EX 7.4.3

lunedì 8 giugno 2020 17:59

Ex 7.4.3

Consider

$$\begin{cases} y' = t(y^2-1)\arctg y \\ y(0) = y_0 \end{cases}$$

1. Local ex. and uniqueness.

Eqn has form

$$y' = f(t,y) := t(y^2-1)\arctg y$$

Clearly f is well defined on

$$D = \mathbb{R} \times \mathbb{R}$$

and $f, \partial_y f \in \mathscr{C}(D)$

2. St. sols.

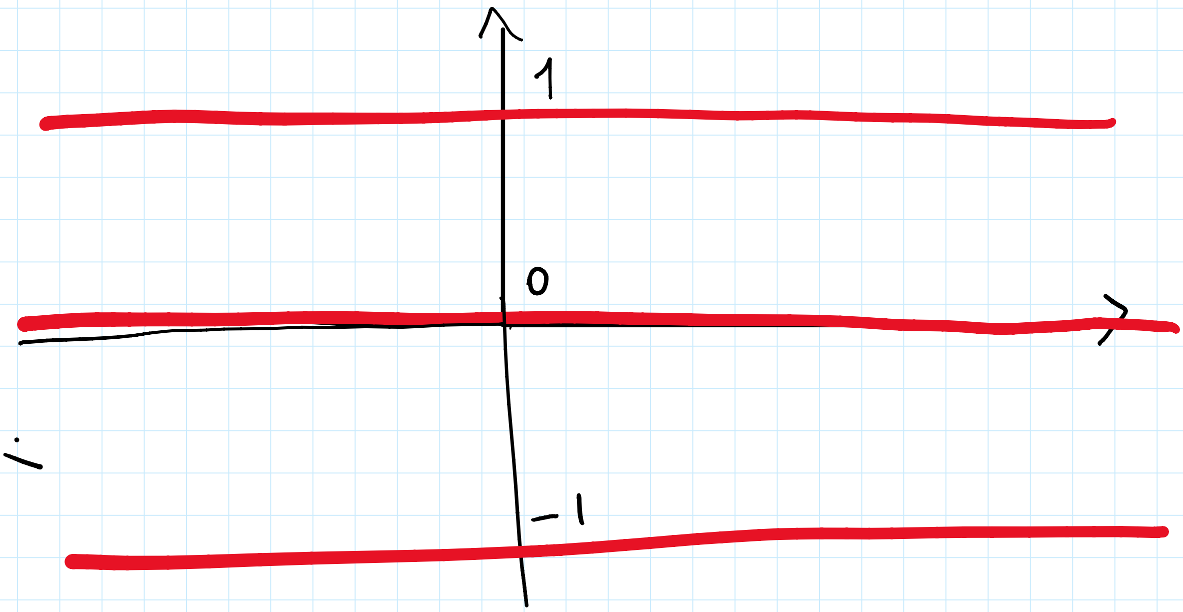
$y \equiv C$ is sol $\Leftrightarrow$ $0 = t^2(C^2-1)\arctg C$ $(\forall t)$

$\Leftrightarrow$ $0 = (C^2-1)\arctg C$

$\Leftrightarrow$ $C^2-1=0 \ \vee \ \arctg C = 0$

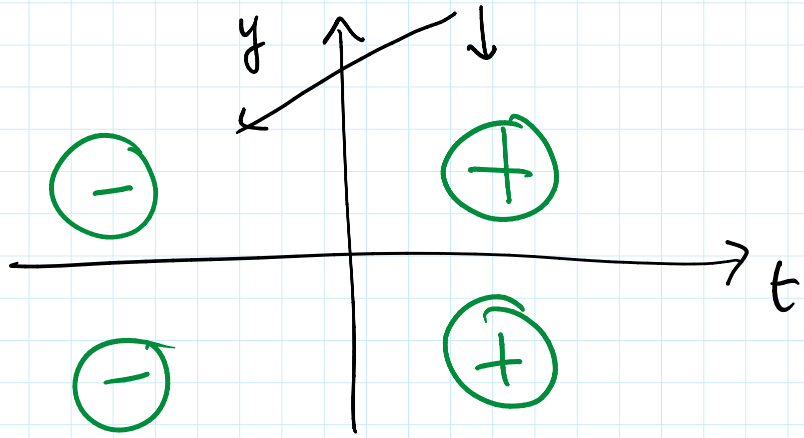
$\Leftrightarrow$ $C = \pm 1 \ \vee \ C = 0.$

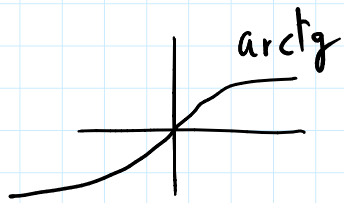
There're three st. sols. (in red here below)



