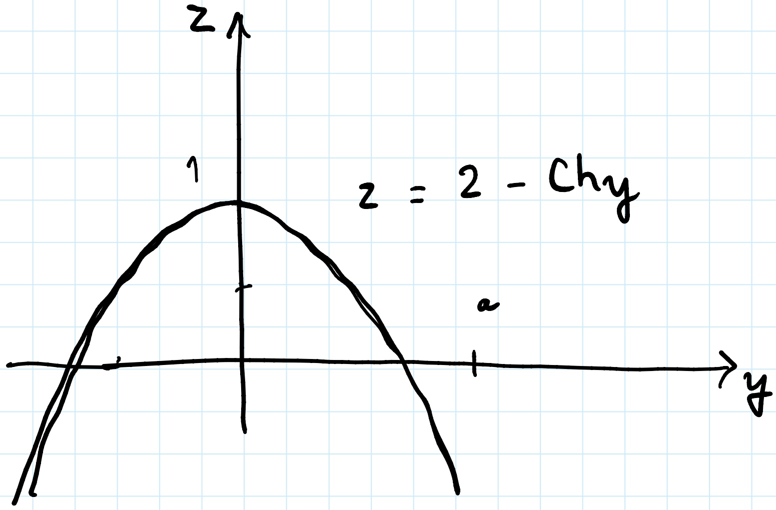
Ex 5.7.1 #4

domenica 14 giugno 2020 08:22

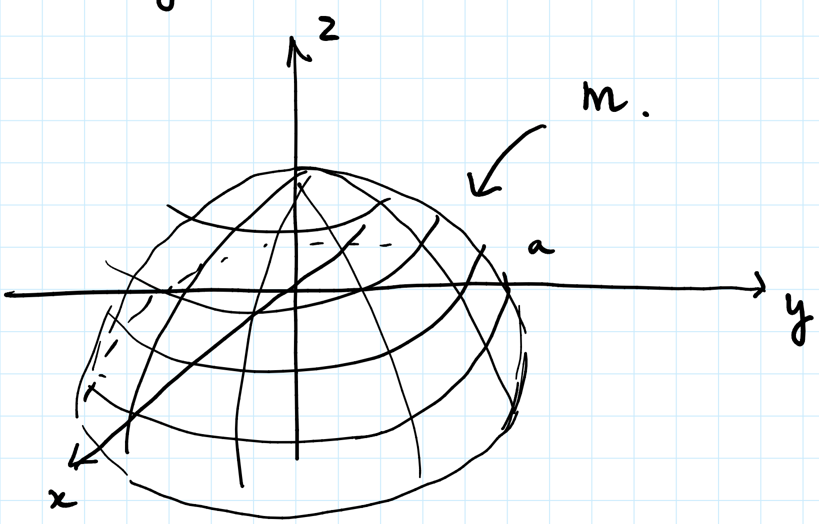
Rotation surface around z - axis $f(y) = 2 - Ch\,y$

$y \in [0, a]$

We start by plotting graph of f:

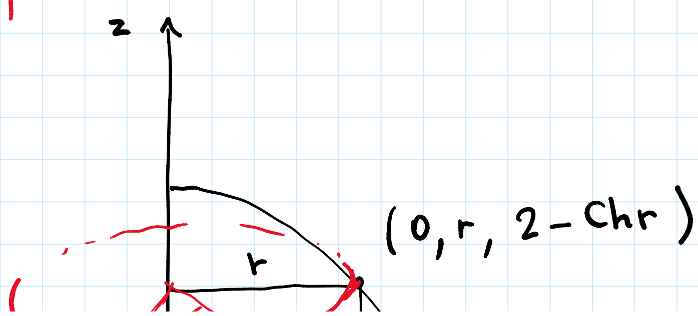


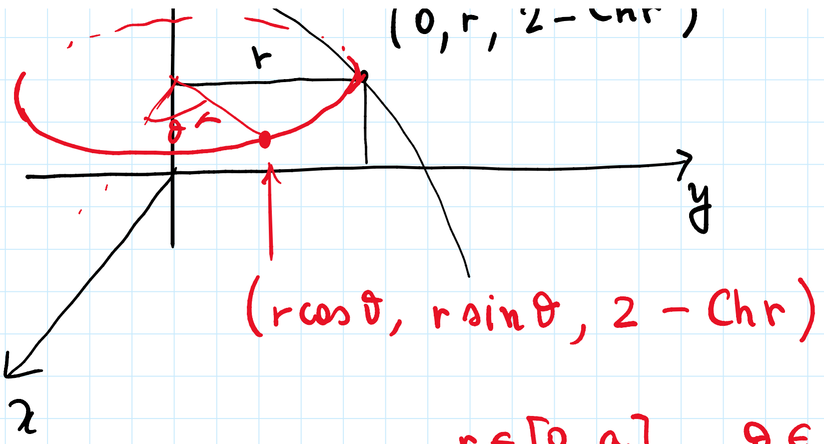
By rotating this around z - axis



Let's move to calculate Area (M).

First: parametrization of M.





$r \in [0, a]$, $\theta \in [0, 2\pi]$

$$\phi(r, \theta) = (r\cos\theta, \; r\sin\theta, \; 2 - \mathrm{Ch}\, r)$$

Second:

$$\mathrm{Area}\,(m) = \int_{[0,a]\times[0,2\pi]} \|\partial_r \phi \wedge \partial_\theta \phi\| \; dr\, d\theta$$

