Ex 4.5.6 #7

giovedì 11 giugno 2020 19:52

Compute

$$\mathrm{Vol}\left(\left\{x^2+y^2+z^2\leq 16,\ \ x^2+y^2\geq 4\right\}\right)$$

Sol:

$$\mathrm{Vol}=\int_{\substack{x^2+y^2+z^2\leq 16\\ x^2+y^2\geq 4}} 1\ dx\,dy\,dz$$

$$\underset{\text{cyl coords}}{=}\int_{\rho\geq 0,\ \theta\in[0,2\pi],\ z\in\mathbb{R}:\ \rho^2+z^2\leq 16,\ \rho^2\geq 4}\rho\ d\rho\, d\theta\, dz$$

$$\begin{cases} x=\rho\cos\theta\\ y=\rho\sin\theta\\ z=z\end{cases}$$

$$\overset{RF}{=}\int_0^{2\pi}\left(\int_{\rho\geq 0,\ z\in\mathbb{R}:\ \underbrace{\rho^2+z^2\leq 16,\ \rho^2\geq 4}_{\rho\geq 2}}\rho\ d\rho\, dz\right)d\theta$$

$$0\leq \rho^2\leq 16-z^2$$

$$\Downarrow$$

$$16-z^2\geq 0 \Leftrightarrow z^2\leq 16$$

$$\overset{RF}{=}2\pi\int_{-4}^{4}\left(\int_{2\leq\rho\leq\sqrt{16-z^2}}\rho\, d\rho\right)dz$$

$$= 2\pi \int_{-4} \left(\int_{2 \le \rho \le \sqrt{16 - z^2}} \rho \, d\rho \right) dz$$

$$= 2\pi \int_{-4}^{4} \left(\int_{2}^{\sqrt{16 - z^2}} \rho \, d\rho \right) dz$$

$$\left. \frac{\rho^2}{2} \right]_{\rho = 2}^{\rho = \sqrt{16 - z^2}}$$

$$= 2\pi \, \frac{1}{2} \int_{-4}^{4} \left(16 - z^2 - 4 \right) dz$$

$$= \pi \int_{-4}^{4} (12 - z^2) \, dz$$

$$= \pi \left(12 \cdot 8 - \left[\frac{z^3}{3} \right]_{-4}^{4} \right)$$

$$\frac{1}{3} \left[4^3 - (-4)^3 \right] = \frac{2}{3} 4^3 = \frac{96}{3}$$

$$= \pi \, \frac{2}{3} \, 96 = 64\pi. \quad \blacksquare$$

$$\mathrm{Vol}\left(\left\{ z \geq x^2+y^2,\ z \leq 18 - x^2 - y^2 \right\}\right)$$

Sol:
$$V = \int_{z \geq x^2+y^2,\ z \leq 18-(x^2+y^2)} 1 \, dx\,dy\,dz$$

$$\underset{\text{cyl coords}}{=} \int_{\rho \geq 0,\ \theta \in [0,2\pi],\ z \in \mathbb{R}\,:\ z \geq \rho^2,\ z \leq 18-\rho^2} \rho \, d\rho\, d\theta\, dz$$

$$\overset{RF}{=} 2\pi \int_{\rho \geq 0,\ z \in \mathbb{R}\,:\ z \geq \rho^2,\ z \leq 18-\rho^2} \rho \, d\rho\, dz$$

$$\boxed{\rho^2 \leq z \leq 18-\rho^2}$$

$$\Downarrow$$

$$\rho^2 \leq 18 - \rho^2 \iff$$

$$2\rho^2 \leq 18 \iff \rho^2 \leq 9$$

$$\overset{\rho \geq 0}{\iff} 0 \leq \rho \leq 3$$

$$= 2\pi \int_{0 \leq \rho \leq 3,\ \rho^2 \leq z \leq 18-\rho^2} \rho \, d\rho\, dz$$

