Hellinger Versus Kullback–Leibler Multivariable Spectrum Approximation

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Abstract—In this paper, we study a matricial version of a generalized moment problem with degree constraint. We introduce a new metric on multivariable spectral densities induced by the family of their spectral factors, which, in the scalar case, reduces to the Hellinger distance. We solve the corresponding constrained optimization problem via duality theory. A highly nontrivial existence theorem for the dual problem is established in the Byrnes–Lindquist spirit. A matricial Newton-type algorithm is finally provided for the numerical solution of the dual problem. Simulation indicates that the algorithm performs effectively and reliably.

Index Terms—Approximation of multivariable power spectra, convex optimization, Hellinger distance, Kullback–Leibler index, matricial descent method.

I. INTRODUCTION

In the past ten years, building on their previous work, Byrnes, Georgiou, Lindquist, and collaborators have developed a broad program on generalized analytic interpolation and generalized moment problems that arise in spectral estimation and robust control [3], [6]–[11], [17], [21]–[27], [42]. While we refer the reader to the cited literature for better motivation, we recall that many problems of spectral estimation and robust control [3], [6]–[11], [17], [21]–[27], [42]. While we refer the reader to the cited literature for better motivation, we recall that many problems of 

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[17], [42]. As in [45] for the Kullback–Leibler case, we prefer to employ matricial descent methods. A number of nontrivial difficulties in the algorithms are overcome by resorting to ideas and results from spectral factorization theory. In our simulation, these iterative schemes (particularly a Newton-type method with backtracking line search) appear to perform effectively and reliably. This method is further analyzed and developed in [47]. There, a global convergence theorem is established. In [47], moreover, the spectrum approximation procedure introduced in this paper is applied to multivariate spectral estimation.

The paper is outlined as follows. Section II is devoted to the formulation of a generalized moment problem in the sense of Byrnes–Georgiou–Lindquist, and to the corresponding existence question. In Section III, two approximation problems for scalar spectral densities are introduced. The first employs a Kullback–Leibler-type criterion while the second features the multivariable spectrum approximation problem with respect to the normalized Lebesgue measure $d\theta/2\pi$. The question of existence of $\Phi \in S^m_{+\times m}(T)$ satisfying (3) and, when existence is granted, the parametrization of all solutions to (3), may be viewed as a generalized moment problem. For instance, in the case $m = 1$, take $G(z)$ with $k$th component $G_k(z) = e^{-k-n-1}$. Take moreover

$$\begin{align*}
A &= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0 
\end{bmatrix}, \quad B = \begin{bmatrix} 0 \\
0 \\
\vdots \\
0 \\
1 
\end{bmatrix}, \\
\Sigma &= \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{m-1} \\
c_1 & c_0 & c_2 & \cdots & c_{m-2} \\
c_2 & c_1 & c_0 & \cdots & c_{m-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
c_{m-1} & c_{m-2} & c_{m-3} & \cdots & c_0 
\end{bmatrix}
\end{align*}$$

where $c_k := E\{y(n)\bar{y}(n+k)\}$. This is the covariance extension problem, where the information available on the process $y$ is the finite sequence of covariance lags $c_0, c_1, \ldots, c_{m-1}$. It is known that the set of densities consistent with the data is nonempty if $\Sigma \geq 0$ and contains infinitely many elements if $\Sigma > 0$ [33] (see also [9], [10], [21], and [22]).

Existence of $\Phi \in S^m_{+\times m}(T)$ satisfying constraint (3) is a nontrivial issue. It was shown in [23] and [24] that such a family is nonempty if and only if there exists $H \in \mathbb{C}^{m \times n}$ such that

$$\Sigma - A\Sigma A^* = BH + H^*B'$$

or, equivalently, the following rank condition holds

$$\text{rank} \left( \begin{bmatrix} \Sigma - A\Sigma A^* & B \\
B^* & 0 \end{bmatrix} \right) = 2m. \quad (6)$$

We wish to give an alternative formulation of this existence result. First of all, notice that, without loss of generality, we can take $\Sigma = I$. Indeed, if $\Sigma \neq I$, it suffices to replace $G$ with $G' := \Sigma^{-1/2}G$ and $(A, B)$ with $(A' = \Sigma^{-1/2}A\Sigma^{1/2}, B' = \Sigma^{-1/2}B)$. Thus, constraint (3) from now on reads

$$\int G\Phi G^* = I. \quad (7)$$

Let $\Pi_B = B(B^*B)^{-1}B^*$ denote the orthogonal projection onto $\text{Range}(B)$.