Let's compute the area element $\|\partial_r \phi \wedge \partial_\theta \phi\|$

We have

$$\partial_r \phi \wedge \partial_\theta \phi = \det \begin{bmatrix} i & j & k \\ \partial_r \phi_1 & \partial_r \phi_2 & \partial_r \phi_3 \\ \partial_\theta \phi_1 & \partial_\theta \phi_2 & \partial_\theta \phi_3 \end{bmatrix}$$

$$= \det \begin{bmatrix} i & j & k \\ \cos\theta & \sin\theta & -\mathrm{Sh}\, r \\ -r\sin\theta & r\cos\theta & 0 \end{bmatrix}$$

$$= \left(+ r\,\mathrm{Sh}\, r \cos\theta, \; + r\,\mathrm{Sh}\, r \sin\theta, \; r(\cos\theta)^2 + r(\sin\theta)^2 \right)$$

$$= r\left(\mathrm{Sh}\, r \cos\theta, \; \mathrm{Sh}\, r \sin\theta, \; 1 \right)$$

$$\Rightarrow \quad \|\partial_r\phi \wedge \partial_\theta \phi\| \underset{(r>0)}{=} r \,\|(\operatorname{Sh} r \cos\theta, \operatorname{Sh} r \sin\theta, 1)\|$$

$$= r\sqrt{(\operatorname{Sh} r)^2(\cos\theta)^2 + (\operatorname{Sh} r)^2(\sin\theta)^2 + 1^2}$$

$$= r\sqrt{1 + (\operatorname{Sh} r)^2}$$

$$\underset{(\operatorname{Ch}^2 - \operatorname{Sh}^2 = 1)}{=} r\sqrt{(\operatorname{Ch} r)^2} \underset{\operatorname{Ch}>0}{=} r \operatorname{Ch} r$$

Therefore

$$\operatorname{Area}(\mathcal{M}) = \int_{r\in[0,a]\ \theta\in[0,2\pi]} r \operatorname{Ch} r \; dr\, d\theta$$

$$\overset{RF}{=} \int_0^{2\pi} \left(\int_0^a r \operatorname{Ch} r \; dr \right) d\theta$$

$$= 2\pi \int_0^a r \underbrace{\operatorname{Ch} r}_{= (\operatorname{Sh} r)'} \; dr$$

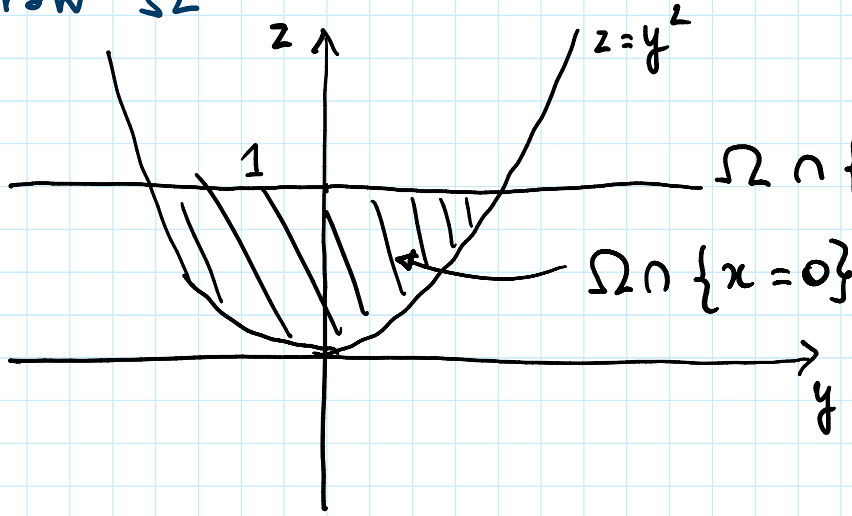
$$= 2\pi \left[r \operatorname{Sh} r \Big|_0^a - \int_0^a \underbrace{\operatorname{Sh} r}_{= (\operatorname{Ch} r)'} \, dr \right]$$

$$= 2\pi \left[a \operatorname{Sh} a - (\operatorname{Ch} a - \underbrace{\operatorname{Ch} 0}_{=1}) \right]$$

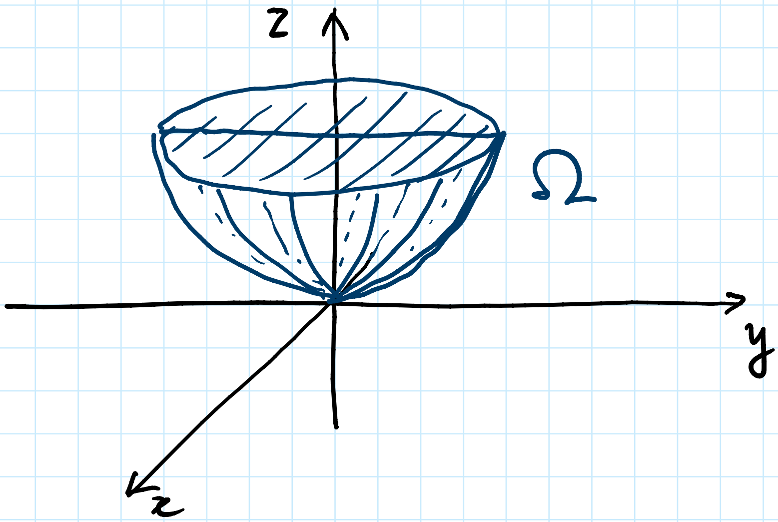
$\boxtimes$

$$\Omega = \{ (x,y,z) \in \mathbb{R}^3 : x^2+y^2 \leq z \leq 1 \}$$

1. Draw Ω



Because Ω is rotation invariant (around z-axis)



2. Vol(Ω)

We have

$$\mathrm{Vol}(\Omega) = \int_\Omega 1 \, dx\,dy\,dz$$

$$= \int_{x^2+y^2 \leq z \leq 1} dx\,dy\,dz$$

(cyl coords: $\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z \end{cases}$ $\quad \rho \geq 0,\ \theta\in[0,2\pi],\ z\in\mathbb{R}$

$$= \int_{\substack{\rho^2 \leq z \leq 1 \\ \rho\geq 0,\ \theta\in[0,2\pi],\ z\in\mathbb{R}}} \rho \, d\rho \, d\theta \, dz$$

$$\stackrel{RF}{=} \int_0^{2\pi} \left(\int_{\rho^2\leq z\leq 1} \rho \, d\rho dz \right) d\theta$$

$$= 2\pi \int_{\underbrace{\rho^2 \leq z \leq 1}} \rho \, d\rho \, dz$$

$$\rho^2 \leq 1 \underset{\rho\geq 0}{\iff} 0 \leq \rho \leq 1 \qquad \rho^2 \leq z \leq 1$$

$$\stackrel{RF}{=} 2\pi \int_0^1 \left(\int_{\rho^2}^1 \rho \, dz \right) d\rho$$

$$= 2\pi \int_0^1 \rho \underbrace{\int_{\rho^2}^1 dz}_{=1-\rho^2} d\rho$$

$$= 2\pi \int_0^1 \rho - \rho^3 \, d\rho = 2\pi \left[\frac{\rho^2}{2}\Big|_0^1 - \frac{\rho^4}{4}\Big|_0^1 \right]$$

