 0$$

$$\iff \begin{cases} y \geq 0 \\ y - 2x \geq 0 \end{cases} \vee \begin{cases} y \leq 0 \\ y - 2x \leq 0 \end{cases}$$



$$\begin{cases} \\ y-2x\geq 0\end{cases} \qquad \begin{cases} \\ y-2x\leq 0\end{cases}$$

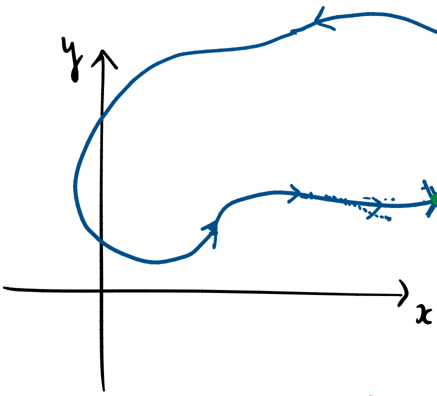


$$\Leftrightarrow \begin{cases} y\geq 0 \\ y\geq 2x\end{cases} \vee \begin{cases} y\leq 0 \\ y\leq 2x\end{cases}$$

Rmks:

- if the orbit is a closed circuit, then the corresponding sol. must be periodic.

In p.



- In part $]\alpha,\beta[=]-\infty,+\infty[$
- if the orbit is contained into a compact set cont in domain $\Rightarrow$ $]\alpha,\beta[=]-\infty,+\infty[$.

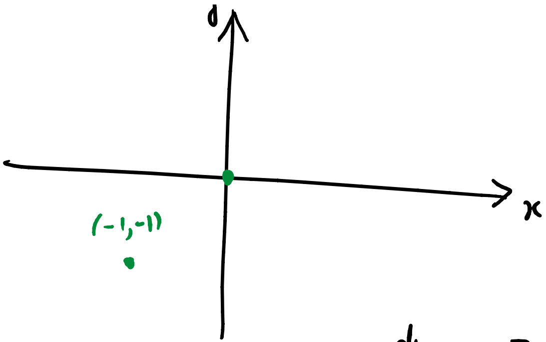
#3. $$\begin{cases} x' = x(1+y) \\ y' = -y(1+x)\end{cases}$$

i) St pts. ii) First Int iii) Orbits of syst.

i) $(x,y)\equiv(a,b)$ is sol $\Leftrightarrow$ $\begin{cases} \boxed{0 = a(1+b)} \\ 0 = -b(1+a)\end{cases}$

$$\Leftrightarrow \begin{cases} a=0 \\ -b=0\end{cases} \vee \begin{cases} 1+b=0 & b=-1 \\ 0 = 1+a\end{cases}$$

$\Updownarrow$ $(0,0)$ $\qquad$ $\Updownarrow$ $(-1,-1)$



ii) The total eqn is $\frac{dy}{dx} = \frac{-y(1+x)}{x(1+y)} = -\frac{y}{1+y}\cdot\frac{1+x}{x}$

it is a sep vars eqn $\Leftrightarrow$

$$\left(\frac{1+y}{y}\right)dy = -\frac{1+x}{x}\,dx$$

$$\Leftrightarrow \left(\frac{1}{y}+1\right)dy = -\left(\frac{1}{x}+1\right)dx$$

$$\Leftrightarrow \int\left(\frac{1}{y}+1\right)dy = -\int\left(\frac{1}{x}+1\right)dx + C$$

$$\Leftrightarrow \boxed{\log|y|+y = -(\log|x|+x)+C}$$

$$E(x,y) = \log|y|+y+\log|x|+x.$$

iii) $\log|y|+y = \log|y| + \log e^y$

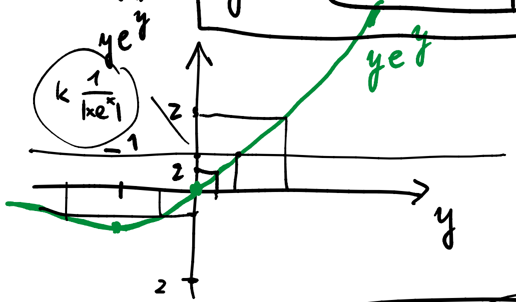
$= \log|ye^y|$

$\log|ye^y| = +\log|xe^x|^{-1} + C$

$|ye^y| = e^{\log|xe^{+x}|^{-1}+C} = \overset{\overset{0}{\wedge}}{k}\,\frac{1}{|xe^x|}$