3. $\nearrow$ $\searrow$

$$y \nearrow \iff 0 \le y' = t\,(y^2-1)\,\mathrm{arctg}\,y$$

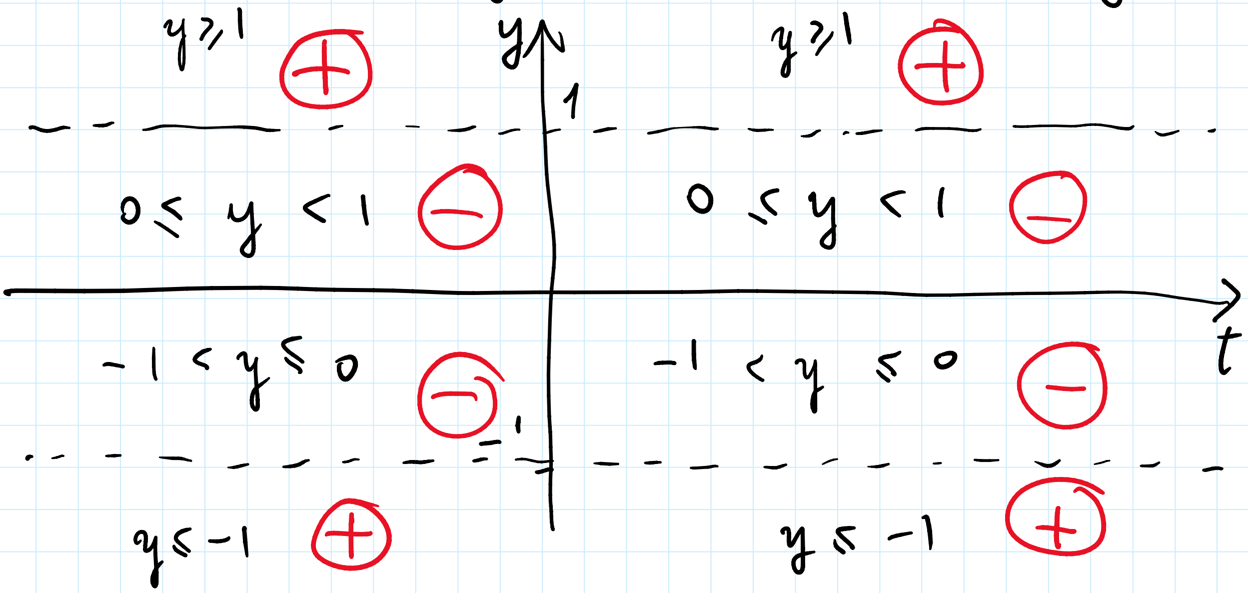




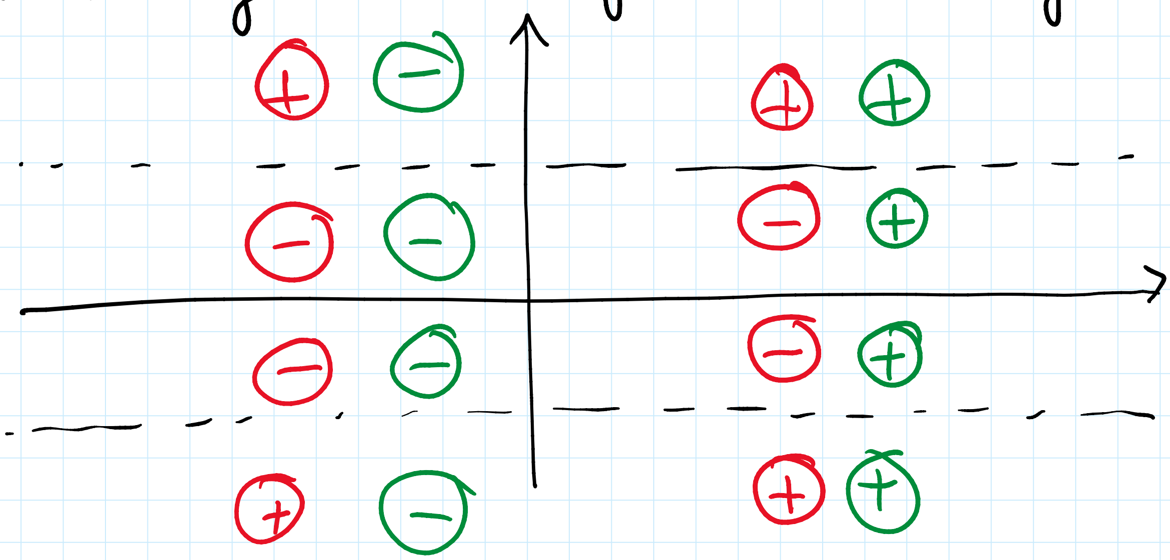
$$(y^2-1)\,\mathrm{arctg}\,y \ge 0 \iff \begin{array}{l} y^2-1 \ge 0 \;\wedge\; \mathrm{arctg}\,y \ge 0 \\ \Updownarrow \\ \boxed{y^2 \ge 1 \;\wedge\; y \ge 0} \iff y \ge 1 \\ \text{OR} \\ y^2-1<0 \;\wedge\; \mathrm{arctg}\,y<0 \\ \Updownarrow \\ \boxed{y^2 \le 1 \;\wedge\; y<0} \iff y \le -1 \end{array}$$

$\iff \quad y \geq 1 \quad \vee \quad y \leq -1$

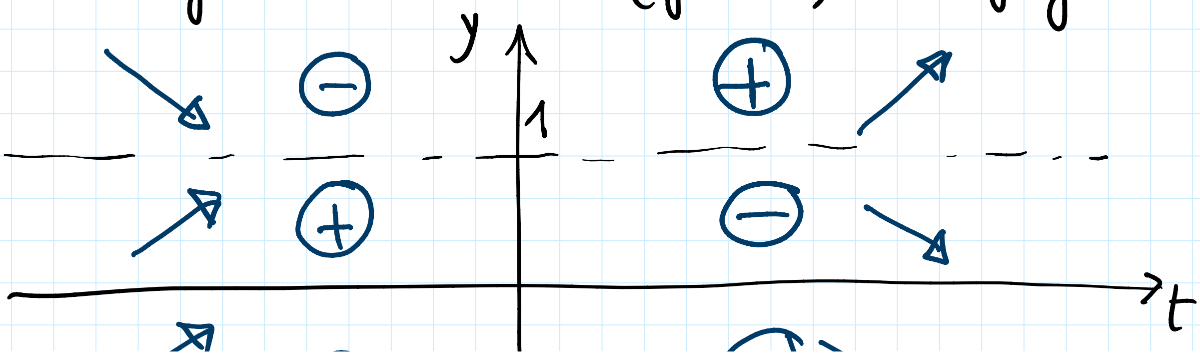
$\operatorname{sgn}(y^2-1)\arctg y$ in plane (t,y)

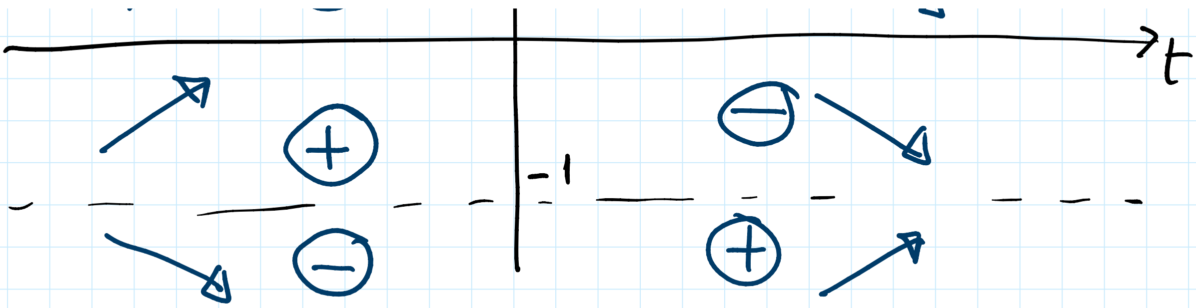


Combining with sgn t we get



thus sgn of $t\cdot(y^2-1)\arctg y$ is





Since now $y_0 \in]0,1[$.

4. y is even.

We've to prove $y(-t) \equiv y(t)$ ($\equiv$ means $\forall t$)

Strategy: take $z(t) := y(-t)$ and prove z solves the same CP solved by y
THEN, by uniqueness, $\boxed{z \equiv y}$

First: $z(0) = y(-0) = y(0) = y_0$ (same passage cond)

Second: $z'(t) = (y(-t))'$

$= -y'(-t)$

$y'(s) = s(y(s)^2-1)\operatorname{arctg} y(s)$

$$= -\left[-t\,(\underbrace{y(-t)}_{=z(t)}{}^2-1)\operatorname{arctg} y(-t)\right]$$

$$z(t)$$

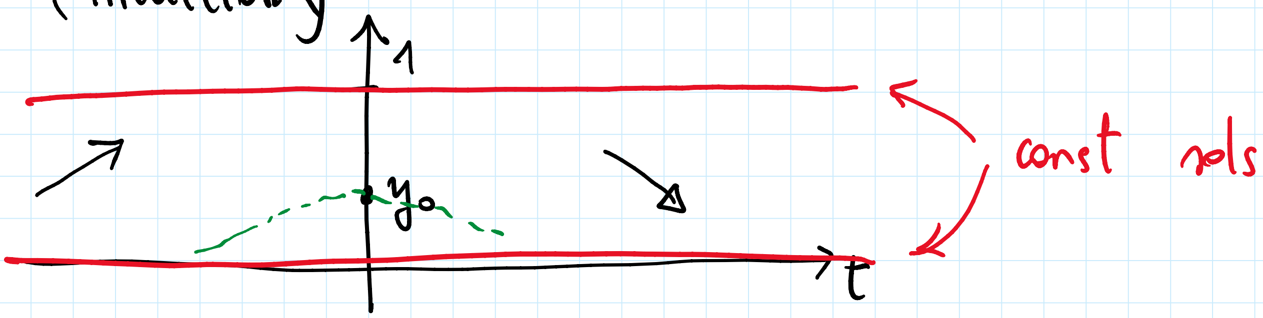
$$= t(z^2-1)\,\mathrm{arctg}\, z$$

$\Rightarrow z$ solves also the same diff eqn of y.

5. Nature of $t=0$

Claim: $t=0$ is a global max for y

(Intuition



Strategy: a) $y \nearrow$ on $]\alpha, 0]$

b) $y \searrow$ on $[0, \beta[$

Both follows from $0 < y < 1 \quad \forall t$ and point #3.

Indeed: IF $y(\hat{t}) \leq 0$ THEN either

- $y(\hat{t}) = 0 \implies y$ crosses at sol $\equiv 0$

$$\Downarrow \text{uniq.}$$

$$y \equiv 0$$

$$\Downarrow$$

$$0 < y_0 = y(0) = 0 \qquad \text{IMPOSSIBLE}$$

- $y(\hat{t}) < 0 \Rightarrow$ int. value thm $\exists \hat{\hat{t}} : y(\hat{\hat{t}}) = 0$

$\Rightarrow$ back to prev case.

$y < 1$ analogous.