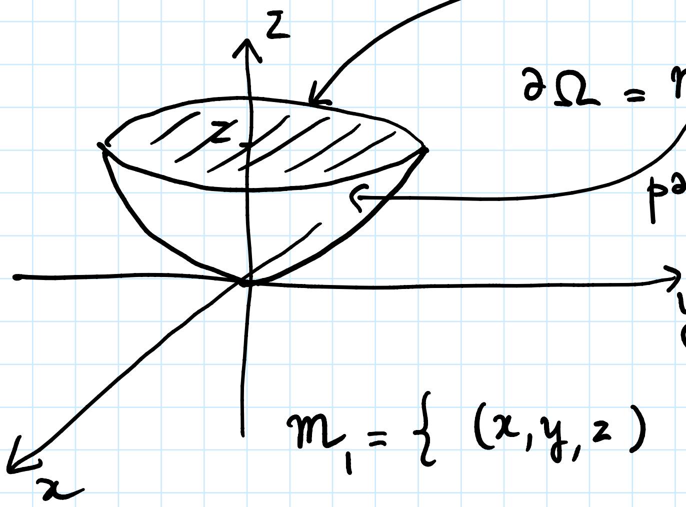
$$= 2\pi\left[\frac{1}{2} - \frac{1}{4}\right] = \frac{\pi}{2}.$$

3. Area of $\partial\Omega$

disk

$\partial\Omega = m_1 + m_2$

paraboloid



$$m_1 = \{(x,y,z) : z = 1,\ x^2+y^2 \leq 1\}$$

$$m_2 = \{(x,y,z) : z = x^2+y^2,\ x^2+y^2 \leq 1\}$$

In both cases, m_1 and m_2 are graphs of functs.

m_1 = graph f $\quad f(x,y) \equiv 1$ on $(x,y) \in \{x^2+y^2 \leq 1\}$

m_2 = graph g $\quad g(x,y) = x^2+y^2$ on $(x,y) \in \{x^2+y^2 \leq 1\} = D$

$$\Rightarrow \text{Area}(\partial\Omega) = \text{Area}(m_1) + \text{Area}(m_2)$$

$$= \int_D \sqrt{1 + \|\nabla f\|^2}\, dx\,dy + \int_D \sqrt{1 + \|\nabla g\|^2}\, dx\,dy$$

$\nabla f \equiv 0 \quad \Rightarrow \quad \int_D \sqrt{1 + \|\nabla f\|^2}\, dx\,dy = \int_D \sqrt{1}\, dx\,dy =$

$$\text{Area}(D) = \pi 1^2 = \pi$$

$$\nabla g = (2x, 2y) \quad \Rightarrow \quad \|\nabla g\|^2 = 4(x^2+y^2).$$

$$\Rightarrow \int_D \sqrt{1+\|\nabla g\|^2}\, dx\, dy = \int_D \sqrt{1+4(x^2+y^2)}\, dx\, dy$$

$$\underset{\text{pol coord}}{=} \int_{\rho^2 \leq 1,\ \theta \in [0,2\pi]} \sqrt{1+4\rho^2}\, \rho\, d\rho\, d\theta$$

$$\overset{\text{RF}}{=} \int_0^{2\pi} \int_0^1 (1+4\rho^2)^{1/2} \rho\, d\rho\, d\theta$$

$$= 2\pi \int_0^1 (1+4\rho^2)^{1/2} \rho\, d\rho$$

$$\left[(1+4\rho^2)^{3/2}\right]' = \frac{3}{2}(1+4\rho^2)^{1/2} 8\rho$$

$$= 12(1+4\rho^2)^{1/2}\rho$$

$$= 2\pi \int_0^1 \left[\frac{1}{12}(1+4\rho^2)^{3/2}\right]' d\rho$$

$$= 2\pi \frac{1}{12} (1+4\rho^2)^{3/2} \Big|_0^1$$

$$= \frac{\pi}{6}\left[5^{3/2} - 1\right].$$

4. Outward flux of $\vec{F} = (x, y, z)$ plus components

We use divergence thm.

$$\int_{\partial\Omega} \vec{F}\cdot\vec{n}_e = \int_{\Omega} \operatorname{div}\vec{F}$$

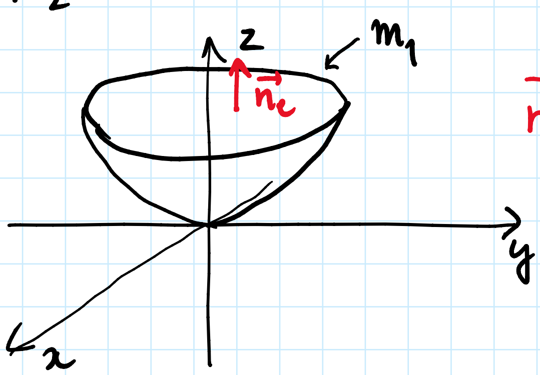
where
$$\operatorname{div}\vec{F} = \partial_x f_1 + \partial_y f_2 + \partial_z f_3$$
$$= \partial_x x + \partial_y y + \partial_z z$$
$$= 3.$$

$$\Rightarrow \int_{\partial\Omega} \vec{F}\cdot\vec{n}_e = \int_{\Omega} 3\, dx\,dy\,dz = 3 \operatorname{Vol}\Omega$$
$$= 3\cdot\frac{\pi}{2}.$$