$ye^y = z$

$$\Leftrightarrow \boxed{ye^y = \pm \overset{\overset{0}{\wedge}}{k}\,\frac{1}{|xe^x|} = k\,\frac{1}{|xe^x|}} \quad k\in\mathbb{R}$$



$g(y) = ye^y$

$g(0) = 0$

$g>0 \Leftrightarrow y>0$

$g(+\infty) = +\infty$

$g(-\infty) = 0$

$g' = e^y(1+y)$

$z\dagger$

$k>0$ $\quad ye^y = k\dfrac{1}{|xe^x|}$

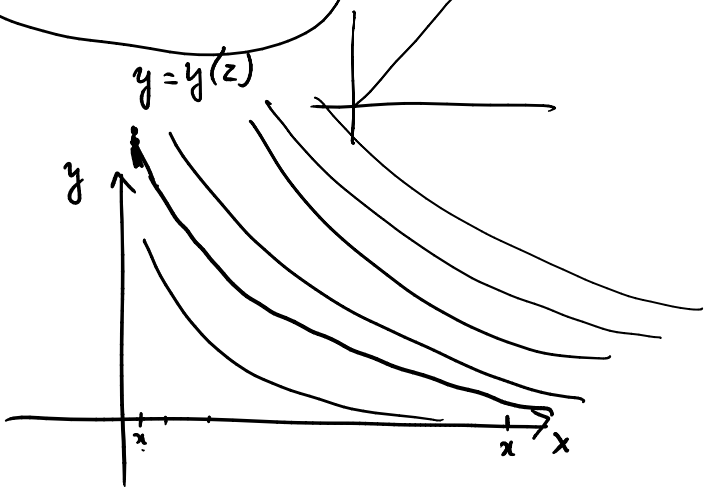
$ye^y = z$ $\quad y = y(z)$

$\dot{z} = k\dfrac{1}{|xe^x|}$

$g(-\infty)=0$

$g' = e^y(1+y)$

$g'>0 \iff y>-1$



Exercise Consider the system

$$\begin{cases} x' = y - x \\ y' = xy - x^2 \end{cases}$$

i) St sols

ii) First int

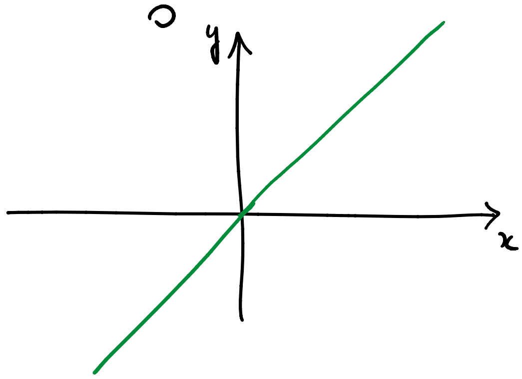
iii) Phase portrait; are there periodic sols? global sols?

iv) Solve $x(0)=1$, $y(0)=\frac{1}{2}$.

Sol: i) $(x,y)\equiv(a,b)$ is sol $\iff \begin{cases} 0 = b-a \\ 0 = ab - a^2 \end{cases}$

$\iff \begin{cases} 0 = b-a \\ 0 = a\underbrace{(b-a)}_{=0}=0 \end{cases} \iff$ $b-a=0$

(a,a), $a\in\mathbb{R}$.



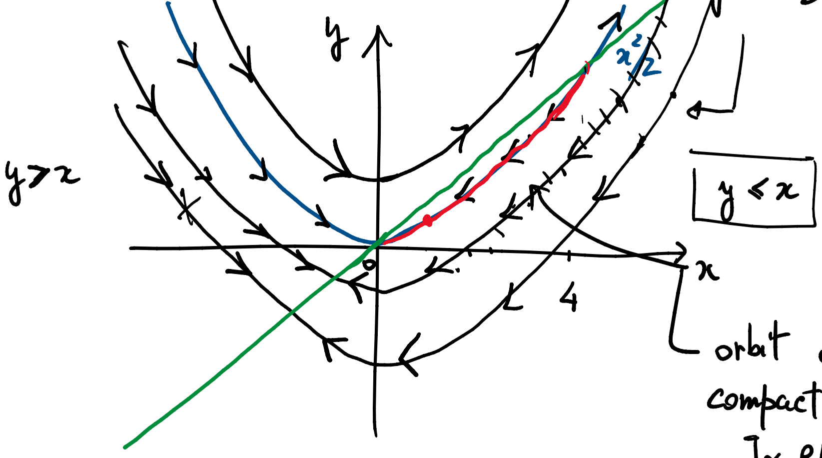
ii) The total eqn is $\dfrac{dy}{dx} = \dfrac{xy-x^2}{y-x} = \dfrac{x\cancel{(y-x)}}{\cancel{y-x}} = x$