Proposition 2.1: A necessary and sufficient condition for the existence of spectra in $S^m_{+\times m}(T)$ satisfying (7) is that the following relation holds

$$\langle I - \Pi_B \rangle (I - AA^*) (I - \Pi_B) = 0. \quad (8)$$

II. GENERALIZED MOMENT PROBLEM

We consider the following basic setup patterned after [25], [27], [31]. Let $S^m_{+\times m}(T)$ be the family of bounded, coercive, $\mathbb{C}^{m \times m}$-valued spectral density functions on the unit circle. Thus, a measurable, bounded matrix-valued function $\Phi$ belongs to $S^m_{+\times m}(T)$ if it satisfies the following properties:

1) the values of $\Phi$ are $m \times m$, Hermitian, nonnegative definite matrices;
2) there exists a positive constant $c_0$ such that $\Phi(e^{i\theta}) - c_0 I$ is positive definite a.e. on $T$.

Notice that $\Phi \in S^m_{+\times m}(T)$ if and only if $\Phi^{-1} \in S^m_{+\times m}(T)$. Let $\Psi \in S^m_{+\times m}(T)$ represent an a priori estimate of the spectrum of an underlying zero-mean, wide-sense stationary $m$-dimensional stochastic process $\{y(n), n \in \mathbb{Z}\}$. We consider a rational transfer function

$$G(z) = (zI - A)^{-1}B, \quad A \in \mathbb{C}^{m \times n}, B \in \mathbb{C}^{n \times m} \quad (1)$$

where $A$ has all its eigenvalues in the open unit disk, $B$ is full column rank, and $(A, B)$ is a reachable pair. Here $G$ models a bank of filters. We consider the situation where new data become available in the form of an asymptotic estimate $\Sigma > 0$ of the state covariance of the system with transfer function $G$ and input the unknown process $y$. In other words, we suppose we can estimate the covariance of the $n$-dimensional stationary process $\{x_k; k \in \mathbb{Z}\}$ satisfying

$$x_{k+1} = Ax_k + By_k, \quad k \in \mathbb{Z}. \quad (2)$$

In general, $\Psi$ is not consistent with $\Sigma$, and it is necessary to find $\Phi$ in $S^m_{+\times m}(T)$ that is closest to $\Psi$ in a suitable sense among spectra consistent with $\Sigma$, namely satisfying

$$\int G\Phi G^* = \Sigma \quad (3)$$

where a star denotes transposition plus conjugation. Here, and throughout the paper, integration takes place on $[-\pi, \pi]$ with respect to the normalized Lebesgue measure $d\theta/2\pi$. The question of existence of $\Phi \in S^m_{+\times m}(T)$ satisfying (3) and, when existence is granted, the parametrization of all solutions to (3), may be viewed as a generalized moment problem. For instance, in the case $m = 1$, take $G(z)$ with $k$th component $G_k(z) = e^{-k-n-1}$. Take moreover
When (8) is satisfied, there exists $\Phi \in S_{m}^{n \times m}(\mathbb{T})$ satisfying (7) of McMillan degree less than or equal to $2n$.

Proof: Necessity: Suppose there exists $y$-dimensional, wide-sense stationary with spectral density $\Phi \in S_{m}^{n \times m}(\mathbb{T})$ satisfying (7). Let $z$ be defined by (2). Taking covariances on both sides of (2), we get

$$I = AA^* + AE\{x_ky_k^*\}B^* + BE\{y_kx_k^*\}A^* + BE\{y_ky_k^*\}B^*.$$  

Now taking $AA^*$ to the left-hand side of the equation, and pre- and postmultiplying each side by $(I - \Pi_B)$, we obtain (8).