Components. Because

$$\int_{\partial\Omega} \vec{F}\cdot\vec{n}_e = \int_{m_1} \vec{F}\cdot\vec{n}_e + \int_{m_2} \vec{F}\cdot\vec{n}_e$$

we have to determine the two terms at r.h.s. We compute $\int_{m_1}$ directly then we deduce $\int_{m_2}$ by difference. On m_1 clearly



$$\vec{n}_e = (0,0,1)$$

$$\Rightarrow \int_{M_1} \vec{F} \cdot \vec{n}_e = \int_{M_1} (x,y,z) \cdot (0,0,1) = \int_{M_1} \overset{\overset{1}{\shortparallel}}{z}$$

$$M_1 = \{ (x,y,z) : x^2 + y^2 \leq 1 \,, \ z = 1 \} .$$

$$= \int_{M_1} 1 = \text{Area}(M_1) = \pi .$$

$$\Rightarrow \int_{M_2} \vec{F} \cdot \vec{n}_e = \frac{3}{2}\pi - \pi = \frac{\pi}{2} \quad \blacksquare$$

Ex 5.7.12

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$$\Omega = \{ (x,y,z) \in [0,+\infty[^3 : \quad 1 \leq x^2+y^2 \leq 4-z^2 \}$$

1. Outward flux of $\vec{F} = (x,y,z^2)$

We start drawing Ω (not required and not necessary). Because conditions on (x,y) depend on x^2+y^2, and because this quantity is invariant by rotations around z-axis, Ω is invariant by rots around z-axis. We can plot a sect of Ω on the yz plane (that is $\Omega \cap \{x=0\}$) then we rotate this figure around z-axis.

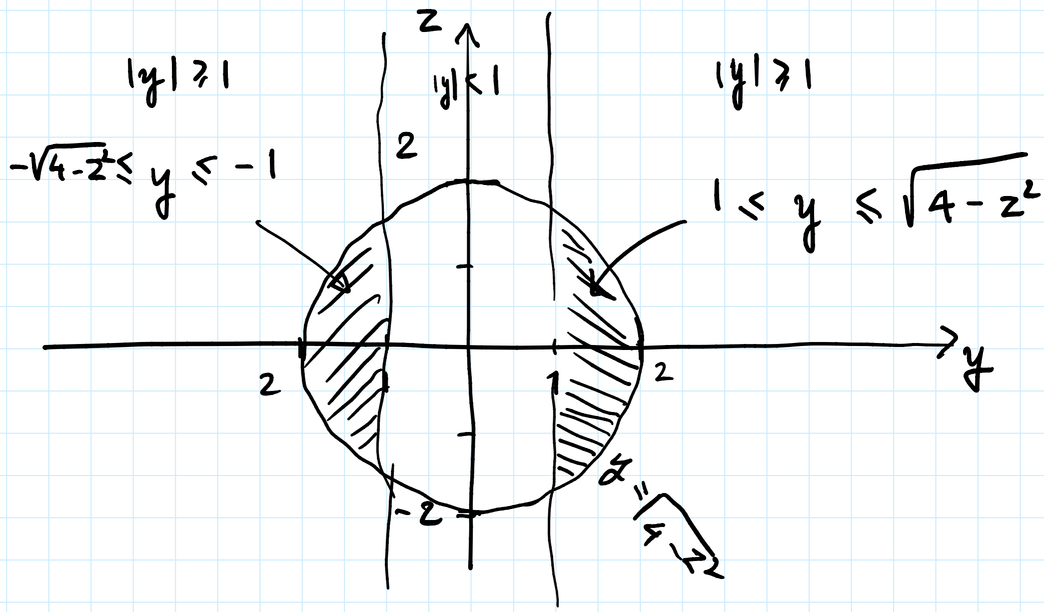
$$\Omega \cap \{x=0\} = \{ \underbrace{1 \leq y^2}_{|y| \geq 1} \underbrace{\leq 4-z^2}_{1 \leq |y| \leq \sqrt{4-z^2}} \}$$

$4-z^2$ must be ≥ 0

$\Rightarrow 4-z^2 \geq 0$

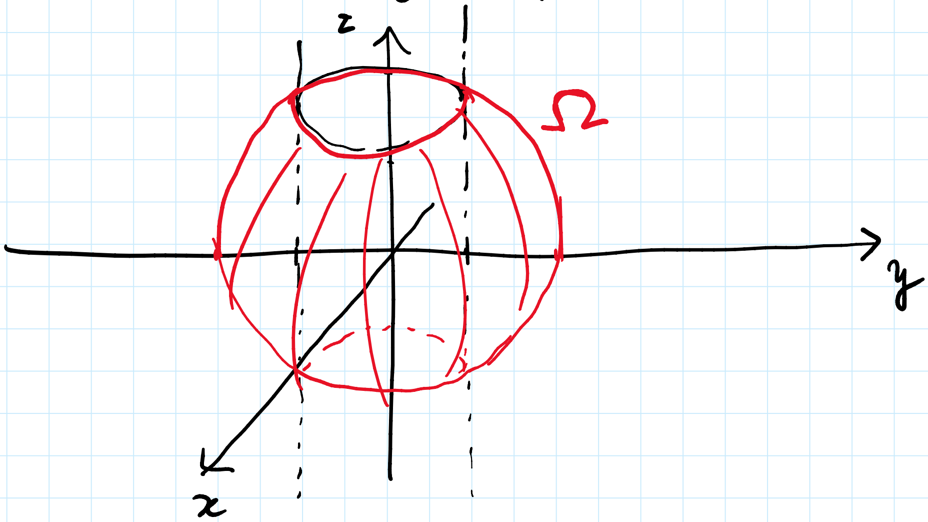
$\Leftrightarrow z^2 \leq 4$

$\Leftrightarrow |z| \leq 2.$



In black $\Omega \cap \{x=0\}$

In black $\Omega \cap \{x=0\}$



According to divergence thm

$$\int_{\partial\Omega} \vec{F}\cdot\vec{n_e} = \int_{\Omega} \operatorname{div}\vec{F}$$