$\iff dy = x\,dx$

$$\Leftrightarrow \quad dy = x\,dx$$

$$\Leftrightarrow \quad y = \frac{x^2}{2} + C \qquad E(x,y) = y - \frac{x^2}{2}$$

iii) Orbits are contained into $y = \frac{x^2}{2} + C$



$\frac{x^2}{2} = 4x$

orbit contained in compact set $\Rightarrow$ $]\alpha,\beta[=]-\infty,+\infty[$

<u>Orientation</u>: $x \nearrow \Leftrightarrow 0 \leq x' = y - x \Leftrightarrow y \geq x$

There're not periodic sols.

iv) We want to solve CP $x(0)=1$, $y(0)=1/2$

Notice that $E(x,y) = E(x(0),y(0)) = E(1, 1/2)$

$E = y - \frac{x^2}{2}$ $\qquad = \frac{1}{2} - \frac{1}{2}^2 = 0$

$$\Rightarrow \quad y = \frac{x^2}{2}$$

$$\Rightarrow \quad x' = y - x = \frac{x^2}{2} - x = \frac{1}{2}(x^2 - 2x)$$

$$\Leftrightarrow \quad \frac{x'}{x^2 - 2x} = 1/2$$

$$\Leftrightarrow \quad \frac{dx}{x^2-2x} = 1/2\, dt \quad \Longrightarrow \quad \int \frac{dx}{x^2-2x} = \frac{t}{2} + C \quad (*)$$

$$\int \frac{dx}{x^2-2x} = \int \frac{1}{x(x-2)}\,dx = \int \left(\frac{1}{x} - \frac{1}{x-2}\right)\left(-\frac{1}{2}\right) dx$$

$$= -\frac{1}{2}\left[\log|x| - \log|x-2|\right]$$

$$= -\frac{1}{2}\left[\log|x| - \log|x-2|\right]$$

$$= -\tfrac{1}{2}\log\left|\frac{x}{x-2}\right|.$$

Therefore $(*)$ becomes $-\frac{1}{2}\log\left|\frac{x}{x-2}\right| = \frac{t}{2} + C$

By imposing initial cond $(x(0)=1)$

$$C = -\tfrac{1}{2}\log\left|\frac{1}{1-2}\right| = 0$$

$$\Rightarrow \quad +\frac{1}{\cancel{2}}\log\left|\frac{x}{x-2}\right| = -t/\cancel{2}$$

$$\Rightarrow \quad \left|\frac{x}{x-2}\right| = e^{-t}$$

$$\Rightarrow \quad \frac{x}{x-2} = \pm e^{-t} \qquad \left(\text{at } t=0 \quad \overset{-1}{\overset{\|}{\frac{1}{1-2}}} = \pm 1 \Rightarrow \ominus\right)$$

$$\Rightarrow \quad \frac{x}{x-2} = -e^{-t}$$

$$\Rightarrow \quad x = -e^{-t}(x-2) \iff x(1+e^{-t}) = 2e^{-t}$$

$$\Rightarrow \quad x(t) = \frac{2e^{-t}}{1+e^{-t}} \qquad y(t) = x(t) - \frac{x(t)^2}{2}.$$

$\boxtimes$

Exercise

$$\begin{cases} x' = y(x^2+y^2) \\ y' = x(x^2+y^2) \end{cases}$$

i) St sols, ii) First int iii) Phase portrait, global sols periodic sols if any

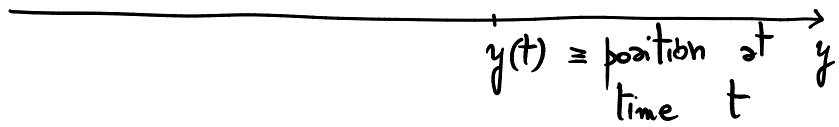
iv) Solve CP $x(0)=0$, $y(0)=1$

Conservative Systems

Newton's second law says

$$m\vec{a} = \vec{F}$$

This is normally a second order diff eqn.