Sufficiency: We adapt the argument in [25, p. 1814]. For a given purely nondeterministic $m$-dimensional process $y$ with spectrum $\Phi$, define the process $w$ as the output of the linear stable system

$$x_{k+1} = Ax_k + By_k,$$  \hspace{1cm} (9)

$$w_k = (B^*B)^{-1}B^*x_{k+1}.$$  \hspace{1cm} (10)

Inverting the system (9)–(10), we get

$$x_{k+1} = (I - \Pi_B)Ax_k + Bw_k$$ \hspace{1cm} (11)

$$y_k = -(B^*B)^{-1}B^*Ax_k + w_k.$$ \hspace{1cm} (12)

Write (8) as a Lyapunov identity

$$I = (I - \Pi_B)AA^*(I - \Pi_B) + \Pi_B.$$ \hspace{1cm} (13)

Since $(A, B)$ is controllable, so is the pair $((I - \Pi_B)A, B(B^*B)^{-1/2})$. It now follows from (13) that $(I - \Pi_B)A$ has all eigenvalues in the open unit disc $\mathbb{D}$. Thus, system (11)–(12) is stable, $(B^*B)^{-1}B^*G(z)$ is minimum phase, and the processes $y$ and $w$ are causally equivalent. It follows that if we choose $w$ to be a white noise sequence with intensity $E\{w_kw_k^*\} = (B^*B)^{-1}$ and $y$ to be defined by (11) and (12), then: 1) (11) and (12) are the innovation representations of $y$; 2) the state covariance of the steady-state Kalman filter (11), (12) satisfies the Lyapunov equation (13), and is, therefore, the identity; and 3) the spectral density of $y$ is given by

$$\Phi_y = W(z)(B^*B)^{-1}W(z)^*.$$  \hspace{1cm} (14)

where

$$W(z) = I - (B^*B)^{-1}B^*A(zI - (I - \Pi_B)A)^{-1}B$$

is the transfer function of (11) and (12). We conclude that if we feed $G$ in (9) with such a process $y$, the filter state $x$ will have the required covariance, namely the identity matrix, and (7) will be satisfied. Moreover, $\Phi_y$ is rational of McMillan degree at most $2n$ and it belongs to $S_{m \times m}(\mathbb{T})$ since its values and the values of $\Phi_y^{-1}$ on $\mathbb{T}$ are positive-definite matrices.

The geometric condition (8) seems more amenable to generalization than (6). The spectrum (14) has been shown in [25, Sec. III] to be the maximum entropy spectrum among those satisfying (7). This is accomplished there in a clever way: by relating the constrained maximum entropy problem to a special one-step-ahead prediction problem.

III. CONSTRAINED SPECTRUM APPROXIMATION: THE SCALAR CASE

A. Kullback–Leibler Criterion

In [31], the Kullback–Leibler measure of distance for spectra in $S_+(\mathbb{T}) := S_{1}^{\infty}(\mathbb{T})$ was introduced

$$\mathbb{D}(\Psi\|\Phi) = \int \Psi \log \left( \frac{\Psi}{\Phi} \right).$$

As is well known, this pseudodistance originates in hypothesis testing, where it represents the mean information for observation for discrimination of an underlying probability density from another [38, p. 6]. It also plays a central role in information theory, identification, stochastic processes, etc.; see, e.g., [2], [12], [13], [15], [20], [36], [46], [50], and references therein. It is also known in these fields as divergence, relative entropy, information distance, etc. If

$$\int \Phi = \int \Psi$$

we have $\mathbb{D}(\Psi\|\Phi) \geq 0$. The choice of $\mathbb{D}(\Psi\|\Phi)$ as a distance measure, even for spectra that have different zeroth moments, is discussed in [31, Sec. III]. It is observed there that the constraint (3) often fixes the zeroth Fourier coefficient of feasible spectra (this happens for sure when $A$ is singular). In that case, rescaling $\Psi$, we are guaranteed that the index is nonnegative and equal to zero if and only if the two spectra are equal. T. Georgiou has kindly informed us [28] that even when $A$ is nonsingular, under a rather mild assumption, it is possible to modify the index so that all $\Phi$ satisfying the constraint have the same zeroth moment. In any case, the method entails a rescaling of the $a \text{ priori}$ density $\Psi$, so that the optimization problem amounts to approximating the “shape” of the $a \text{ priori}$ spectrum. This is, of course, sensible to pursue in several engineering applications such as speech processing.

We mention, for the benefit of the reader, that in the same spirit, Georgiou has very recently investigated other distances for power spectra, [29], [30]. Motivated by classical prediction theory, where the optimal one-step-ahead predictor does not depend on the $L^1$ norm of the spectrum, he seeks natural distances between rays of spectral densities. Considering the degradation of performance when an optimal predictor for one stochastic process is employed to predict a different stochastic process, he is naturally led to introduce a certain metric on rays.