where $\operatorname{div}\vec{F} = \partial_x x + \partial_y y + \partial_z z^2$

$$= 1+1+2z = 2(1+z)$$

$$\Rightarrow \int_{\partial\Omega} \vec{F}\cdot\vec{n_e} = \int_{\Omega} 2+2z$$

$$= 2\int_{1\leq x^2+y^2\leq 4-z^2} (1+z)\, dx\,dy\,dz$$

$$\underset{\text{cyl coords}}{=} 2\int_{\substack{1\leq\rho^2\leq 4-z^2 \\ \rho\geq 0,\ \theta\in[0,2\pi],\ z\in\mathbb{R}}} (1+z)\,\rho\, d\rho\, d\theta\, dz$$

$$\overset{\text{RF}}{=} 2.\int^{2\pi} \Big(\int (1+z)\rho\, d\rho\, dz \Big) d\theta$$

$$\overset{RF}{=} 2. \int_0^{2\pi} \left(\int_{\substack{1 \leq \rho^2 \leq 4 - z^2 \\ \rho \geq 0,\ z \in \mathbb{R}}} (1+z)\rho \, d\rho \, dz \right) d\theta$$

$$= 4\pi \int_{\left[\substack{1 \leq \rho^2 \leq 4 - z^2 \\ \rho \geq 0,\ z \in \mathbb{R}}\right.} (1+z)\rho \, d\rho \, dz.$$

$$|z| \leq 2, \quad 1 \leq \rho \leq \sqrt{4 - z^2}$$

but <u>warning</u>: don't miss

$$1 \leq 4 - z^2 \iff z^2 \leq 3$$

$$\iff |z| \leq \sqrt{3}$$

this bypass

$$\overset{RF}{=} 4\pi \int_{-\sqrt{3}}^{\sqrt{3}} \left(\int_1^{\sqrt{4-z^2}} (1+z)\rho \, d\rho \right) dz$$

$$= 4\pi \int_{-\sqrt{3}}^{\sqrt{3}} (1+z) \underbrace{\int_1^{\sqrt{4-z^2}} \rho \, d\rho}_{\left.\frac{\rho^2}{2}\right|_1^{\sqrt{4-z^2}} = \frac{1}{2}\left[4 - z^2 - 1\right]} dz$$

$$= 4\pi \int_{-\sqrt{3}}^{\sqrt{3}} (1+z) \frac{1}{2} (3 - z^2) \, dz$$

$$= 2\pi \int_{-\sqrt{3}}^{\sqrt{3}} 3 + 3z - z^2 - z^3 \, dz$$

$$\int_{-\sqrt{3}}^{\sqrt{3}} z = 0 \qquad \int_{-\sqrt{3}}^{\sqrt{3}} z^3 = 0$$

$$= 2\pi \left[3 \cdot 2\sqrt{3} - \frac{z^3}{3} \Big|_{-\sqrt{3}}^{\sqrt{3}} \right]$$

$$= 2\pi \left[6\sqrt{3} - \frac{1}{\cancel{3}} 2 \cdot \cancel{3}\sqrt{3} \right] = 8\pi\sqrt{3}.$$

2. Component of $\int_{\partial\Omega} \vec{F} \cdot \vec{n}_e$ on $\partial\Omega \cap \{x^2+y^2+z^2=4\}$

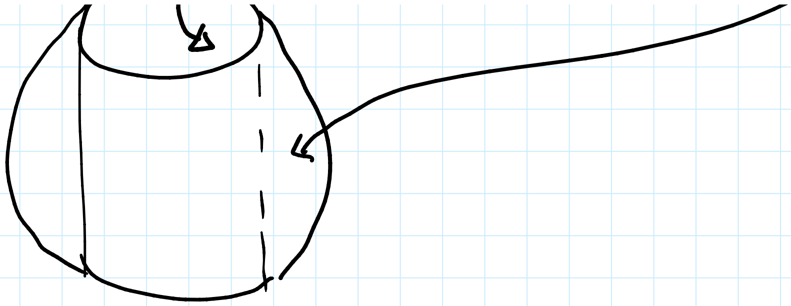
We may proceed in two alternative ways:

#1 , Compute $\displaystyle\int_{\partial\Omega \cap \{x^2+y^2+z^2=4\}} \vec{F} \cdot \vec{n}_e$ directly

#2. Compute $\displaystyle\int_{\partial\Omega \cap \{x^2+y^2=1\}} \vec{F} \cdot \vec{n}_e$ and deduce

indirectly what we wish being

$$\int_{\partial\Omega} \vec{F} \cdot n_e = \int_{\partial\Omega \cap \{x^2+y^2=1\}} + \int_{\partial\Omega \cap \{x^2+y^2+z^2=4\}}$$



Which way should we choose? In both cases we've to compute a flux by using the direct formula (hence we need a parametrization). We have to be careful when parametrizing because a param. might not induce an outward normal.

#1. In this case, because $\partial\Omega \cap \{x^2+y^2+z^2=4\}$ belongs to a sphere of radius 2, the natural standard param is

$$\phi(\theta,\varphi) = (2\cos\theta\sin\varphi,\ 2\sin\theta\sin\varphi,\ 2\cos\varphi)$$

We have to det ranges for θ, φ. Since

$$\Omega = \{1 \leq x^2+y^2 \leq 4-z^2\}$$

$$\partial\Omega \cap \{x^2+y^2+z^2=4\}$$

$$= \{x^2+y^2 \geq 1,\quad x^2+y^2+z^2=4\}$$

$$\Updownarrow$$

$$4(\cos\theta)^2(\sin\varphi)^2 + 4(\sin\theta)^2(\sin\varphi)^2 \geq 1$$

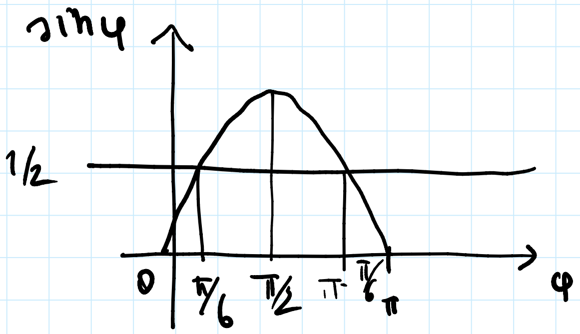
$$\Updownarrow$$

$$\Downarrow$$

$$(\sin \varphi)^2 \geq 1/4 \iff \sin\varphi \geq 1/2$$

$$\underset{\varphi\in[0,\pi]}{\Updownarrow}$$

$$\iff \pi/6 \leq \varphi \leq \pi - \pi/6 = \frac{5}{6}\pi$$



Thus:

$$\partial\Omega \cap \{x^2+y^2+z^2=4\} = \Phi(D)$$

where

$$D = \left\{ (\theta,\varphi) : \theta\in[0,2\pi],\ \varphi\in\left[\pi/6, \tfrac{5}{6}\pi\right] \right\}$$

$$\Rightarrow \int_{\partial\Omega\cap\{x^2+y^2+z^2=4\}} \vec{F}\cdot\vec{n}_\Phi =$$

$$= \int_D \det \begin{bmatrix} \vec{F} \\ \partial_\theta \Phi \\ \partial_\varphi \Phi \end{bmatrix} (\Phi(\theta,\varphi))\, d\theta\, d\varphi$$

$$= \int_{\substack{\theta\in[0,2\pi] \\ \varphi\in[\pi/6,\,5/6\,\pi]}} \det \begin{array}{c} \begin{array}{ccc} x & y & z^2 \end{array} \\ \begin{bmatrix} 2Cs & 2Ss & (2c)^2 \\ -2Ss & 2Cs & 0 \\ 2Cc & -2Sc & -2s \end{bmatrix} \end{array}$$

$C = \cos\theta \qquad S = \sin\theta, \qquad c = \cos\varphi \qquad s = \sin\varphi$

$$= \int_{\substack{\theta \in [0,2\pi] \\ \varphi \in [\frac{\pi}{6}, \frac{5}{6}\pi]}} 4c^2 \det \begin{bmatrix} 2Cs & 2Ss \\ 2Cc & -2Sc \end{bmatrix} - 2s \det \begin{bmatrix} 2Cs & 2Ss \\ -2Ss & 2Cs \end{bmatrix}$$

$$= \int_{\substack{\theta \in [0,2\pi] \\ \varphi \in [\frac{\pi}{6}, \frac{5}{6}\pi]}} 4c^2 4\left(-Cs\,Sc - Ss\,Cc\right) - 2s \cdot 4 \underbrace{\left(C^2 s^2 + S^2 s^2\right)}_{= s^2}$$

$$= \int_{\substack{\theta \in [0,2\pi] \\ \varphi \in [\frac{\pi}{6}, \frac{5}{6}\pi]}} -32\, Cs\, sc^3 - 8s^3$$

$$\overset{RF}{=} \int_0^{2\pi} \left(\int_{\pi/6}^{5/6\pi} -32 \cos\theta \sin\theta \sin\varphi (\cos\varphi)^3 - 8 (\sin\varphi)^3 d\varphi \right) d\theta$$

$$= -16 \underbrace{\int_0^{2\pi} \sin(2\theta)\, d\theta}_{=0} \int_{\pi/6}^{5/6\pi} \sin\varphi (\cos\varphi)^3 d\varphi$$

$$- 8 \cdot 2\pi \int_{\pi/6}^{5/6\pi} (\sin\varphi)^3 d\varphi$$

$$= -16\pi \int_{\pi/6}^{5/6\pi} (\sin\varphi)^3 \, d\varphi.$$

$$(\sin\varphi)^3 = \sin\varphi \, (\sin\varphi)^2$$

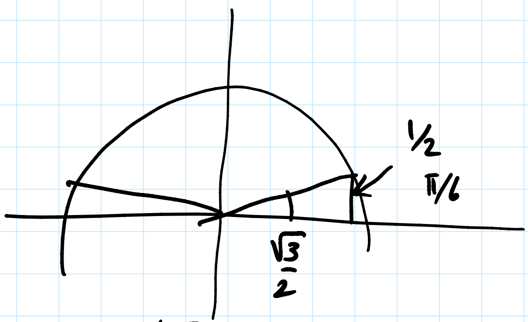
$$(\sin\varphi)^2 = 1 - (\cos\varphi)^2$$

$$\shortparallel$$
$$1-(\cos\varphi)^2$$

$$= -16\pi\left[\int_{\pi/6}^{5/6\pi} \underbrace{\sin\varphi}_{\shortparallel\ (-\cos\varphi)'}\, d\varphi + \int_{\pi/6}^{5/6\pi} \underbrace{-\sin\varphi\,(\cos\varphi)^2}_{\left(\frac{1}{3}(\cos\varphi)^3\right)'}\, d\varphi\right]$$

$$= -16\pi\left[-\cos\varphi\Big|_{\pi/6}^{5/6\pi} + \frac{(\cos\varphi)^3}{3}\Big|_{\pi/6}^{5/6\pi}\right]$$

$$= -16\pi\left[-\left(-\frac{\sqrt3}{2}-\frac{\sqrt3}{2}\right) + \frac{1}{3}\left[\left(-\frac{\sqrt3}{2}\right)^3 - \left(\frac{\sqrt3}{2}\right)^3\right]\right]$$



$$= -16\pi\left[\sqrt3 - \frac{1}{\cancel{3}}\cancel{2}\cdot\frac{\cancel{3}\sqrt3}{\cancel{8}4}\right] = -16\pi\,\frac{3}{4}\sqrt3$$
$$= -12\pi\sqrt3 .$$