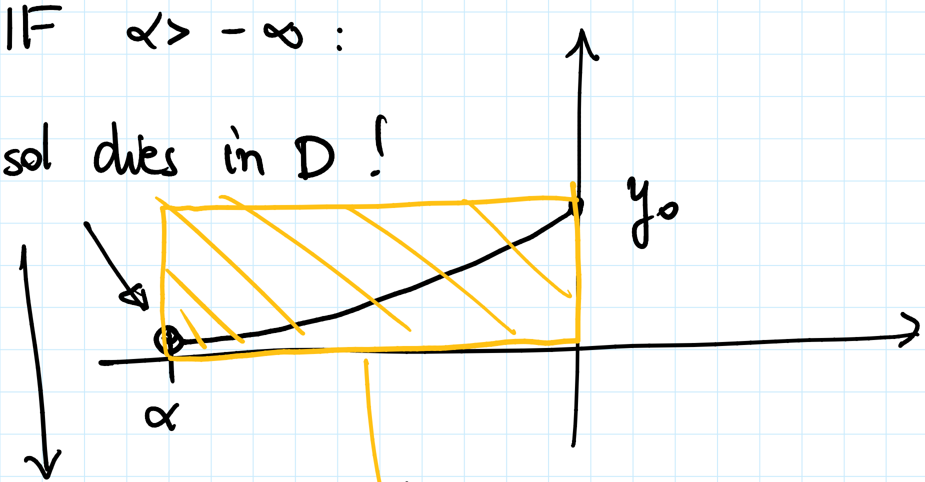
6. α, β ?

Claim: $\alpha = -\infty, \quad \beta = +\infty.$

Because y is even we prove $\alpha = -\infty$.

IF $\alpha > -\infty$:

sol dies in D !



PRECISELY : take $K = [\alpha, 0] \times [0, 1]$

or $[0, y_0]$

closed & bdd and $\subset D$.

But $\nexists \tau$: $(t, y(t)) \notin K \quad \alpha < t < \tau$.

CONTRADICTION

7. limits of u as $t \to \alpha$, $t \to \beta$.

7. limits of y as $t \to \alpha$, $t \to \beta$.

First: let $\ell = \lim_{t \to \alpha} y(t)$.

- It exists, because $y \nearrow$ on $]-\infty, 0[$
- it is $\in [0,1]$, because we proved (# 5) $0 < y < 1 \Rightarrow 0 \leq \ell \leq 1$

But how can we find ℓ precisely?

Strategy: pass to $t \to -\infty$ into diff eqn.

Here we have to be a little more careful than usual:

$$y' = \underbrace{t}_{\downarrow \, -\infty} \, \underbrace{(y^2 - 1) \arctg y}_{\downarrow \, (\ell^2-1) \arctg \ell}$$

Now: what is $-\infty \cdot (\ell^2 - 1) \arctg \ell$?

It depends! However, we may have two alternatives:

- $(\ell^2 - 1) \arctg \ell \neq 0$

$\neq 0$

• $(\ell^2-1)\,\mathrm{arctg}\,\ell \neq 0$

$$\Longrightarrow \quad y' \longrightarrow -\infty \cdot \overset{\neq 0}{C} = \pm\infty$$

What goes wrong with this?

Think about:

$$y \longrightarrow \ell \in \mathbb{R} \qquad (\in [0,1])$$

$$\wedge \quad y' \longrightarrow \infty$$

$\Longrightarrow$ IMPOSSIBLE (asymptote test)

• $(\ell^2-1)\,\mathrm{arctg}\,\ell = 0$ (this is the unique possibility)

$$\Downarrow$$

$$\ell^2 - 1 = 0 \quad \vee \quad \mathrm{arctg}\,\ell = 0$$

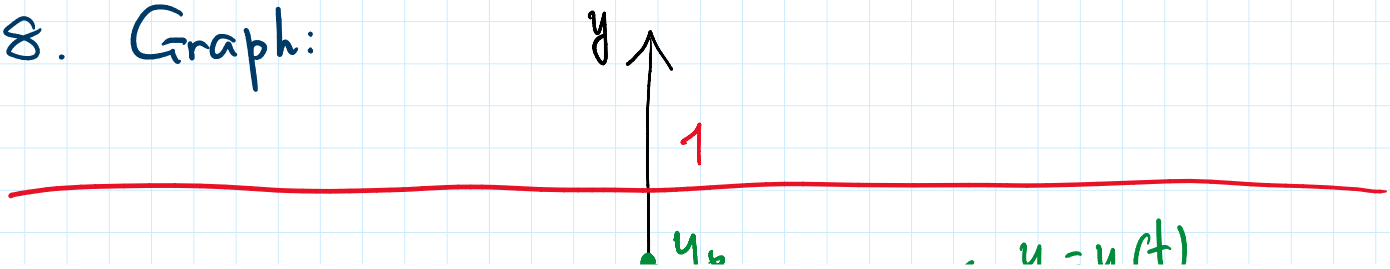
$$\Downarrow$$

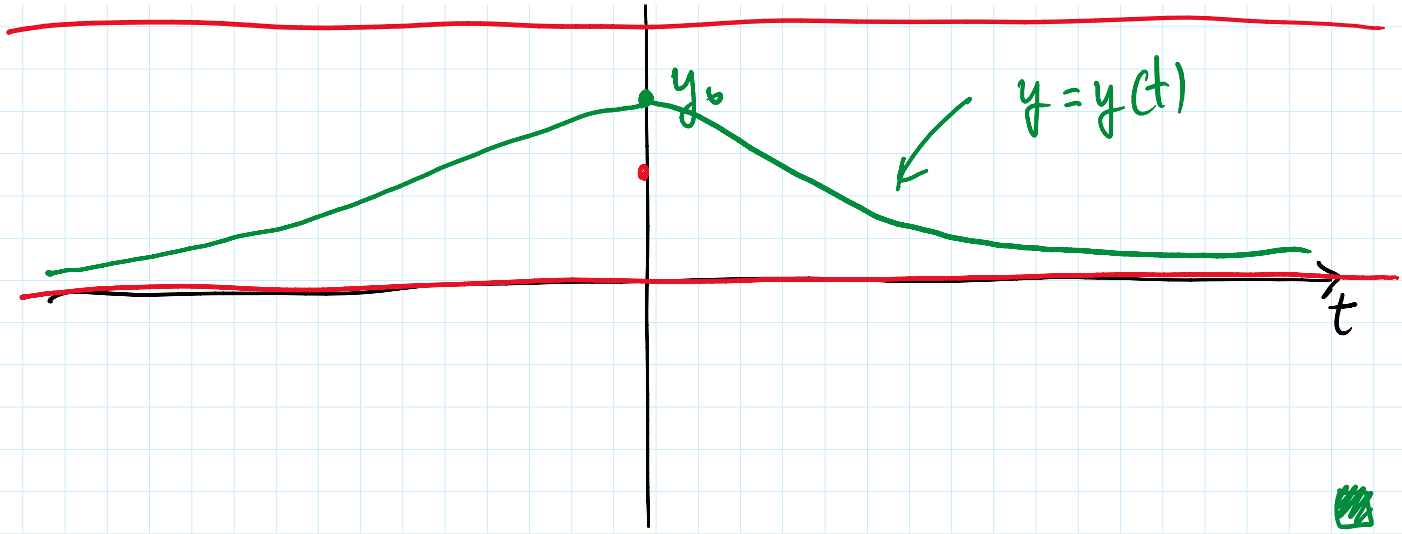
$$\ell = \pm 1 \quad \vee \quad \ell = 0$$

and because $\ell \in [0,1]$ (actually, $\ell \in [0, y_0]$)