$$\overset{RF}{=} 2\pi \int^{3} \left(\int^{18-\rho^2} \rho \, dz \right) d\rho$$

$$= 2\pi \int_0 \left(\int_{\rho^2} \rho \, dz \right) d\rho$$

$$= 2\pi \int_0^3 \rho \left(18 - \rho^2 - \rho^2 \right) d\rho$$

$$= 2\pi \int_0^3 18\rho - 2\rho^3 \, d\rho$$

$$= 2\pi \left[18 \left. \frac{\rho^2}{2} \right|_0^3 - 2 \left. \frac{\rho^4}{4} \right|_0^3 \right]$$

$$= \cancel{2}\pi \left[\frac{18 \cdot 9}{\cancel{2}} - \frac{1}{\cancel{2}} 81 \right]$$

$$= \pi \left[162 - 81 \right] = 81\pi. \quad \blacksquare$$

Ex 4.5.7 #4

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Compute $\displaystyle\int_{\mathbb{R}^2} \frac{1}{1+(x^2+2y^2)^2}\,dx\,dy$

Sol: It is convenient to introduce adapted polar coords:

$$\Phi^{-1}\begin{cases} x = \rho\cos\vartheta \\ y = \frac{\rho}{\sqrt{2}}\sin\vartheta \end{cases}$$

(In such a way that $x^2+2y^2 = \rho^2(\cos\vartheta)^2+2\frac{\rho^2}{2}(\sin\vartheta)^2$
$= \rho^2$

$$|\det(\Phi^{-1})'| = \frac{1}{\sqrt{2}}\rho$$

$$\Rightarrow \int_{\mathbb{R}^2} \frac{1}{1+(x^2+2y^2)^2}\,dx\,dy =$$

$$= \int_{\rho\geq 0,\ \theta\in[0,2\pi]} \frac{1}{1+\rho^4}\frac{\rho}{\sqrt{2}}\,d\rho\,d\vartheta$$

RF

$$= \frac{\cancel{2}\pi}{\sqrt{2}} \int_0^{+\infty} \frac{2\rho}{1+\rho^4}\,d\rho$$

$$\frac{1}{1+(\rho^2)^2}(\rho^2)' = \left[\operatorname{arctg}\rho^2\right]'$$

$$= \frac{\pi}{\sqrt{2}}\operatorname{arctg}\rho^2 \Big|_0^{+\infty} = \frac{\pi}{\sqrt{2}}\left(\frac{\pi}{2}-0\right) = \frac{\pi^2}{2\sqrt{2}}$$

$$= \frac{\pi}{\sqrt{2}} \operatorname{arctg} \rho^2 \Big|_0^{+\infty} = \frac{\pi}{\sqrt{2}} \left(\frac{\pi}{2} - 0 \right) = \frac{\pi}{2\sqrt{2}}$$ $\blacksquare$

Ex 4.5.7 #6

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Compute $\displaystyle\int_{\mathbb{R}^3} \sqrt{x^2+y^2+z^2}\; e^{-(x^2+y^2+z^2)}\, dx\,dy\,dz$

Sol: We use spherical coords (thus $x^2+y^2+z^2=\rho^2$)

We have

$$\text{integral} = \int_{\rho\geq 0,\ \theta\in[0,2\pi],\ \varphi\in[0,\pi]} \sqrt{\rho^2}\, e^{-\rho^2} \cdot \rho^2 \sin\varphi \; d\rho\, d\theta\, d\varphi$$

$$\overset{RF}{=} \int_0^{2\pi} \left(\int_0^{\pi} \left(\int_0^{+\infty} \rho^3 e^{-\rho^2} d\rho \right) \sin\varphi \, d\varphi \right) d\theta$$

$$= 2\pi \left(\int_0^{\pi} \sin\varphi \, d\varphi \right) \left(-\frac{1}{2} \int_0^{+\infty} \rho^2 \underbrace{(-2)\rho\, e^{-\rho^2}}_{= (e^{-\rho^2})'} d\rho \right)$$