Thus $\displaystyle\int_{\partial\Omega\cap\{x^2+y^2+z^2=4\}} \vec F\cdot\vec n_\phi = -12\pi\sqrt3$

this coincides with what we wish $\Leftrightarrow \vec n_\phi = \vec n_e$ (otherwise we've just to change sign).
Now, since we choose the standard param of sphere it is easy to check that

of sphere it is easy to check that

$$\vec{n}_{\phi} = \vec{n}_{e}.$$

$\blacksquare$

$$\Omega = \{(x,y,z)\in\mathbb{R}^3 : z^2+6 \le x^2+y^2 \le 5z\}$$

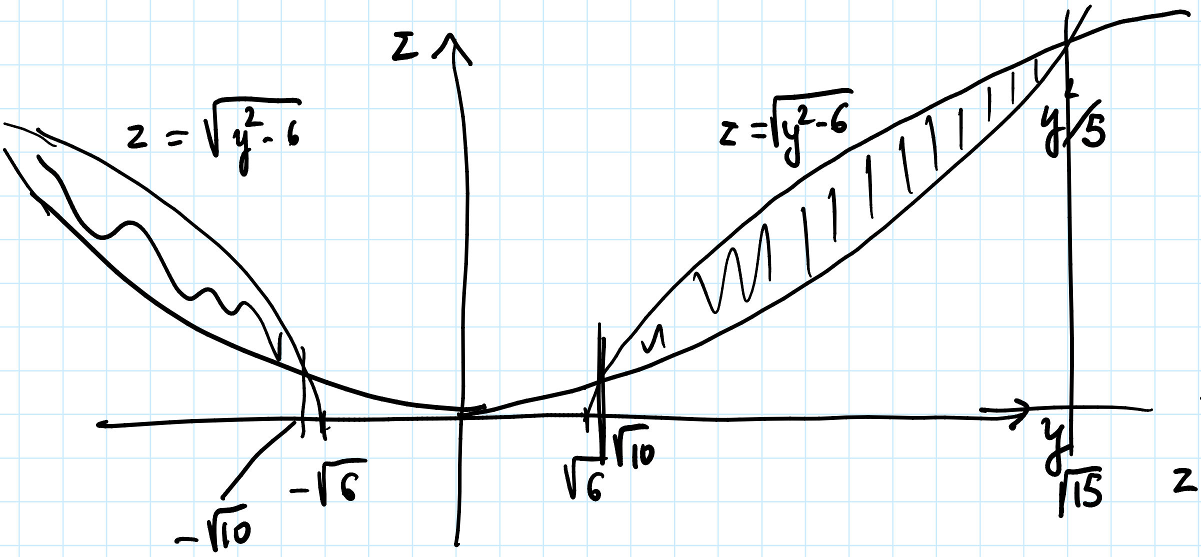
1. Is Ω a rotation solid? Draw Ω

Yes it is! This because Ω depends on (x,y) through x^2+y^2. To draw Ω we start drawing a section of Ω on yz plane, that is

$$\Omega\cap\{x=0\} = \{(y,z) : z^2+6 \le \underbrace{y^2 \le 5z}\}$$

$$\downarrow$$

$$z \geq \frac{y^2}{5}$$



$$z^2 \le \underbrace{y^2-6}_{\oplus}$$

$$\Updownarrow$$

$$y^2 \geq 6$$

$$\Updownarrow$$

$$|y| \geq \sqrt{6}$$

$$\underbrace{\frac{y^2}{5} \le z \le \sqrt{y^2-6}}_{\le}$$

$$\Updownarrow$$

$$\frac{y^2}{5} \le \sqrt{y^2-6} \iff y^2 \le 5\sqrt{y^2-6}$$

$$\iff y^4 \le 25(y^2-6)$$

$$\Leftrightarrow \quad y^4 \leqslant 25\,(y^2-6)$$

$$\Leftrightarrow \quad y^4 - 25y^2 + 150 \leqslant 0$$

$$\frac{25-\sqrt{625-600}}{2} \leqslant y^2 \leqslant \frac{25+\sqrt{625-600}}{2}$$

$$\Updownarrow$$

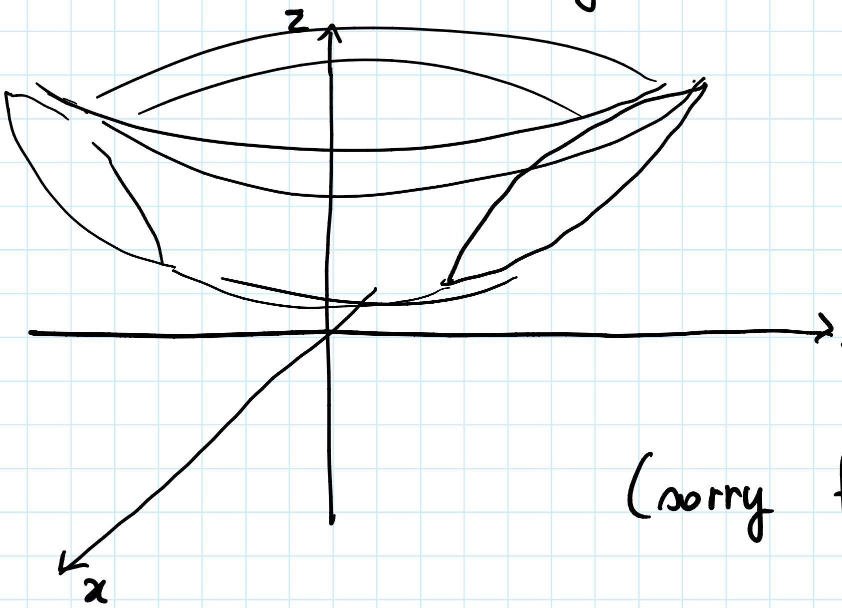
$$\frac{25-5}{2} \leqslant y^2 \leqslant \frac{25+5}{2}$$

$$\Updownarrow$$

$$10 \leqslant y^2 \leqslant 15$$

$$\Updownarrow$$

$$\sqrt{10} \leqslant |y| \leqslant \sqrt{15}$$



(sorry for bad figure.)