$\Rightarrow \quad a = y''(t)$

In Physics $\quad F = F(t, y(t), y'(t))$. Let's focus on the important case when

$$F = F(y(t))$$

The eqn of motion becomes

$$\boxed{m y''(t) = F(y(t))}$$

We say that <u>F is conservative</u> if $F = \partial_y f$ where $f = f(y)$ is called <u>potential</u>. So

$$\boxed{\cancel{m} y'' = \partial_y f(y)}$$

We show how that there's a standard (and natural) way to transform a conservative system into a 2x2 syst. of diff eqns that can be studied with methods seen above.

So, how can we reduce a second order eqn to a first order one?

$$\boxed{y'' = \partial_y f(y)} \quad , \qquad y'' = (y')'$$

$$\boxed{y'' = \partial_y f(y)} \quad , \quad y'' = (y')' , \quad y' = v$$

$$\Updownarrow$$

$$\begin{cases} v = y' \\ v' = \partial_y f(y) \end{cases} \iff (S) \begin{cases} y' = v & = a(y,v) \\ v' = \partial_y f(y) & = b(y,v) \end{cases}$$

The couple (y,v) is the mechanical state of the system

position, velocity

System (S) has always a ~~first integral~~

Indeed, the tot. eqn is

$$\frac{dv}{dy} = \frac{\partial_y f(y)}{v} \qquad \text{sep vars eqn}$$

$$\iff v\,dv = \partial_y f(y)\,dy$$

$$\iff \int v\,dv = \int \partial_y f(y)\,dy + C$$

$$\frac{v^2}{2} = f(y) + C$$

$$\Rightarrow \boxed{E(y,v) = \frac{v^2}{2} - f(y)} \leftarrow \text{mechanical energy}$$

kinetic energy, potential energy

We can use methods developed for systems, the phase portrait in this case will be plotted in the plane (y,v)

plotted in the plane (y,v)

Example 8.5.3

Consider $\qquad y'' = y^2 - y = \partial_y f$

i) Find st sols

ii) Det a potential and energy for the equiv syst

iii) Plot orbits for the equiv syst

iv) Find sol of CP $\quad y(0)=3,\ y'(0)=3$.

i) $y \equiv C$ is sol of eqn $\Leftrightarrow 0 = C^2 - C = C(C-1)$

$\Leftrightarrow C = 0, 1$.

ii) Potential is $f = f(y)$: $\partial_y f = y^2 - y \Rightarrow f = \frac{y^3}{3} - \frac{y^2}{2}$

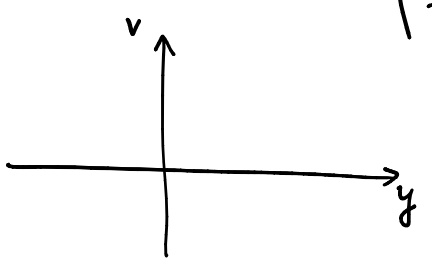
$$\Rightarrow E(y,v) = \frac{v^2}{2} - f(y) = \frac{v^2}{2} - \left(\frac{y^3}{3} - \frac{y^2}{2}\right)$$

$$\left(\text{equiv sys} \quad \begin{cases} y' = v \\ v' = y^2 - y \end{cases}\right)$$

iii) We have to plot lines

$$E(y,v) = C \Leftrightarrow \frac{v^2}{2} - \left(\frac{y^3}{3} - \frac{y^2}{2}\right) \equiv C$$

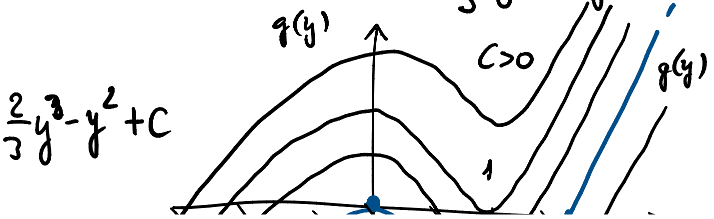
$$\Leftrightarrow v^2 - \left(\frac{2}{3}y^3 - y^2\right) \equiv C$$



$$\Updownarrow$$

$$v = \pm\sqrt{\frac{2}{3}y^3 - y^2 + C}$$

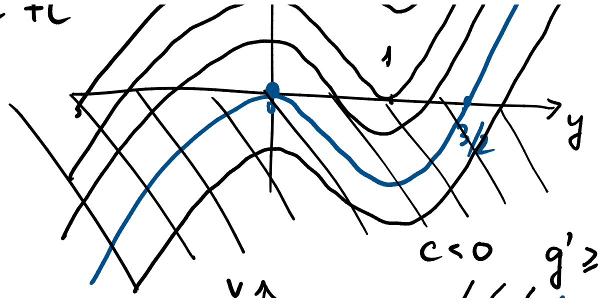
Plot of $\quad g(y) = \frac{2}{3}y^3 - y^2 = y^2\left(\frac{2}{3}y - 1\right)$