As observed in the introduction, minimizing $\Phi \rightarrow \mathbb{D}(\Psi\|\Phi)$ rather than $\Phi \rightarrow \mathbb{D}(\Phi\|\Psi)$ is unusual with respect to the statistics-probability-information theory world. Besides leading to a more tractable form of the optimal solution, however, it also includes as special case (7) maximization of entropy [25]. In [31], the following problem is considered.

Problem 3.1 (Approximation problem 1): Given $\Psi \in S_+(\mathbb{T})$, find $\Phi_{KL}$ that solves

$$\text{minimize} \quad \mathbb{D}(\Psi\|\Phi) \quad \text{over} \quad \{\Phi \in S_+(\mathbb{T}) \mid \int G\Phi G^* = I\}.$$  \hspace{1cm} (16)
Remark 3.2: In the context of the covariance extension problem (4), the minimizers in Problem 3.1, when \( \Psi \) ranges over positive trigonometric polynomials of degree \( n \), are precisely the coercive spectra consistent with the first \( n \) covariance lags and of degree at most \( 2n \), [9], [10], [21], [22]. This illustrates the role of the “a priori parameter” \( \Psi \) in obtaining a description of all solutions to the moment problem of prescribed complexity.

B. Hellinger Criterion

In this paper, we consider a different metric on spectral density functions. Given \( \Phi \) and \( \Psi \) in \( S_+ (\mathbb{T}) \), the Hellinger distance is defined by

\[
d_H (\Phi, \Psi) := \left[ \int_{-\pi}^{\pi} \left( \sqrt{\Phi (e^{i\theta})} - \sqrt{\Psi (e^{i\theta})} \right)^2 \frac{d\theta}{2\pi} \right]^{1/2}.
\]

It is a bona fide distance on \( S_+ (\mathbb{T}) \). Moreover, it satisfies the following properties.

Proposition 3.3: Consider \( \Phi, \Psi \in S_+ (\mathbb{T}) \). Then
1) \( d_H (\Phi, \Psi) \leq \sqrt{\| \Phi \|_1 + \| \Psi \|_1} \);
2) \( d_H (\Phi, \Psi)^2 \leq \| \Phi - \Psi \|_1 \);
3) \( \| \Phi - \Psi \|_1 \leq \left( \sqrt{\| \Phi \|_1} + \sqrt{\| \Psi \|_1} \right) d_H (\Phi, \Psi) \).

These extend well-known properties of the Hellinger distance in the case of probability density functions. The straightforward proof may be found in [19].

Remark. On a finite-dimensional statistical manifold, endowed with the Fisher information as the metric tensor, both the Hellinger distance and the Kullback–Leibler pseudodistance can be viewed as instances of the broader concept of \( \alpha \)-divergences between two points, which arise from the so-called Amari connections. In particular, the 0-divergence, which indeed is the Hellinger distance, arises from the Levi–Civita connection. See [1, p. 66 and following].

We consider the following approximation problem.

Problem 3.4: (Approximation problem 2) Given \( \Psi \in S_+ (\mathbb{T}) \), find \( \Phi_H \) that solves

\[
\min_{\Phi \in S_+ (\mathbb{T})} d_H^2 (\Phi, \Psi) \quad \text{over} \quad \left\{ \Phi \in S_+ (\mathbb{T}) \mid G \Phi G^* = I \right\}.
\]

IV. OPTIMALITY CONDITIONS

A. Kullback–Leibler Approximation

Consider first Problem 3.1. The variational analysis in [31] is outlined as follows (see also [45]). For \( \Lambda \in \mathbb{C}^{n \times n} \) Hermitian satisfying \( G^* \Lambda G > 0 \) on all of \( \mathbb{T} \), consider the Lagrangian function

\[
L (\Phi, \Lambda) = \mathbf{D}(\Psi | \Phi) + \text{tr} \left( \Lambda \left( \int G \Phi G^* - I \right) \right)
\]

\[
= \mathbf{D}(\Psi | \Phi) + \int G^* \Lambda G \Phi - \text{tr} (\Lambda)
\]

where “tr” denotes the trace operator. Consider the unconstrained minimization of the strictly convex functional \( L (\Phi, \Lambda) \)

\[
\min L (\Phi, \Lambda) \| \Phi \in S_+ (\mathbb{T}) \}
\]

This is a convex optimization problem. The variational analysis yields the following result.