$$\ell \neq -1, +1 \quad \Rightarrow \quad \boxed{\ell = 0}$$

8. Graph:




y_0
$y = y(t)$
t

EX 7.4.5
lunedì 8 giugno 2020 18:35

Consider the CP

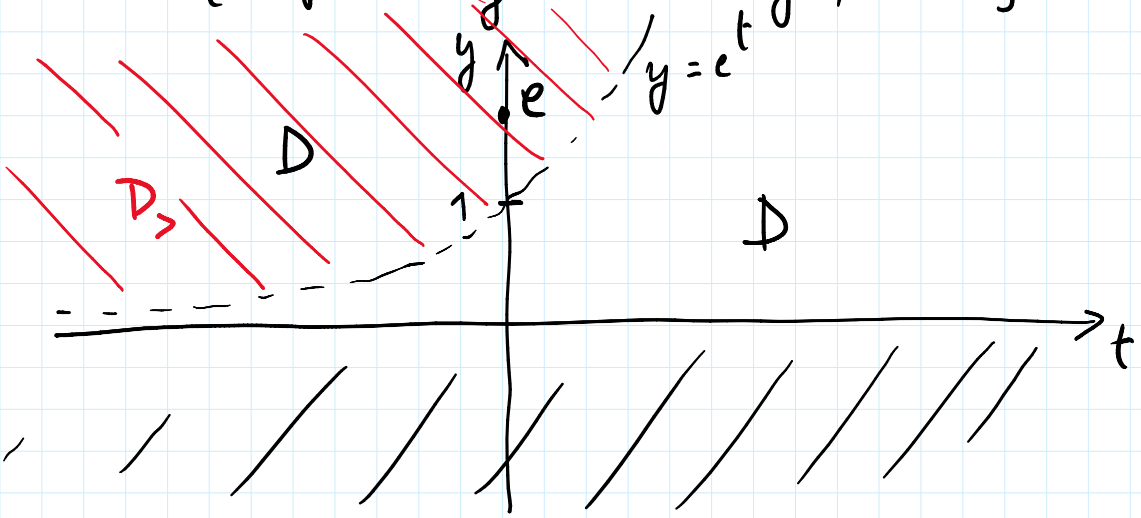
$$\begin{cases} y' = \dfrac{1}{t-\log y} \\ y(0) = e \end{cases}$$

1. Local $\exists !$

Let $f(t,y) = \dfrac{1}{t-\log y}$. f is well defined on

$$D = \{(t,y) : t - \log y \neq 0\}$$

$$= \{(t,y) : y > 0 \wedge y \neq e^t\}$$



2. Sol of CP is contained in $D_> = \{(t,y) : y > e^t\}$.

Indeed, if false, $\exists\, \hat{t} : y(\hat{t}) \leq e^{\hat{t}}$.

But: $y(\hat{t}) = e^{\hat{t}} \Rightarrow (\hat{t}, y(\hat{t})) \notin D$ (impossible)

$$y(\hat{t}) < e^{\hat{t}}, \text{ being also } y(0) = e = e^1 > e^0$$

$$\Downarrow \text{ int. val. thm}$$

$$\exists \hat{\hat{t}} : y(\hat{\hat{t}}) = e^{\hat{\hat{t}}} \Rightarrow \text{prev. point}$$

3. $y \searrow$ (warning: text contains an error)

By 2. $(t, y(t)) \in D_>$ $\forall t$. $\Rightarrow y > e^t \Leftrightarrow \log y > t$

$$\Rightarrow y' = \frac{1}{t - \log y} < 0 \Rightarrow y \searrow.$$

4. Concavity

Strategy: study y'' by deriving eqn.

$$y'' = \left(\frac{1}{t - \log y}\right)'$$

$$= -\underbrace{\frac{1}{(t - \log y)^2}}_{\oplus} \left(1 - \frac{y'}{y}\right)$$

Now: $y > 0$ (because of D)

$y' < 0$ (by 3) $\Rightarrow 1 - \dfrac{y'}{y} > 0$

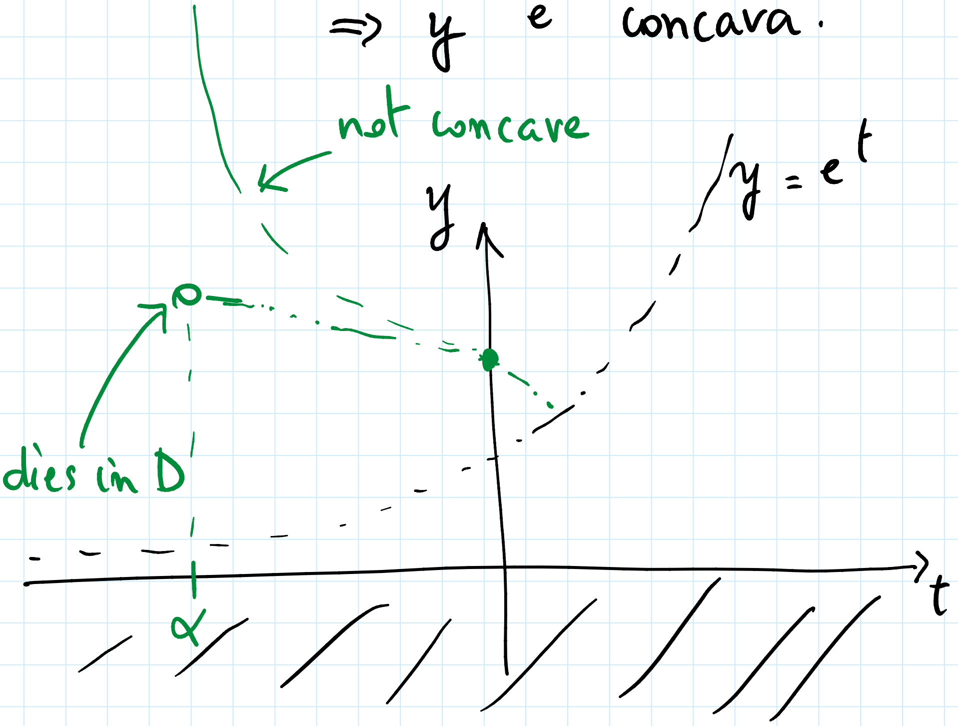
Thus $y' \geq 0$ never $\Rightarrow$ $y'' \leq 0$ always.

$\Rightarrow$ y è concava.

5. $\alpha = ?$

Guess:

$\alpha = -\infty$.



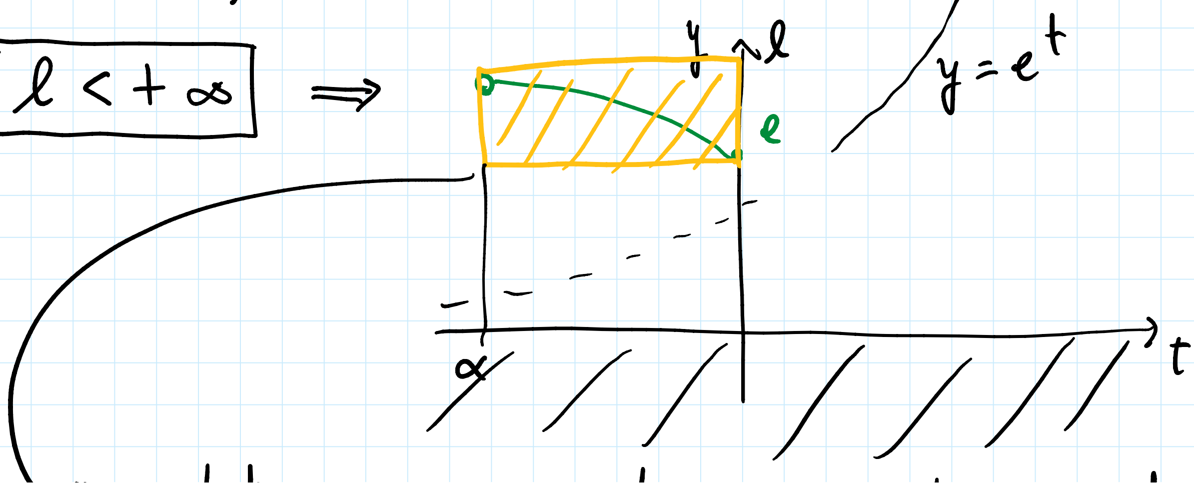
Let's prove the two cases cannot be possible

Assume $\alpha > -\infty$ and let

$$\ell = \lim_{t \to \alpha} y(t)$$

Limit $\exists$ because $y \searrow$ and $\ell > e$ (same reason). Now we have the alternative

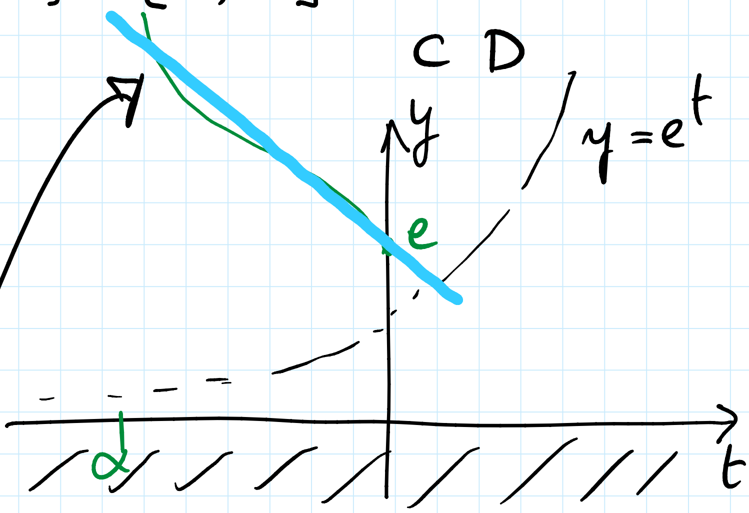
- $\boxed{\ell < +\infty}$ $\Rightarrow$



Δ solution never leaves in the past set

$K = [\alpha, 0] \times [e, \ell]$ closed and bdd $\subset D$

impossible.