2. Vol Ω

$$\mathrm{Vol}\,\Omega = \int_\Omega 1 \; dx\, dy\, dz$$

$$\underset{\text{(cyl coords)}}{=} \int_{\substack{z^2+6 \leq \rho^2 \leq 5z \\ \rho \geq 0,\ \theta \in [0,2\pi],\ z \in \mathbb{R}}} \rho \, d\rho \, d\theta \, dz$$

$$\overset{\text{RF}}{=} \int_0^{2\pi} \left(\int_{\substack{z^2+6 \leq \rho^2 \leq 5z \\ \rho \geq 0,\ z \in \mathbb{R}}} \rho \, d\rho \, dz \right) d\theta$$

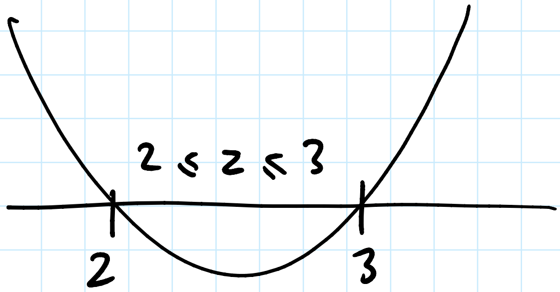
$$= 2\pi \int_{\substack{\boxed{z^2+6 \leq \rho^2 \leq 5z} \\ \rho \geq 0,\ z \in \mathbb{R}}} \rho \, d\rho \, dz$$

by $\rho^2 \leq 5z \Longrightarrow \boxed{z \geq 0}$

by $z^2+6 \leq \cancel{\rho^2} \leq 5z$

$$\Downarrow$$

$$z^2+6 \leq 5z \iff z^2 - 5z + 6 \leq 0$$



$$z_{1,2} = \frac{5 \pm \sqrt{25-24}}{2} = \frac{5 \pm 1}{2} = 2, 3$$

$$\Rightarrow \text{Vol}\,\Omega = 2\pi \int \rho \, d\rho \, dz$$

$$2 \leqslant z \leqslant 3, \quad z^2+6 \leqslant \rho^2 \leqslant 5z \quad \rho \geqslant 0$$

$$\overset{RF}{=} 2\pi \int_2^3 \left(\int_{\sqrt{z^2+6}}^{\sqrt{5z}} \rho \, d\rho \right) dz$$

$$= \cancel{2}\pi \int_2^3 \frac{\rho^2}{\cancel{2}} \Bigg|_{\sqrt{z^2+6}}^{\sqrt{5z}} dz$$

$$= \pi \int_2^3 5z - (z^2+6) \, dz$$

$$= \pi \left[\frac{5}{2} z^2 \Big|_2^3 - \frac{z^3}{3} \Big|_2^3 - 6 \cdot 1 \right]$$

$$= \pi \left[\frac{5}{2} \cdot 5 - \frac{1}{3} (27-8) - 6 \right]$$

$$= \pi \, \frac{75 - 38 - 36}{6}$$

$$= \frac{\pi}{6}.$$

3. Outward flux of $\vec{F} = (x^2, y, z^2)$.

We apply divergence thm:

$$\int_{\partial\Omega} \vec{F} \cdot \vec{n}_e = \int_{\Omega} \operatorname{div} \vec{F}$$

$$= \int_\Omega \Big(\underset{\substack{\| \\ 2x}}{\partial_x x^2} + \underset{\substack{\| \\ 1}}{\partial_y y} + \underset{\substack{\| \\ 2z}}{\partial_z z^2} \Big)\, dx\,dy\,dz$$

$$= \int_\Omega 1 + 2(x+z)\, dx\,dy\,dz$$

$$= \underset{\substack{\| \\ \pi/6}}{\mathrm{Vol}\,\Omega} + 2 \int_\Omega (x+z)\, dx\,dy\,dz$$

About

$$\int_\Omega x \quad dx\,dy\,dz \underset{\text{cyl coords}}{=} \int_{\substack{z^2+6 \le \rho^2 \le 5z \\ \rho \ge 0,\ \theta \in [0,2\pi],\ z \in \mathbb{R}}} \overbrace{\rho \cos\theta}^{x} \cdot \rho\, d\rho\, d\theta\, dz$$

$$\overset{RF}{=} \int_0^{2\pi} \Big(\int_{\substack{z^2+6 \le \rho^2 \le 5z \\ \rho \ge 0 \quad z \in \mathbb{R}}} \rho^2 \cos\theta \, d\rho\, dz \Big)\, d\theta$$

$$= \underset{\substack{\| \\ 0}}{\int_0^{2\pi} \cos\theta \, d\theta} \cdot \int_{z^2+6 \le \rho^2 \le 5z} \rho^2 \, d\rho\, dz$$

$= 0.$

$$\int_{\Omega} z \, dx\,dy\,dz \underset{\text{cyl coords}}{=} \int_{\substack{z^2+6 \le \rho^2 \le 5z \\ \rho \ge 0 \;\; \theta \in [0,2\pi] \;\; z \in \mathbb{R}}} \rho z \, d\rho\, dz\, d\theta$$

$$\overset{RF}{=} 2\pi \int_{\substack{z^2+6 \le \rho^2 \le 5z \\ \rho \ge 0 \;\; z \in \mathbb{R}}} \rho z \, d\rho\, dz$$

$$\underset{\text{as above}}{\overset{RF}{=}} 2\pi \int_2^3 \left(\int_{\sqrt{z^2+6}}^{\sqrt{5z}} \rho z \, d\rho \right) dz$$

$$= 2\pi \int_2^3 z \left(5z - (z^2+6) \right) dz$$

$$5z^2 - z^3 - 6z \quad dz$$

$$= 2\pi \left[5 \frac{z^3}{3} \Big|_2^3 - \frac{z^4}{4} \Big|_2^3 - 6 \frac{z^2}{2} \Big|_2^3 \right]$$

$$= 2\pi \left[\frac{5}{3} \underbrace{(27 - 8)}_{19} - \frac{1}{4} \underbrace{(81 - 16)}_{65} - 3 \underbrace{(9 - 4)}_{5} \right]$$

$$= \cancel{2}\pi \, \frac{380 - 195 - 180}{\cancel{12}\, 6} = \frac{5}{6}\pi$$

Conclusion: outward flux $= \frac{\pi}{6} + 2\,\frac{5}{6}\pi = \frac{11}{6}\pi$. $\blacksquare$