venerdì 28 febbraio 2020 11:12

Do 8.7.3 (similar to last pb)

8.7.5 Consider

$$y'' = (\mathrm{Ch}\, y)(\mathrm{Sh}\, y),$$

i) st sols

ii) Show that if $y = y(t)$ is a sol $\Rightarrow$ $y(-t)$ and $-y(t)$ are sol of the eqn.

For which values of $a, b \in \mathbb{R}$ are the sols of CP $y(0) = a$, $y'(0) = b$ are even? for which a, b sols are odd?

iii) Det the energy/first int for the system.

Plot phase portrait

iv) Solve CP $y(0) = \log(1+\sqrt{2})$, $y'(0) = 1$

Sol: i) $y \equiv C$ is a sol $\Leftrightarrow$ $0 = (\mathrm{Ch}\, C)(\mathrm{Sh}\, C)$

$$\mathrm{Ch}\, C \geq 1$$

$\Leftrightarrow \mathrm{Sh}\, C = 0 \Leftrightarrow C = 0$

ii) Let $y = y(t)$ be a sol of diff eqn

$$y''(t) = (\mathrm{Ch}\, y(t))(\mathrm{Sh}\, y(t))$$

We prove $\varphi(t) = y(-t)$ is a sol for the eqn.

that is

$$\varphi'' = (\mathrm{Ch}\,\varphi)(\mathrm{Sh}\,\varphi)$$

We've

$$\varphi''(t) = \big(y(-t)\big)'' = \big[[y(-t)]'\big]'$$

$$= [-y'(-t)]' = y''(-t)$$

$$= \mathrm{Ch}\,\underbrace{y(-t)}_{\varphi(t)}\ \mathrm{Sh}\,\underbrace{y(-t)}_{\varphi(t)}$$

$$\varphi'' = \mathrm{Ch}\,\varphi\ \mathrm{Sh}\,\varphi$$

Let now $\psi(t) = -y(t)$. and let's check that

$$\psi'' = \mathrm{Ch}\,\psi\ \mathrm{Sh}\,\psi$$

$$\psi''(t) = [-y(t)]'' = -y''(t)$$

$$= -\ \mathrm{Ch}\,y(t)\ \mathrm{Sh}\,y(t)$$

$$\mathrm{Ch}\,\psi(t)\ \mathrm{Sh}\,\psi(t) = \mathrm{Ch}(-y(t))\ \mathrm{Sh}(-y(t))$$

$$\| $$

$$\mathrm{Ch}\,y(t)\,[-\mathrm{Sh}\,y(t)]$$

Next question is . for which a, b sols of

$$\begin{cases} y'' = \mathrm{Ch}\,y\ \mathrm{Sh}\,y \\ y(0) = a \\ y'(0) = b \end{cases}$$

are even

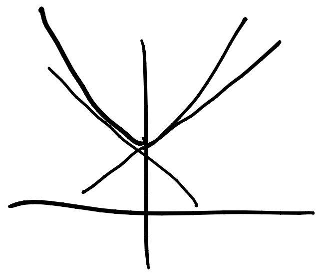
$$\begin{cases} y(0) = a & \text{are} \quad \text{even} \\ y'(0) = b \end{cases}$$

Sol y is even $\iff$ $y(t) \equiv y(-t)$

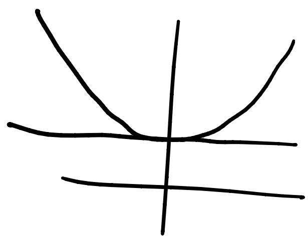
Notice that $\varphi(t) = y(-t)$ is a sol of the eqn

$$\begin{cases} \varphi'' = \operatorname{Ch}\varphi \operatorname{Sh}\varphi \\ \varphi(0) = y(-0) = a \\ \varphi'(0) = -y'(-0) = -b \end{cases}$$

$\varphi'(t) = -y'(-t)$



$\Rightarrow$ $y \equiv \varphi$ iff $b = 0$



The last question asks when sols of

$$\begin{cases} y'' = \operatorname{Ch} y \operatorname{Sh} y \\ y(0) = a \\ y'(0) = b \end{cases}$$

are odd.