Theorem 4.1: The unique solution \( \hat{\Phi}_{KL} \) to problem (20) is given by

\[
\hat{\Phi}_{KL} = \frac{\Psi}{G^* \Lambda G}.
\]

Moreover, suppose \( \hat{\Lambda} = \Lambda^* \) is such that

\[
G^* \Lambda G > 0, \quad \forall e^{i\theta} \in \mathbb{T},
\]

\[
\int G \Psi G^* = I.
\]

Then, \( \hat{\Phi}_{KL} \) given by

\[
\hat{\Phi}_{KL} = \frac{\Psi}{G^* \Lambda G}
\]

is the unique solution of the approximation Problem (3.1).

Thus, the original Problem 3.1 is now reduced to finding \( \Lambda \) satisfying (22) and (23). This is accomplished via duality theory. Consider the dual functional

\[
\Lambda \rightarrow \inf \{ L (\Phi, \Lambda) \mid \Phi \in S_+ (\mathbb{T}) \}.
\]

For \( \Lambda \) satisfying (22), the dual functional takes the form

\[
\Lambda \rightarrow L \left( \frac{\Psi}{G^* \Lambda G}, \Lambda \right) = \int \Psi \log G \Lambda G - \text{tr} (\Lambda) + \int \Psi.
\]

Consider now the maximization of the dual functional (25) over the set

\[
\mathcal{L}^{KL} := \{ \Lambda = \Lambda^* \mid G^* \Lambda G > 0, \forall e^{i\theta} \in \mathbb{T} \}.
\]

Let, as in [31],

\[
J_\Psi (\Lambda) := - \int \Psi \log G \Lambda G + \text{tr} (\Lambda).
\]

The dual problem is then equivalent to

\[
\min \{ J_\Psi (\Lambda) \mid \Lambda \in \mathcal{L}^{KL} \}.
\]

The dual problem is also a convex optimization problem. In [31], \( \Lambda \) is further restricted to belong to the range of the operator \( \Gamma \) defined on the set \( \mathcal{D}_H (\mathbb{T}) \) of Hermitian-valued continuous functions defined on \( \mathbb{T} \) by

\[
\Gamma (\Psi) = \int G \Phi G^*, \quad \Phi \in \mathcal{D}_H (\mathbb{T}).
\]

As mentioned in Section II (5)

\[
\text{Range}(\Gamma) = \{ \Sigma = \Sigma^* : \exists H \in \mathbb{C}^{m \times n} \text{ s.t. } \Sigma - A \Sigma A^* = BH + H^* B^* \}.
\]

The problem then becomes

\[
\min \{ J_\Psi (\Lambda) \mid \Lambda \in \mathcal{L}^{KL}_H \}, \quad \mathcal{L}^{KL}_H = \mathcal{L}^{KL} \cap \text{Range}(\Gamma).
\]

The reason is that the orthogonal complement of \( \text{Range}(\Gamma) \) is given by

\[
\text{Range}(\Gamma)^\perp = \{ M = M^* \mid G^* MG \equiv 0 \text{ on } \mathbb{T} \}.
\]
This follows from the fact that $M \in \text{Range}(\Gamma)^\perp$ iff $\forall \Phi \in \mathcal{C}_H(\mathbb{T})$

$$0 = \text{tr} \left( \int G\Phi G^* M \right) = \int G^* M \text{G} \Phi.$$  


The dual functional is shown in [31] to be strictly convex on the restricted domain $L^1_{KL}$. It is also shown in [11] that $J_\psi$ has a unique minimum point in $L^1_{KL}$. This result implies that, under assumption (6), there exists a (unique) $\hat{\Lambda}$ in $L^1_{KL}$ satisfying (23). Such a $\hat{\Lambda}$ then provides the optimal solution of the primal problem (15) and (16) via (24).

**B. Hellinger Approximation**

The variational analysis for Problem 3.4 is very similar (see [19] for details). We state without proof the following result: it will be proven in Section VII in the (more general) multivariable case.

**Theorem 4.2:** Assume that Problem 3.4 is feasible, namely that condition (6) (or, equivalently, condition (8)) is satisfied. Then, there exists $\hat{\Lambda} = \Lambda^* \in \mathbb{C}^{n \times n}$ such that

$$1 + G^* \hat{\Lambda} G > 0, \quad \forall \epsilon^{i\theta} \in \mathbb{T}, \quad \int \frac{G^*}{1 + G^* \hat{\Lambda} G} G = I. \quad (32)$$

In this case, Problem 3.4 admits a unique solution which is given by

$$\hat{\Phi}_H = \frac{\Psi}{1 + G^* \hat{\Lambda} G}. \quad (33)$$

**Remark 4.3:** Suppose the a priori density $\Psi$ is rational. Then, the solution in (33) has, in general, degree $2n$ higher than the solution in (24).