$g' = 2y^2 - 2y$

$= 2(y^2 - y)$

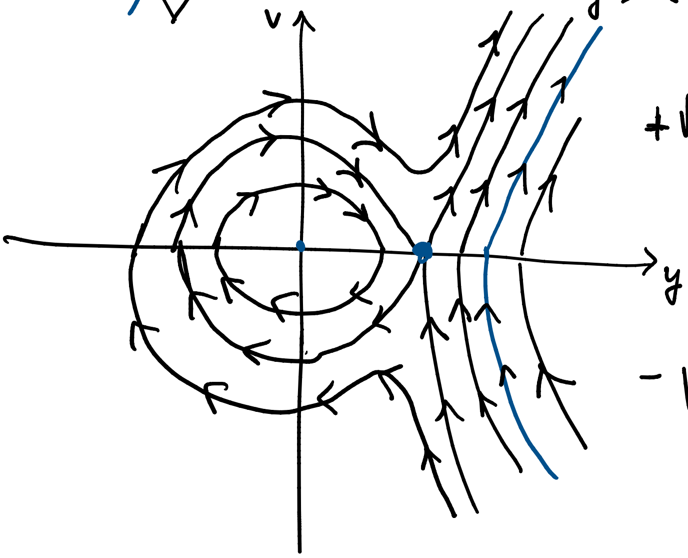
$\frac{\;}{3}y^3 - y^2 + c$



$= 2(y^2 - y)$

$= 2y(y-1)$

$c<0 \qquad g' \geq 0 \Leftrightarrow y \leq 0 \vee y \geq 1$

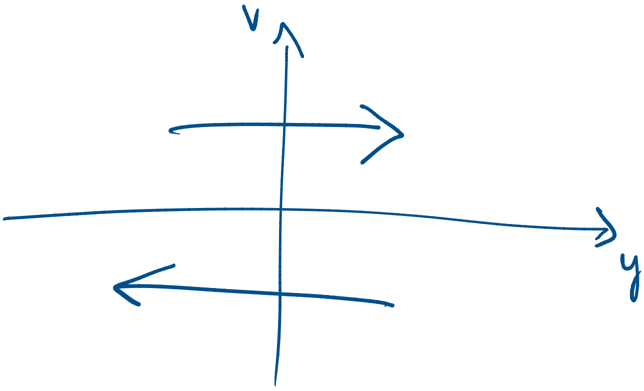


$+\sqrt{\frac{2}{3}y^3 - y^2 + c}$

$-\sqrt{\qquad}$

Orientation:

$$\begin{cases} \boxed{y' = v} \\ v' = \_\_\_\_\_\_ \end{cases} \qquad y \nearrow \Leftrightarrow 0 \leq y' \Leftrightarrow v \geq 0$$



iv) We have to solve CP

$$y(0) = 3, \quad \underset{\displaystyle v(0) = 3}{\underset{\shortparallel}{y'(0)}} = 3$$

Because of the conservation of E,

$$E(y,v) \equiv C = E(\underset{\underset{3}{\shortparallel}}{y(0)}, \underset{\underset{3}{\shortparallel}}{v(0)}) = E(3, 3)$$

$$E(y,v) := \frac{v^2}{2} - \left(\frac{y^3}{3} - \frac{y^2}{1}\right)$$

$$E(y,v) = \frac{v^2}{2} - \left(\frac{y^3}{3} - \frac{y^2}{2}\right) \quad 3 \quad \overset{\shortparallel}{3}$$

$$9 - 9/2 = 9/2$$

$$\Rightarrow E(\overset{y}{3}, \overset{v}{3}) = \frac{9}{2} - \underbrace{\left(\frac{27}{3} - \frac{9}{2}\right)}_{} = 9/2 - 9/2 = 0$$

$$\Rightarrow \quad \frac{v^2}{2} - \left(\frac{y^3}{3} - \frac{y^2}{2}\right) \equiv 0$$

$$\Rightarrow \quad \frac{v^2}{\cancel{2}} = 2\frac{y^3}{3} - \frac{y^2}{\cancel{2}} = \frac{2}{3}y^3 - y^2 = y^2\left(\frac{2}{3}y - 1\right)$$