y odd $\iff$ $y(-t) \equiv -y(t) \iff y(t) \equiv \underbrace{-y(-t)}_{\psi(t)}$

ψ $\qquad -y(-t) \qquad -y(-t)$

y sol $\Rightarrow$ $\boxed{y(-t)} = \hat{y}(t)$

$\Downarrow$

$\Rightarrow$ $-y(t)$ $\qquad -\hat{y}(t) = -y(-t)$

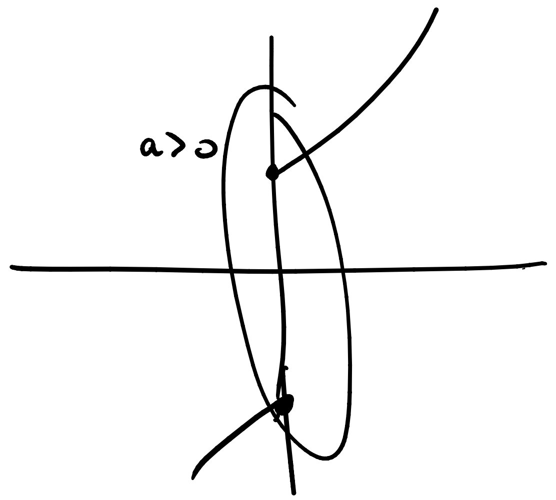
$\Rightarrow \psi(t) = -y(-t)$ solves $\qquad (\psi'(t) = [-y(-t)]'$

$\Rightarrow \psi(t) = -y(-t)$ solves $\qquad \left(\psi'(t) = [-y(-t)]' = -y'(-t)\cdot(-1) = y'(-t)\right)$

$$\begin{cases} \psi'' = \mathrm{Ch}\,\psi\,\mathrm{Sh}\,\psi \\ \psi(0) = -y(-0) = -y(0) = -a \\ \psi'(0) = y'(-0) = y'(0) = b \end{cases}$$

ψ and y are sols of the same CP $\iff a = -a$

$\iff a = 0$



$a>0$

iii) Let's first check that our eqn $y'' = \mathrm{Ch}\,y\,\mathrm{Sh}\,y$

$= \partial_y f(y)$

where $\quad f(y) = \int \underbrace{\mathrm{Ch}\,y\,\mathrm{Sh}\,y\,dy}_{(\mathrm{Ch}\,y)'} = \dfrac{(\mathrm{Ch}\,y)^2}{2}$

$$\Rightarrow E(y,v) = \frac{v^2}{2} - \frac{(\mathrm{Ch}\,y)^2}{2}$$

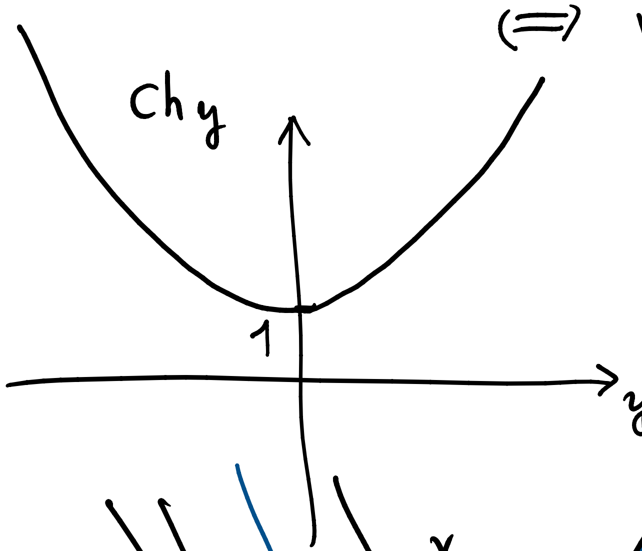
$v = y'$

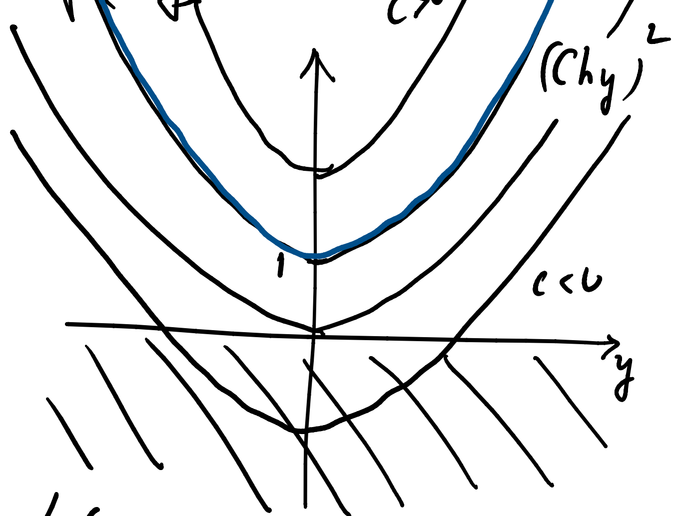
To plot the phase portrait of the system we've to plot lines we've to plot lines