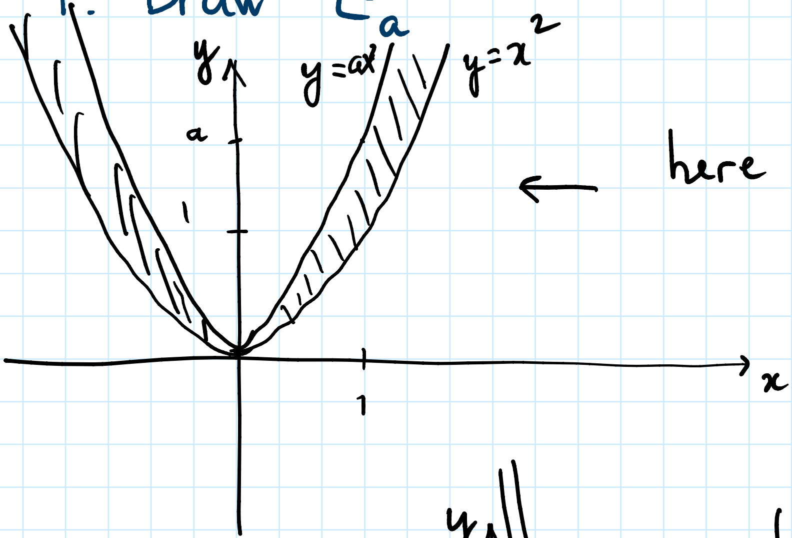
$$= 2\pi \cdot \cancel{2} \cdot \left(-\frac{1}{\cancel{2}} \right) \left[\rho^2 e^{-\rho^2} \Big|_0^{+\infty} + \int_0^{+\infty} -2\rho\, e^{-\rho^2} d\rho \right]$$

$$= -2\pi \left[0 + e^{-\rho^2} \Big|_0^{+\infty} \right] = 2\pi \quad \blacksquare$$