**V. GENERALIZING TO THE MULTIVARIABLE CASE: FIRST RESULTS AND DIFFICULTIES**

In this section, we state and derive some results on multivariable spectrum approximation where a “natural” generalization of the scalar Kullback–Leibler and Hellinger distance, respectively, is employed. We also point out the difficulties involved in these approaches that bring to a sudden stop the variational analysis.

**A. Kullback–Leibler Approximation**

Multivariable Kullback–Leibler approximation has been investigated in [25] and [27], whereas [3] deals with the multivariate Nevanlinna–Pick problem. In statistical quantum mechanics, the state of an $n$-level system is represented by a density matrix $\rho$, namely a Hermitian, positive-semidefinite matrix in $\mathbb{C}^{n \times n}$ with unit trace [49]. The convex set of density matrices has as extreme points the 1-D projections. The latter can be identified with the pure states of the system $|\psi\rangle$, where $\psi$ is a unit vector in $\mathbb{C}^n$, via $\rho = \langle \psi, \cdot \rangle \psi$. Quantum analogs of entropy-like functionals have been considered since the early days of quantum mechanics [51]. Recently, renewed interest has originated in quantum information applications [44]. The quantum relative entropy between two density matrices is defined by

$$D(\rho||\sigma) := \text{tr}(\rho(\log \rho - \log \sigma)). \quad (34)$$

Klein’s inequality yields that $D(\rho||\sigma) \geq 0$, and $D(\rho||\sigma) = 0$ if and only if $\rho = \sigma$. Moreover, as in the classical case, the quantum relative entropy is jointly convex in its arguments. We are then led to the following definition: Given $\Phi$ and $\Psi$ in $\mathbb{S}^n_{\times n}(\mathbb{T})$, the relative entropy $D(\Psi||\Phi)$ is given by

$$D(\Psi||\Phi) = \int \text{tr}(\Psi(\log \Psi - \log \Phi)). \quad (35)$$

First of all, we need to worry about nonnegativity of $D(\Psi||\Phi)$ and whether it is zero iff $\Psi = \Phi$.

**Proposition 5.1:** Let $\Phi, \Psi \in \mathbb{S}^n_{\times n}(\mathbb{T})$. Define $\Psi_1 = \Psi/\text{tr}\Psi$ and $\Phi_1 = \Phi/\text{tr}\Phi$.

$$D(\Psi||\Phi) = D(\text{tr}\Psi||\text{tr}\Phi) + \int (\text{tr}\Psi)(\Psi_1(\log \Psi_1 - \log \Phi_1)). \quad (36)$$

It follows that when $\int \text{tr}\Psi = \int \text{tr}\Phi$, then $D(\Psi||\Phi) \geq 0$. Moreover, $D(\Psi||\Phi) = 0$ if and only if the two spectra coincide.

**Proof:**

$$D(\Psi||\Phi) = \text{tr} \int \Psi (\log \Psi - \log \Phi)$$

$$= \text{tr} \int \text{tr}(\Psi)(\Psi_1(\log \Psi_1 - \log \Phi_1) + \text{tr}(\Phi))$$

$$= \text{tr} \int \text{tr}(\Psi)(\Psi_1(\log \Psi_1 - \log \Phi_1)) + \text{tr}(\Psi)$$

$$= \text{tr} \int \text{tr}(\Psi)(\Psi_1(\log \Psi_1 - \log \Phi_1)) + D(\text{tr}\Psi||\text{tr}\Phi).$$

Since $\text{tr}\Psi(e^{i\theta}) = \text{tr}\Phi(e^{i\theta}) = 1, \forall \theta \in [\pi, \pi]$, it follows from Klein’s inequality that

$$\text{tr}\Psi_1(e^{i\theta})(\log \Psi_1(e^{i\theta}) - \log \Phi_1(e^{i\theta})) \geq 0, \forall \theta.$$  

The latter implies that

$$\int (\text{tr}\Psi)(\log \Psi_1 - \log \Phi_1) \geq 0.$$  

When $\int \text{tr}\Psi = \int \text{tr}\Phi$, we also have $D(\text{tr}\Psi||\text{tr}\Phi) \geq 0$. Thus, when $\int \text{tr}\Psi = \int \text{tr}\Phi$, $D(\Psi||\Phi)$ is the sum of two nonnegative terms and the conclusion follows.