$$\Rightarrow \quad v = \pm\sqrt{y^2\left(\frac{2}{3}y-1\right)} = \pm \underset{\pm y}{|y|}\sqrt{\frac{2}{3}y-1}$$

$$v = \pm y\sqrt{\frac{2}{3}y - 1}$$

To det if $\pm$ use initial cond:

$$3 = v(0) = \pm \underset{3}{y(0)}\sqrt{\frac{2}{3}\underset{3}{y(0)} - 1} = \pm 3 \Rightarrow +$$

$$\Rightarrow \quad v = y\sqrt{\frac{2}{3}y - 1}.$$

Because

$$\frac{dy}{dt} = y' = v = y\sqrt{\frac{2}{3}y - 1} \qquad \text{(sep vars eqn)}$$

$$\Leftrightarrow \quad \frac{dy}{y\sqrt{2/3\,y - 1}} = dt$$

$$\Leftrightarrow \int \frac{dy}{y\sqrt{\frac{2}{3}y-1}} = \int dt + C = t+C$$

$$\int \frac{dy}{y\sqrt{\frac{2}{3}y-1}} \underset{u=\sqrt{\frac{2}{3}y-1}}{=} \int \frac{3\cancel{u}\,du}{\frac{\cancel{3}}{2}(u^2+1)\cancel{u}} = 2\int \frac{1}{1+u^2}\,du$$

$$= 2\,\mathrm{arctg}\, u$$

$$u^2 = \frac{2}{3}y-1$$

$$y = \frac{3}{2}(u^2+1) \qquad dy = 3u\,du$$

$$= 2\,\mathrm{arctg}\sqrt{\frac{2}{3}y-1}$$

$$\Rightarrow \quad 2\,\mathrm{arctg}\sqrt{\frac{2}{3}y-1} = t + C$$

$$= 2\,\mathrm{arctg}\,1 = 2\cdot\frac{\pi}{4}$$

Taking $t=0 \Rightarrow C = 2\,\mathrm{arctg}\sqrt{\frac{2}{\cancel{3}}\cdot\cancel{3}-1} = \pi/2$

$$\Rightarrow \mathrm{arctg}\sqrt{} = \frac{t+C}{2}$$

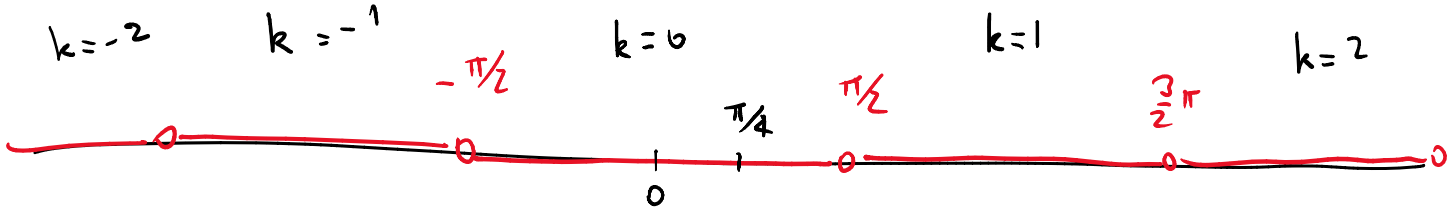
$$\Rightarrow \sqrt{\frac{2}{3}y-1} = \mathrm{tg}\,\frac{t+C}{2}$$

$$\Rightarrow \frac{2}{3}y - 1 = \left(\mathrm{tg}\,\frac{t+C}{2}\right)^2$$

$$\Rightarrow \quad y(t) = \frac{3}{2}\left[\left(tg\,\frac{t+\pi/2}{2}\right)^2 + 1\right] \qquad \blacksquare$$

$$-\frac{\pi}{2} + k\pi < \frac{t+\pi/2}{2} < \frac{\pi}{2} + k\pi$$

if we want $t=0$ be in this int $\Rightarrow$ $-\frac{\pi}{2}+k\pi < \frac{\pi}{4} < \frac{\pi}{2}+k\pi$



$$\Rightarrow \quad k=0 \quad \Rightarrow \quad -\cancel{\pi/2} < \frac{t+\pi/2}{\cancel{2}} < \pi/\cancel{2}$$

$$\Leftrightarrow \quad -\pi - \pi/2 < t < \pi - \pi/2$$

$$\Leftrightarrow \quad \underset{\alpha}{-\frac{3}{2}\pi} < t < \underset{\beta}{\pi/2}$$