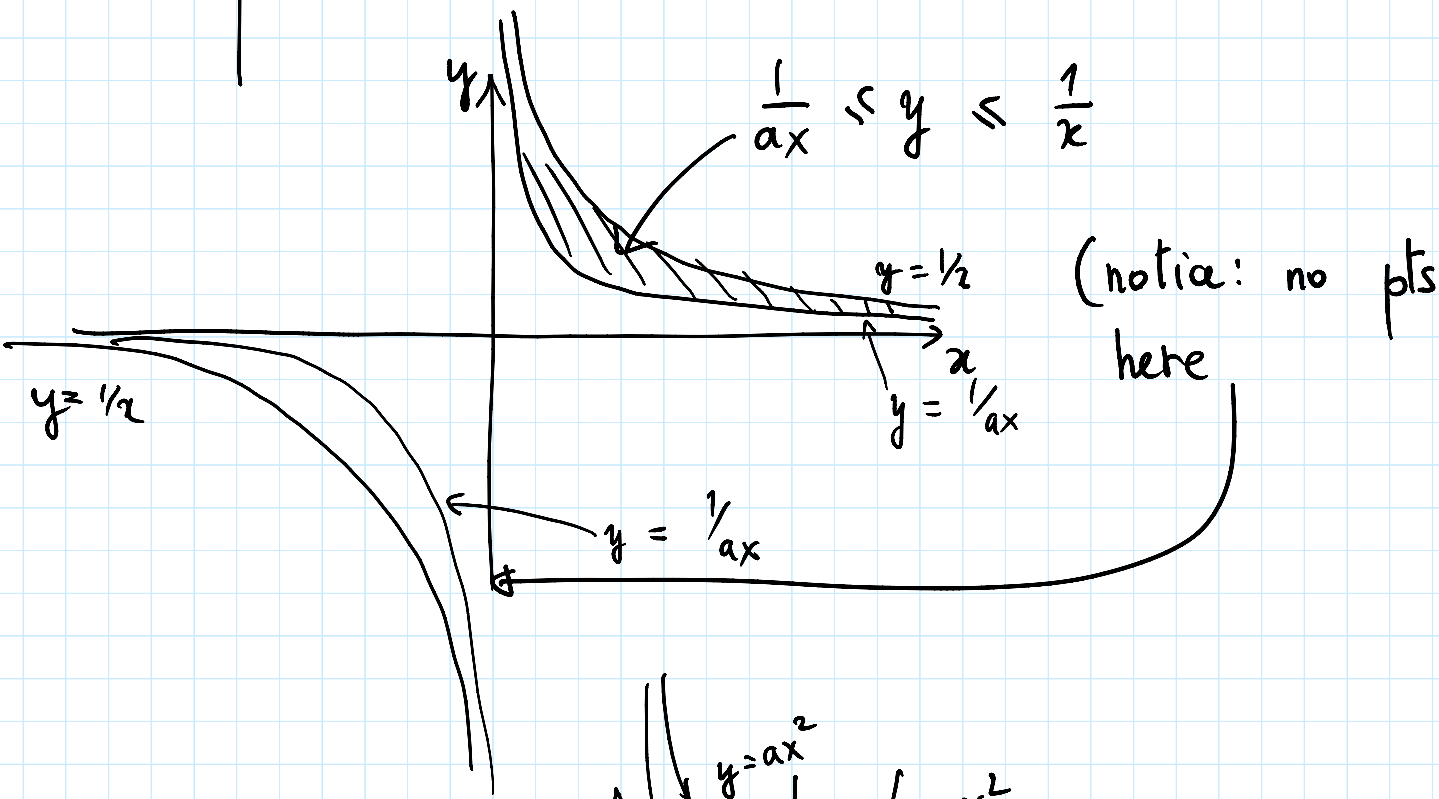
Let $a > 1$ and

$$E_a = \left\{ (x,y) \in \mathbb{R}^2 : \frac{1}{ax} \leq y \leq \frac{1}{x}, \ x^2 \leq y \leq ax^2 \right\}$$