• $\boxed{\ell = +\infty}$

by concavity

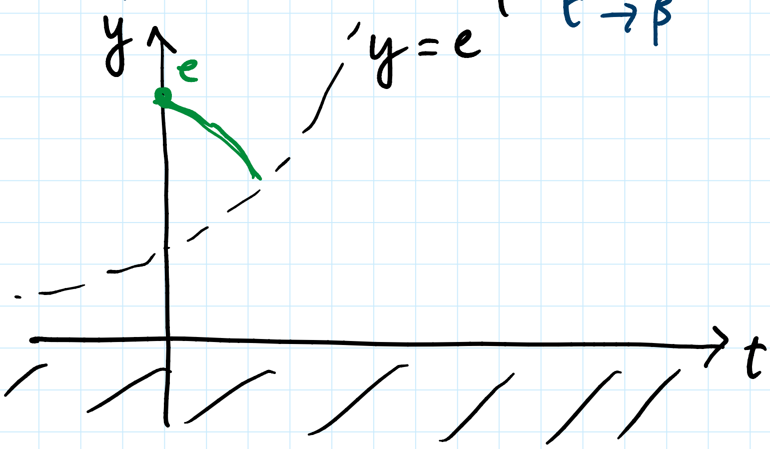
$$y(t) \leq y(0) + y'(0)t = e - t$$

$$\Downarrow t \to \alpha$$

$$+\infty \leq e - \alpha < +\infty$$ impossible

Conclusion: $\alpha = -\infty$

6. $\beta < +\infty$ and $\lim_{t \to \beta} y'(t) = ?$



$y \searrow \Rightarrow y(t) \leq e$

$\forall \; t \in [0, \beta[$

at same time

$y(t) \geq e^t \quad \forall t$

$y(t) > e^t \quad \forall t.$

If $\beta = +\infty$

$$e^t < y(t) \leq e$$

$$\downarrow t \to \beta$$

$$+\infty \leq e \qquad \text{impossible}$$

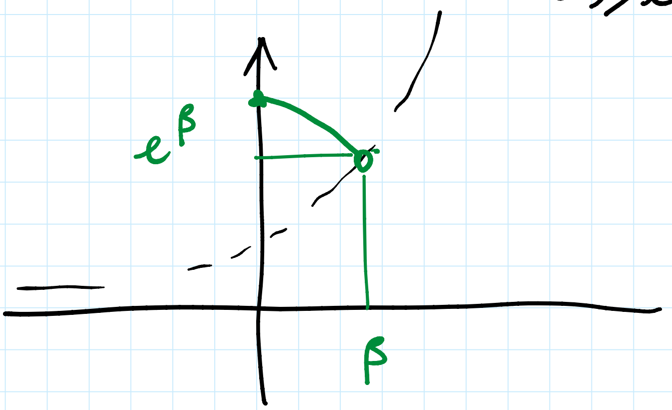
Clearly, when $t \to \beta$, $y(t) \searrow \ell$,

and because $y(t) > e^t$

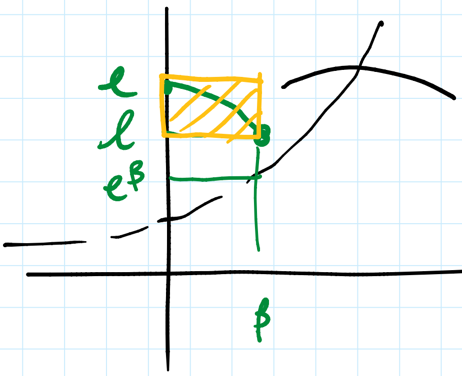
$$\downarrow \qquad \downarrow$$

$$\ell \geq e^\beta$$

<u>Claim:</u> $\boxed{\ell = e^\beta}$



If false, $\ell > e^\beta$



solution never leaves $K = [0, \beta] \times [\ell, e]$ in future

$\Rightarrow$ impossible.

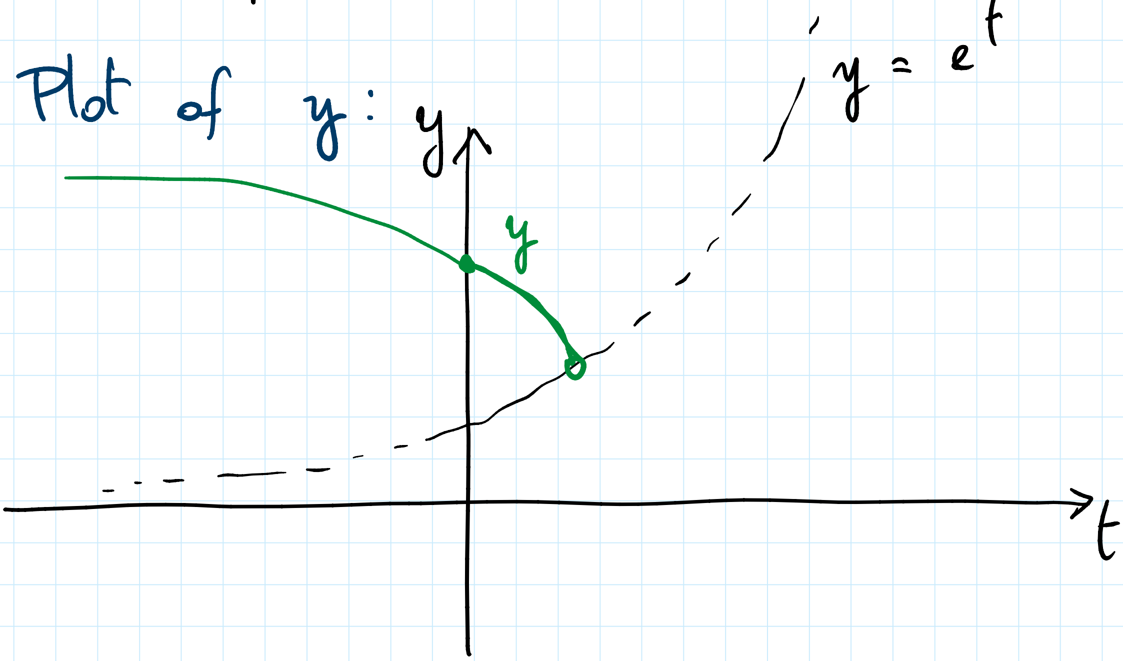
Finally,

$$y' = \frac{1}{t - \log y(t)} \longrightarrow \frac{1}{\beta - \log e^\beta} = \frac{1}{0_-} = -\infty$$

$$y' = \frac{\quad}{t - \log y(t)} \to \frac{\quad}{\beta - \log e^{\beta}} = 0_-$$

$t \downarrow \beta$, $\ \wedge\ 0$, $\ y(t) \downarrow e^{\beta}$; $\ \beta - \log e^{\beta} \to 0_-$

7. Plot of y:



EX 8.7.2

lunedì 8 giugno 2020 20:09

#2.