Consider now the multivariable version of Problem 3.1.
Problem 5.2 (Approximation problem I): For \( \Psi \in S^x_+ (T) \)

\[
\text{minimize} \quad D(\Psi \| \Phi)
\]

over \( \left\{ \Phi \in S^x_+ (T) \mid G\Phi G^* = I \right\} \) \hspace{1cm} (37)

where \( D(\Psi \| \Phi) \) is defined by (35). As in the scalar case, an \textit{a posteriori} rescaling of the prior density is, in general, necessary.

In the light of Proposition 5.1, if \( \Phi \) is the solution of (5.2), the new prior is

\[
\hat{\Psi} = \frac{\int \text{tr} \hat{\Phi}}{\int \text{tr} \Psi} \Psi.
\]

For \( \Lambda \in \mathbb{C}^{n \times n} \) Hermitian such that \( G^* \Lambda G \) is positive definite on all of \( T \), define again the Lagrangian

\[
L(\Phi, \Lambda) = D(\Psi \| \Phi) + \text{tr} \left( \Lambda \left( \int G\Phi G^* - I \right) \right) = D(\Psi \| \Phi) + \text{tr} \left( G^* \Lambda G \Phi - \text{tr} (\Lambda) \right). \hspace{1cm} (39)
\]

The following step, entailing the \textit{unconstrained} minimization of the strictly convex functional \( L(\Phi, \Lambda) \) on \( \Psi \in S^x_+ (T) \), is a stumbling block. The optimality condition reads [27, Sec. IV]

\[
\int_0^\infty (\hat{\Phi}_{KL} + \tau I)^{-1} \Psi (\hat{\Phi}_{KL} + \tau I)^{-1} d\tau = G^* \Lambda G. \hspace{1cm} (40)
\]

In general, an explicit expression for \( \hat{\Phi}_{KL} \) in terms of \( \Psi \) and \( \Lambda \) cannot be obtained, and the variational analysis ends here. We mention that the minimization with respect to the first argument of the relative entropy can, instead, be carried out explicitly, leading to a solution of the exponential form

\[
\Phi^\omega = e^{\text{exp} [\log \Psi - G^* \Lambda G]}
\]

see [27, Sec. IV]). Homotopy-like methods are described in [27] to find \( \Lambda \), when it exists, such that \( \Phi^\omega \) satisfies the constraint.

B. Hellinger Approximation

Recall that, for a positive semidefinite Hermitian matrix \( M, M^{1/2} \) is the square root of \( M \), namely the unique positive semidefinite Hermitian matrix whose square is \( M \). If \( V \) is a unitary matrix that diagonalizes \( M \) so that \( M = V^* \text{diag}(\alpha_1, \ldots, \alpha_n) V \), then simply \( M^{1/2} = V^* \text{diag}(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_n}) V \). Motivated by the analogy with the Kullback–Leibler case, and by the scalar case, we define the Hellinger distance for \( \Phi \) and \( \Psi \) in \( S^x_+ (T) \) to be

\[
d^2_H (\Phi, \Psi) := \int_{-\pi}^\pi \left[ \Phi^{1/2} (e^{i\theta}) - \Psi^{1/2} (e^{i\theta}) \right]^2 d\theta. \hspace{1cm} (41)
\]

Notice that (41) appears also as the natural generalization of the Hellinger distance for density operators of statistical quantum physics introduced in [41]. Consider again the strictly convex Problem 3.4

\[
\text{minimize} \quad d^2_H (\Phi, \Psi) \hspace{1cm} (42)
\]

over \( \left\{ \Phi \in S^x_+ (T) \mid G\Phi G^* = I \right\} \) \hspace{1cm} (43)

where \( d^2_H (\Phi, \Psi) \) is now given by (41). Define \( L^H \) by

\[
L^H (\Phi, \Lambda) = d^2_H (\Phi, \Psi) + \text{tr} \left( \Lambda \left( \int G\Phi G^* - I \right) \right).
\]