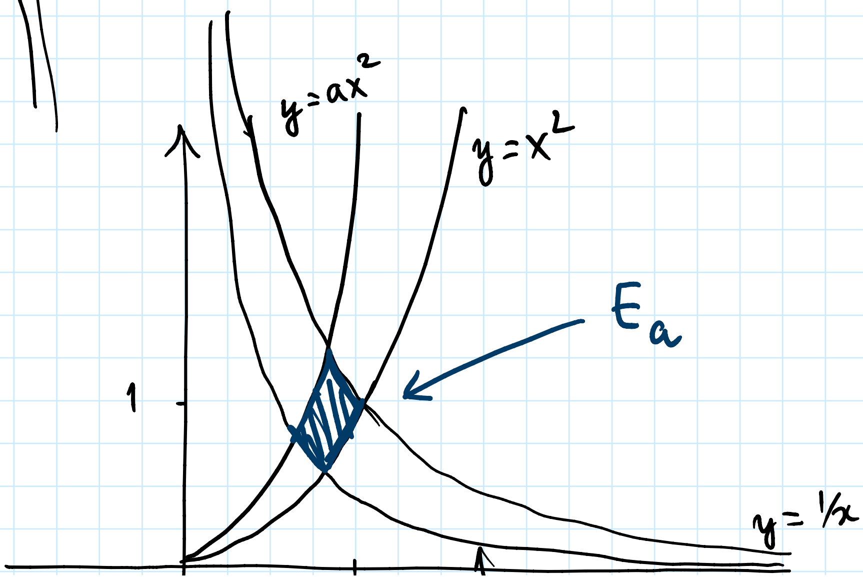
1. Draw E_a

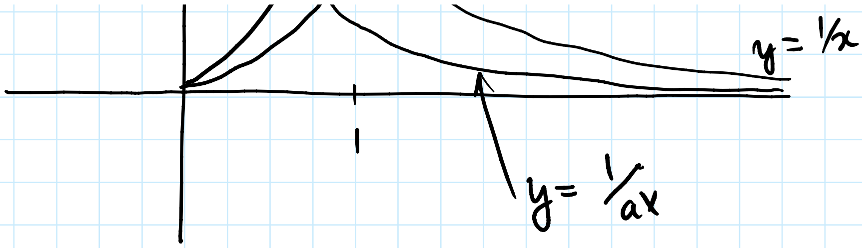


here $x^2 \leq y \leq ax^2$



Thus:





2. $\phi(x,y) = (xy, y/x^2)$. Compute

$$I(a) = \int_{E_a} \frac{x^2}{y} e^{xy} \, dx dy$$

Setting $\quad u = xy \quad v = y/x^2 \quad (u,v) = \phi(x,y)$

we have

$$I(a) = \int_{\{(u,v) \,:\, (x,y) \in E_a\}} \frac{1}{v} e^{u} \, |\det(\phi^{-1})'(u,v)| \, du dv$$

To compute the det notice that

$$|\det \phi'(x,y)| = \left| \det \begin{bmatrix} y & x \\ -2\frac{y}{x^3} & \frac{1}{x^2} \end{bmatrix} \right|$$

$$= \left| \frac{y}{x^2} + 2\frac{y}{x^2} \right| = 2 \left| \frac{y}{x^2} \right|$$

$$\equiv 2|v|$$

$$\Rightarrow \quad |\det (\phi^{-1})'(u,v)| = \frac{1}{2|v|}$$

$$\Rightarrow \quad I(a) = \int_{\underbrace{\{(u,v)\,:\,(x,y)\in E_a\}}_{?}} \frac{1}{v}\, e^{u}\, \frac{1}{2|v|}\, du\, dv$$

$$(x,y)\in E_a \iff \frac{1}{ax} \leq y \leq \frac{1}{x}\,, \quad x^2 \leq y \leq a x^2$$

$$\iff \quad \frac{1}{a} \leq xy \leq 1, \quad 1 \leq \frac{y}{x^2} \leq a$$

(we know $x > 0$ on E_a)

$$\iff \quad \frac{1}{a} \leq u \leq 1, \quad 1 \leq v \leq a$$

$$\Rightarrow \quad I(a) = \int_{1/a \leq u \leq 1,\ 1 \leq v \leq a} \frac{1}{v}\, e^{u}\, \frac{1}{2\underset{v}{\underset{\shortparallel}{|v|}}}\, du\, dv$$

$$= \int_{\frac{1}{a} \leq u \leq 1,\ 1 \leq v \leq a} \frac{1}{2v^2}\, e^{u}\, du\, dv$$

$$\overset{RF}{=} \int_{1/a}^{1} \left(\int_{1}^{a} \frac{1}{2v^2}\, e^{u}\, dv \right) du$$

$$= \frac{1}{2} \left(\int_{1/a}^{1} \frac{1}{v^2}\, dv \right) \left(\int_{1}^{a} e^{u}\, du \right)$$

$$= \frac{1}{2} \left[-\frac{1}{\ } \right]^{1} \left[e^{u} \right]^{a}$$

$$= \frac{1}{2} \left[-\frac{1}{v} \right]_{1/a}^{1} \left[e^{u} \right]_{1}^{a}$$

$$= \frac{1}{2} \left[a - 1 \right] \left(e^{a} - e \right) .$$

3. $\lim\limits_{a \to +\infty} I(a)$

Easy now: $\lim\limits_{a \to +\infty} \frac{1}{2} (a - 1)(e^{a} - e) = +\infty$

$\square$