$$\begin{cases} x' = 2y(y-2x) \\ y' = (1-x)(y-2x) \end{cases}$$

1. St. sols.

$(x,y) \equiv (a,b)$ is a solution $\Leftrightarrow$

$$\begin{cases} 0 = 2b(b-2a) \\ 0 = (1-a)(b-2a) \end{cases} \Leftrightarrow \begin{cases} b=0 \\ // \end{cases} \vee \begin{cases} b=2a \\ // \end{cases}$$

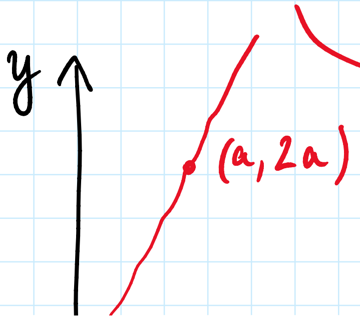
$$\Leftrightarrow \begin{cases} b=0 \\ (1-a)a=0 \end{cases} \vee \begin{cases} b=2a \\ \end{cases}$$

$$\Leftrightarrow \begin{cases} b=0 \\ a=1 \end{cases} \vee \begin{cases} b=0 \\ a=0 \end{cases} \vee \begin{cases} b=2a \\ \end{cases}$$

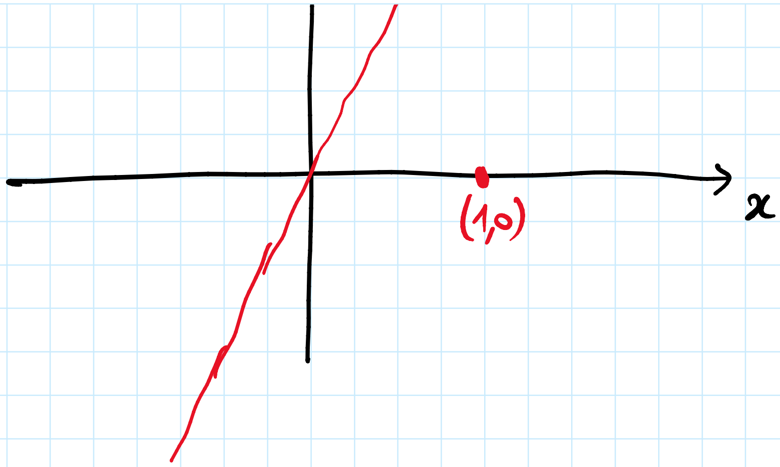
$$\Updownarrow \qquad\qquad \Updownarrow \qquad\qquad \Updownarrow$$

$(1,0)$ $\qquad$ $(0,0)$ $\qquad$ $(a,2a)$ $\quad a \in \mathbb{R}$

already included



• = st. pts.



• = st. pts.

2. Prime/First Integral.

We write the total eqn

$$\frac{dy}{dx} = \frac{(1-x)\cancel{(y-2x)}}{2y\cancel{(y-2x)}} = \frac{1-x}{2y}$$

sep. var. eqn.

$$\iff 2y\,dy = (1-x)\,dx$$

$$\iff y^2 = -\frac{(1-x)^2}{2} + c$$

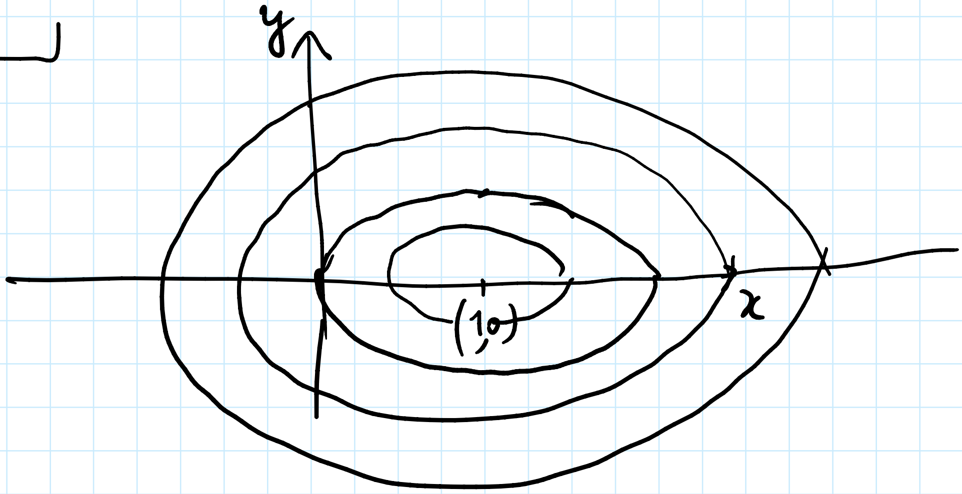
$$\Rightarrow E(x,y) = y^2 + \frac{(x-1)^2}{2}$$

3. Phase portrait.

1st step: plot of level sets of E

$$\frac{(x-1)^2}{2} + y^2 = C$$

$$\underbrace{\frac{\quad}{2} + y^{\,}}_{\text{ellypsis}} = C$$



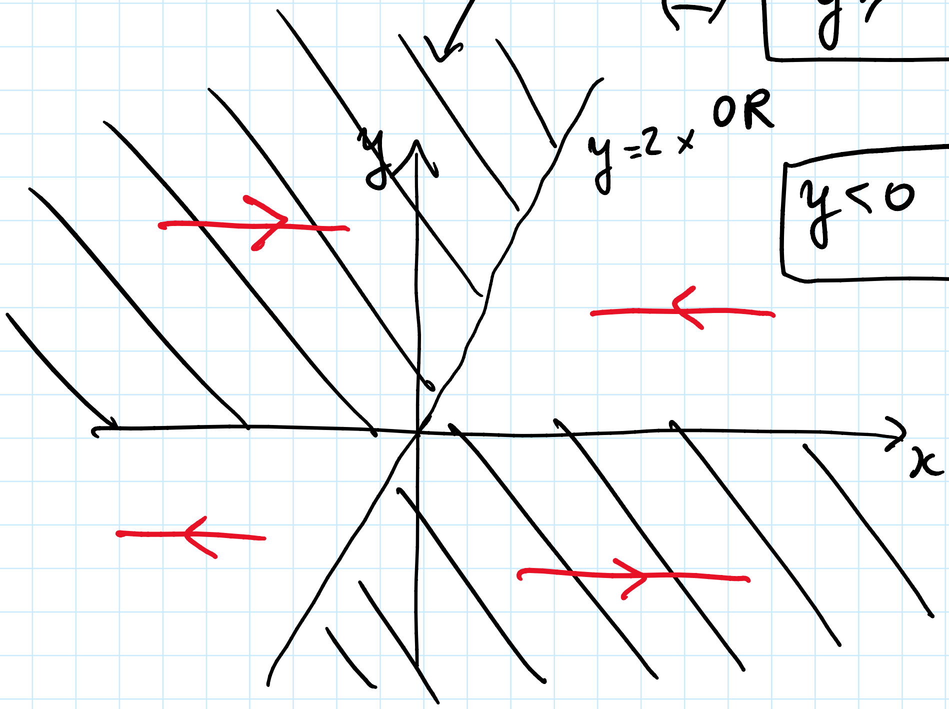
2nd: Orientation:

$$x \nearrow \iff x' \geq 0 \iff \cancel{2}y\,(y-2x) \geq 0$$

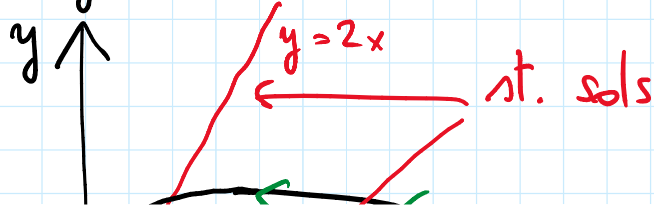
$$\iff \boxed{y \geq 0 \;\wedge\; y-2x \geq 0}$$