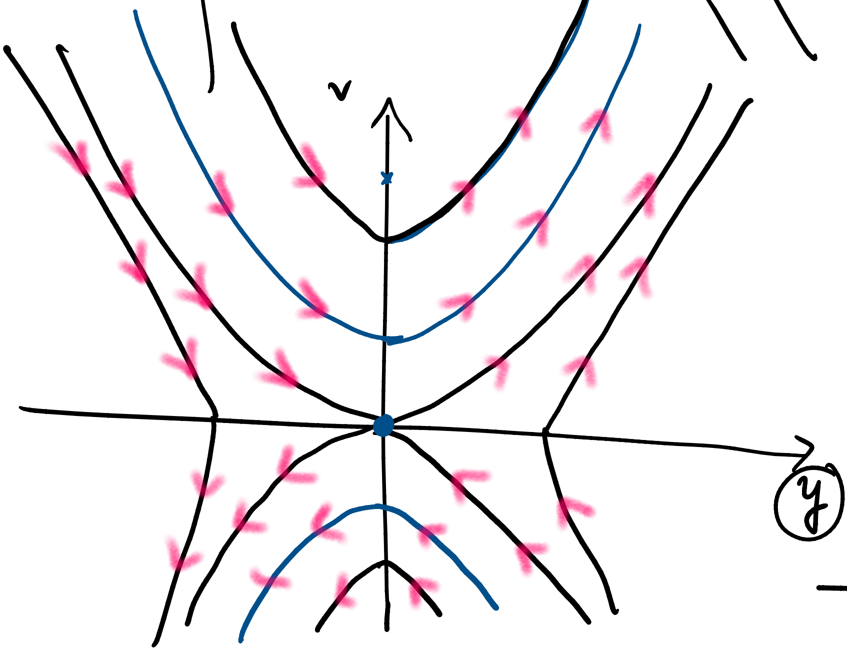
$$E(y,v) = C \iff \frac{v^2}{2} - \frac{(\mathrm{Ch}\, y)^2}{2} = C$$

$$\iff v^2 = (\mathrm{Ch}\, y)^2 + C$$

$$\iff v = \pm\sqrt{(\mathrm{Ch}\, y)^2 + C}$$







$\sqrt{(\mathrm{Ch}\, y)^2 + C}$

$-\sqrt{}$

$$\begin{cases} y' = v \\ v' = \partial_y f(y) \end{cases}$$

$$(y,v) \equiv (a,b) \iff \begin{cases} 0 = b \\ 0 = \partial_y f(a) \end{cases}$$

iv) Solution of CP $\begin{cases} y(0) = \log(1+\sqrt{2}) \\ y'(0) = 1 \end{cases}$

$$\left\{ \; y'(0) = 1 \right.$$

Because of cons. of energy

$$E(y,v) \equiv E\left(\log(1+\sqrt{2}), 1\right) = \frac{1}{2} - \frac{\left(\mathrm{Ch}\,\log(1+\sqrt{2})\right)^2}{2}$$

$$\mathrm{Ch}\,u = \frac{e^{u} + e^{-u}}{2} = \frac{e^{\log(1+\sqrt{2})} + e^{\log(1+\sqrt{2})^{-1}}}{2}$$

$$= \frac{1+\sqrt{2} + \frac{1}{1+\sqrt{2}}}{2} = \frac{1}{2}\,\frac{(1+\sqrt{2})^2 + 1}{1+\sqrt{2}}$$

$$= \frac{1 + 2\sqrt{2} + 2 + 1}{2(1+\sqrt{2})} = \frac{2+\sqrt{2}}{1+\sqrt{2}}$$

$$(\mathrm{Ch}\,u)^2 = \frac{4 + 4\sqrt{2} + 2}{1 + 2\sqrt{2} + 2} = \frac{6 + 4\sqrt{2}}{3 + 2\sqrt{2}}$$