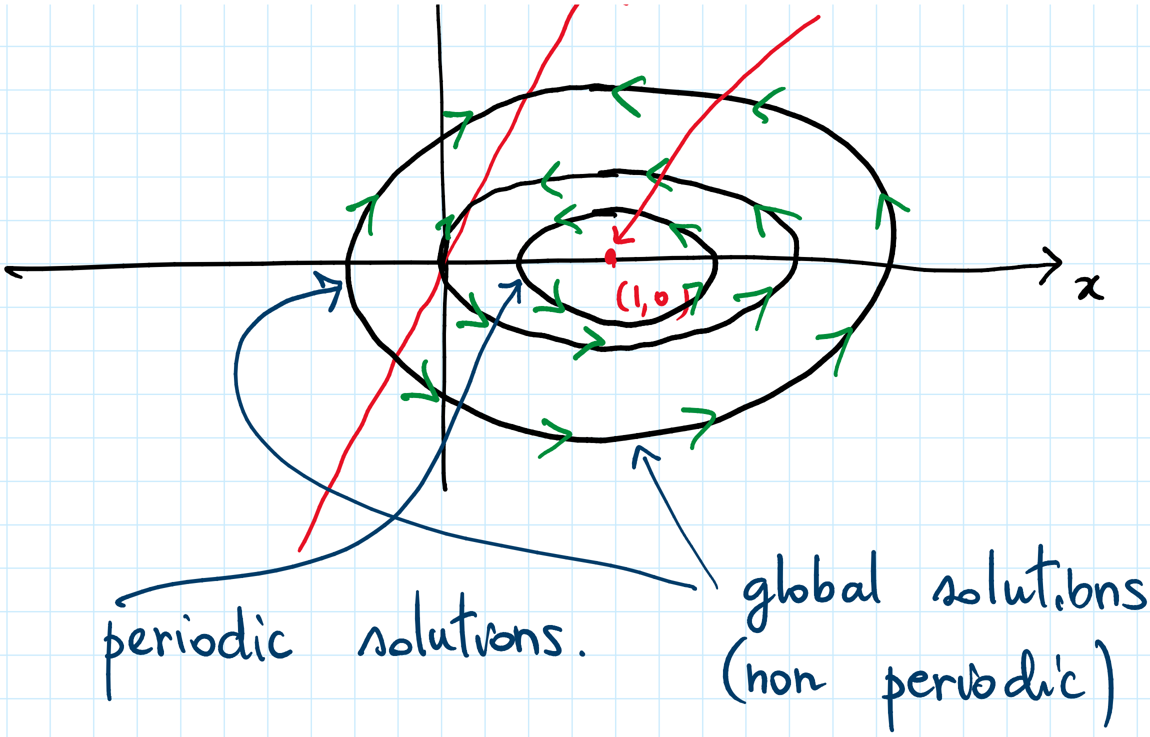
OR

$$\boxed{y<0 \;\wedge\; y-2x<0}$$



3rd: we put together infos





\#4.

$$\begin{cases} x' = 2x^2 y \\ y' = y^2 x + x \end{cases}$$

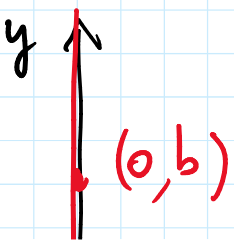
1. St. sols.

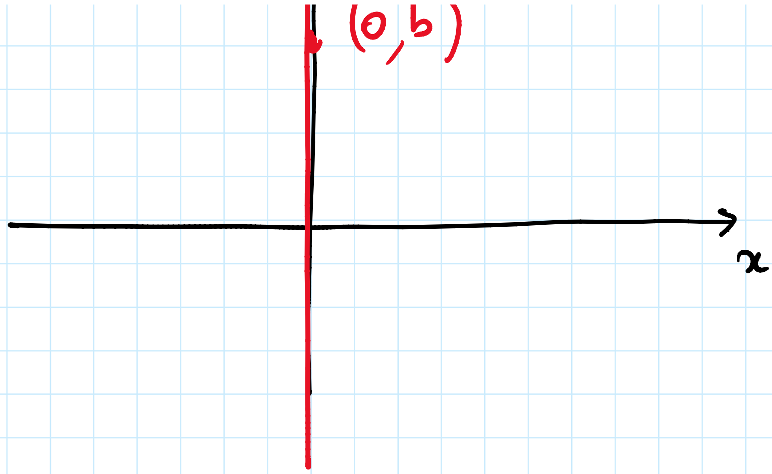
$$(x,y) \equiv (a,b) \iff \begin{cases} 0 = 2a^2 b \\ 0 = b^2 a + a = (b^2+1)a \end{cases}$$

$$\iff \begin{cases} \parallel \\ a = 0 \end{cases} \iff \begin{cases} 0 = 0 \\ a = 0 \end{cases}$$

$$\Updownarrow$$

$(0,b) \quad \forall b \in \mathbb{R}$





2. First / Prime Integral

Let's write the total eqn:

$$\frac{dy}{dx} = \frac{(y^2+1)\cancel{x}}{2x^{\cancel{3}}y} = \frac{\frac{y^2+1}{y}}{2x} \qquad \text{sep var eq}$$

$$\Leftrightarrow \quad \frac{2y}{y^2+1}\,dy = \frac{1}{x}\,dx$$

$$\Leftrightarrow \quad \log(y^2+1) = \log|x| + c$$

$$\Leftrightarrow \quad \log\frac{y^2+1}{|x|} = C \Leftrightarrow \frac{y^2+1}{|x|} = C$$

$$E(x,y) = \frac{y^2+1}{|x|}$$

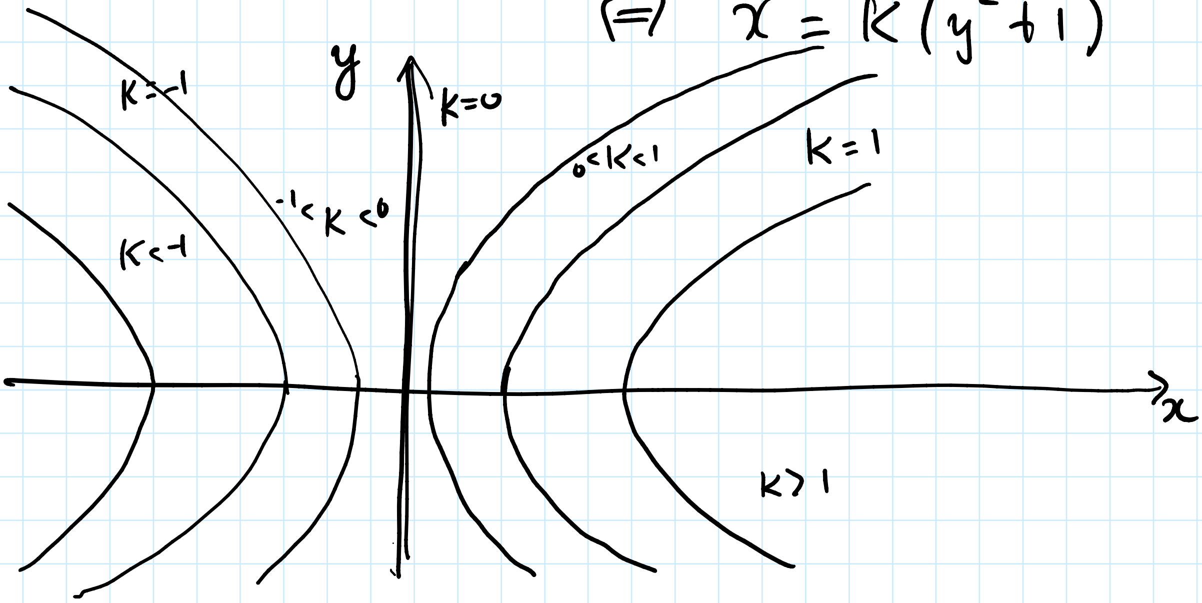
3. Phase portrait

We plot lines $E(x,y) = C$

$$\Updownarrow$$

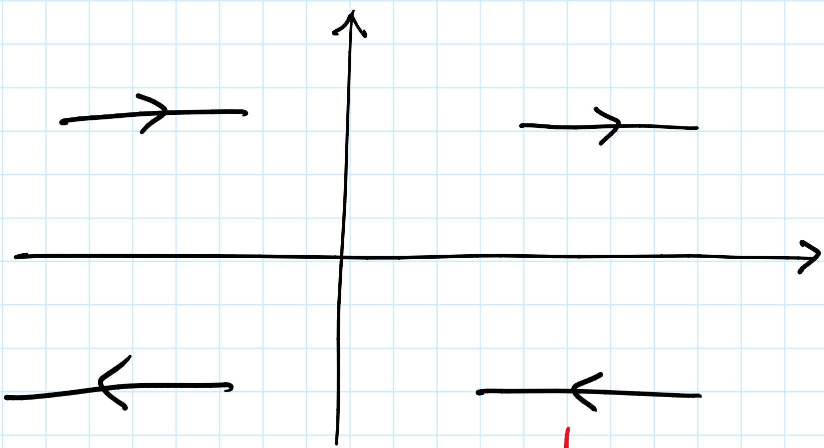
$$C|x| = y^2+1 \iff |x| = K(y^2+1)$$

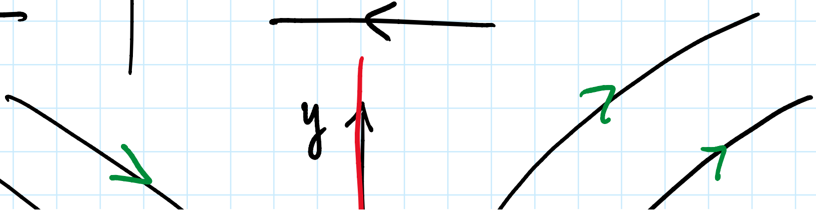
$$\iff x = K(y^2+1) \qquad K \in \mathbb{R}$$



Orientation: $x \nearrow \iff x' \geq 0 \iff 2x^2y \geq 0$

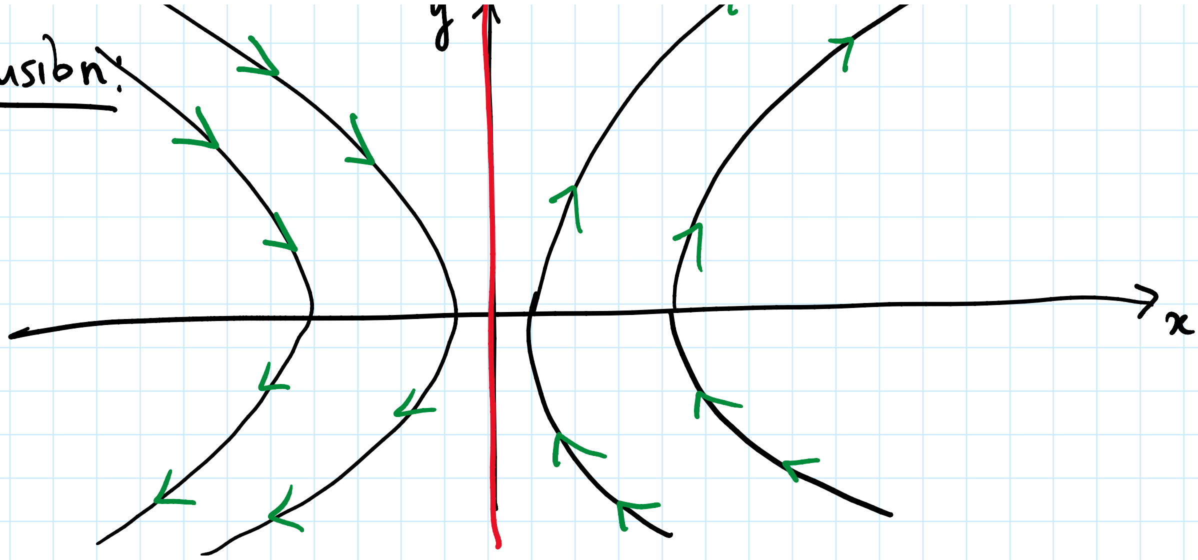
$$\iff y \geq 0$$





Conclusion:

Conclusion:



No periodic sols

No evident non const global sols.

EX 8.7.3

lunedì 8 giugno 2020 20:09

Consider

$$y'' = y^3 - y$$

1. St. sols.

$y \equiv c$ sol $\iff$ $0 = c^3 - c = c(c^2-1)$

$\iff$ $c = 0, \pm 1$ (three const sols)

2. First Int.

We check first that eqn is conservative

that is $\exists\, V = V(y)$ such that

$$y'' = \partial_y V(y)$$

We need

$$\partial_y V = y^3 - y \implies V(y) = \frac{y^4}{4} - \frac{y^2}{2}$$

$\Rightarrow$ eqn is conservative with energy

$$E(y, y') = \frac{1}{2} y'^2 - V(y)$$

$$= \frac{1}{2} y'^2 - \frac{1}{\ } \left(\frac{y^4}{\ } - y^2 \right)$$

$$= \frac{1}{2}y'^2 - \frac{1}{2}\left(\frac{y^4}{2} - y^2\right)$$

3. Sol of CP $y(0) = 2$, $y'(0) = 2$.

Because E is constant along sols $\Rightarrow$

$$E(y,y') \equiv E(y(0), y'(0)) = E(2,2)$$

$$= \frac{1}{2}\cdot 4 - \frac{1}{2}\left(\frac{16}{2} - 4\right)$$

$$= 2 - \frac{1}{2} 4 = 0$$

Thus, for the sol. of our CP we have

$$\cancel{\frac{1}{2}}y'^2 - \cancel{\frac{1}{2}}y^2\left(\frac{y^2}{2} - 1\right) \equiv 0$$

$$\Leftrightarrow \quad y'^2 = y^2\left(\frac{y^2}{2} - 1\right)$$

$$\Leftrightarrow \quad y' = \pm |y| \sqrt{\frac{y^2}{2} - 1} = \pm y \sqrt{\frac{y^2}{2} - 1}$$

? at $t=0$

$$\left.\begin{array}{l} y'(0) = 2 \\ y\sqrt{\frac{y^2}{2} - 1} = 2 \cdot \sqrt{\frac{4}{2} - 1} = 2 \end{array}\right\} \Rightarrow \oplus$$

$$\Rightarrow \quad y' = y\sqrt{\frac{y^2}{2} - 1}$$

We solve by sep of vars:

$$\frac{dy}{y\sqrt{\frac{y^2}{2} - 1}} = dt \quad \Longrightarrow$$

$$\underbrace{\int \frac{dy}{y\sqrt{\frac{y^2}{2} - 1}}} = t + C.$$

to compute this recall that

$$Ch^2 - Sh^2 \equiv 1 \quad \Leftrightarrow \quad Ch^2 - 1 = Sh^2$$

Setting $\frac{y}{\sqrt{2}} = Ch\, u$ $(u>0)$, $dy = \sqrt{2}\, Sh\, u\, du$

$$\Rightarrow \int \frac{\cancel{\sqrt{2}}\, Sh\, u\, du}{\cancel{\sqrt{2}}\, Ch\, u \sqrt{(Sh\, u)^2}}\, du = \int \frac{1}{Ch\, u}\, du$$

$$\quad = |Sh\, u| \overset{u>0}{=} Sh\, u$$

$$" \quad |\operatorname{Sh} u| \overset{u>0}{=} \operatorname{Sh} u$$

$$= \int \frac{1}{\frac{e^{u} + e^{-u}}{2}} \, du = 2 \int \frac{e^{u}}{e^{2u}+1} \, du$$

$$= 2 \operatorname{arctg} e^{u}$$

$$\Longrightarrow \quad 2 \operatorname{arctg} e^{u} = 2 \operatorname{arctg} e^{\operatorname{Ch}^{-1} \frac{y}{\sqrt{2}}} = t + C$$

Setting $t = 0$ we det. $C = 2 \operatorname{arctg} e^{\operatorname{Ch}^{-1} \sqrt{2}}$.

$$\Rightarrow \quad \operatorname{Ch}^{-1} \frac{y}{\sqrt{2}} = \log\left(\operatorname{tg} \frac{t+C}{2}\right)$$

$$\Rightarrow \quad y(t) = \sqrt{2} \operatorname{Ch}\left(\log\left(\operatorname{tg} \frac{t+C}{2}\right)\right) \qquad \blacksquare$$