$$= 2\,\frac{3+2\sqrt{2}}{3+2\sqrt{2}} = 2$$

$$\Rightarrow E(y,v) \equiv \frac{1}{2} - \frac{2}{2} \equiv -\frac{1}{2}$$

$$\Rightarrow \frac{v^2}{2} - \frac{(\mathrm{Ch}\,y)^2}{2} \equiv -\frac{1}{2}$$

$$\mathrm{Ch}^2 - \mathrm{Sh}^2 = 1$$

$$\mathrm{Ch}^2 - 1 = \mathrm{Sh}^2$$

$$\Rightarrow v^2 = -1 + (\mathrm{Ch}\,y)^2 = (\mathrm{Sh}\,y)^2$$

$$\Rightarrow \quad v = \pm \,\mathrm{Sh}\, y \qquad \Big(t = 0 \qquad \begin{array}{l} v(0) = y'(0) = 1 \\ y(0) = \log(1+\sqrt{2}) \end{array}$$

$$\Downarrow \; t = 0$$

$$1 = \pm \underbrace{\mathrm{Sh}}_{>0} \; \underbrace{\log(\overset{>1}{1+\sqrt{2}})}_{>0} \quad \Rightarrow \quad +$$

$$\Rightarrow \quad v = \mathrm{Sh}\, y$$

$$\Rightarrow \quad y' = v = \mathrm{Sh}\, y \quad \Rightarrow \quad \boxed{y' = \mathrm{Sh}\, y}$$

$$\Rightarrow \quad \frac{dy}{\mathrm{Sh}\, y} = dt \quad \Rightarrow \quad \boxed{\int \frac{dy}{\mathrm{Sh}\, y} = t + c}$$

$$\int \frac{1}{\mathrm{Sh}\, y}\, dy = \int \frac{1}{\frac{e^{y} - e^{-y}}{2}}\, dy \;\; e^{-y} = \tfrac{1}{e^{y}} \quad = 2 \int \frac{1}{\frac{e^{2y}-1}{e^{y}}}\, dy$$

$$= 2 \int \frac{e^{y}}{e^{2y}-1}\, dy \underset{u = e^{y} \quad y = \log u \quad dy = \frac{du}{u}}{=}$$

$$= 2 \int \frac{\cancel{u}\; 1}{u^2 - 1}\, \frac{du}{\cancel{u}}$$

$$u^2 - 1 = (u-1)(u+1) \qquad \frac{1}{u^2-1} = \left(\frac{1}{u-1} - \frac{1}{u+1} \right) \frac{1}{2}$$

$$= \int \frac{1}{u-1} - \frac{1}{u+1}\, du = \log|u-1| - \log|u+1|$$

$$= \log \left| \frac{e^{y}-1}{e^{y}+1} \right| \qquad \begin{array}{l} y(0) = \log(1+\sqrt{2}) \\ e^{y(0)} = 1+\sqrt{2} \end{array}$$

$$\Rightarrow \quad \log \left| \frac{e^{y}-1}{\ldots} \right| = t + C$$

$e^{y(0)} = 1+\sqrt{2}$

$$\Rightarrow \quad \log\left|\frac{e^y-1}{e^y+1}\right| = t + C$$

Setting $t=0 \Rightarrow \quad C = \log\left|\frac{e^{y(0)}-1}{e^{y(0)}+1}\right| = \log\left|\frac{\sqrt{2}}{2+\sqrt{2}}\right|$

$$\Rightarrow \quad \left|\frac{e^y-1}{e^y+1}\right| = e^t\,\frac{\sqrt{2}}{2+\sqrt{2}} = \frac{e^t}{1+\sqrt{2}}$$

$$\Leftrightarrow \quad \frac{e^y-1}{e^y+1} = \pm\,\frac{e^t}{1+\sqrt{2}}$$

$t=0$

$$\frac{1}{1+\sqrt{2}} = \pm\,\frac{1}{1+\sqrt{2}}$$

$\Downarrow$

$+$

$$\Rightarrow \quad \frac{e^{y}\,{}^{+1-1}-1}{e^y+1} = \frac{e^t}{1+\sqrt{2}}$$

$$\Rightarrow \quad 1 - \frac{1}{e^y+1} = \frac{e^t}{1+\sqrt{2}}$$

$$\Rightarrow \quad \frac{1}{e^y+1} = 1 - \frac{e^t}{1+\sqrt{2}}$$

$$\Rightarrow \quad e^y\cancel{+1} = \frac{1}{1-e^t/(1+\sqrt{2})} - 1$$

$$\Rightarrow \quad y(t) = \log\left(\frac{1}{1-e^t/(1+\sqrt{2})} - 1\right) \quad \blacksquare$$

Do 8.7.4, 8.7.6