For \( \Lambda \in \mathcal{L}^H \), consider the Lagrangian

\[
\delta S(P, \delta P) := \lim_{\varepsilon \to 0} \frac{(P + \varepsilon \delta P)^{1/2} - P^{1/2}}{\varepsilon}.
\]

Employing the chain rule, it is easy to see that

\[
\delta S(P, \delta P) P^{1/2} + P^{1/2} \delta S(P, \delta P) = \delta P
\]

so that

\[
\delta S(P, \delta P) = \int_0^\infty \exp(-P^{1/2} t) \delta P \exp(-P^{1/2} t) dt. \hspace{1cm} (45)
\]

Taking (45) into account, we get the optimality condition

\[
\int_0^\infty \left[ \exp(-\hat{\Phi}_{H1} (t)) (\hat{\Phi}_{H1} (t) - \Psi^{1/2}) \exp(-\hat{\Phi}_{H1} (t)) \right] dt + \frac{1}{2} G^* \Lambda G = 0. \hspace{1cm} (46)
\]

The integral in (45) is the unique solution of the Lyapunov equation

\[
\hat{\Phi}_{H1}^{1/2} X + X \hat{\Phi}_{H1}^{1/2} = \hat{\Phi}_{H1}^{1/2} - \Psi^{1/2}. \hspace{1cm} (47)
\]

Equations (46) and (47) now yield

\[
-\frac{1}{2} \hat{\Phi}_{H1}^{1/2} (G^* \Lambda G) - \frac{1}{2} (G^* \Lambda G) \hat{\Phi}_{H1}^{1/2} = \hat{\Phi}_{H1}^{1/2} - \Psi^{1/2}
\]

which, in turn, gives

\[
\hat{\Phi}_{H1}^{1/2} (I + G^* \Lambda G) + (I + G^* \Lambda G) \hat{\Phi}_{H1}^{1/2} = 2\Psi^{1/2}. \hspace{1cm} (48)
\]

Since \( I + G^* \Lambda G > 0 \) almost everywhere on \( T \), we finally get

\[
\hat{\Phi}_{H1}^{1/2} = 2 \int_0^\infty \exp \left[ -(I + G^* \Lambda G) t \right] \Psi^{1/2} \exp \left[ -(I + G^* \Lambda G) t \right] dt. \hspace{1cm} (49)
\]

The maximization of the dual functional \( \Lambda \to L^H (\hat{\Phi}, \Lambda) \), however, appears quite problematic.

We show in the next section that, differently from the Kullback–Leibler case, it is possible to define a sensible Hellinger distance for matrical functions that leads to a full unraveling of the optimization problem. This will be accomplished by connecting this problem to a most classical topic at the hearth of systems and control theory, namely the spectral factorization problem.
VI. HELLINGER DISTANCE AND SPECTRAL FACTORIZATION

Let $F$ be a measurable function defined on the unit circle $\mathbb{T}$ and taking values in $\mathbb{C}^{m \times p}$. Then, $F$ belongs to the Hilbert space $L_2^{m \times p}$ if it satisfies $\int \text{tr}(FF^*) < \infty$. For $F, G$ in $L_2^{m \times p}$, the scalar product is defined by

$$\langle F, G \rangle_2 = \int \text{tr}(FG^*)$$

so that $\|F\|_2^2 = \int \text{tr}(FF^*)$. Let $\Phi \in S_+^{m \times m}(\mathbb{T})$. Then, a measurable $\mathbb{C}^{m \times p}$-valued function $W$ is called a spectral factor of $\Phi$ if it satisfies

$$W(e^{i\theta})W(e^{i\theta})^* = \Phi(e^{i\theta}), \quad \text{a.e. on } \mathbb{T}.$$ 

Notice that necessarily $p \geq m$ and $W(e^{i\theta})$ is a.e. full row rank. Moreover, $W$ is bounded on $\mathbb{T}$, and, therefore, it belongs to $L_2^{m \times p}$. When $p = m$, $W^{-1}$ is also bounded, and consequently, $W^{-1} \in L_2^{m \times m}$. Any $\Phi \in S_+^{m \times m}(\mathbb{T})$ satisfies the Szeg"{o} condition

$$\int_{-\pi}^{\pi} \log \det \Phi(e^{i\theta}) \frac{d\theta}{2\pi} > -\infty$$

and admits therefore spectral factors $W$ in $H_2^{m \times m}$, namely the Hardy space of functions in $L_2^{m \times m}$ that possess an analytic extension in $|z| > 1$ (see, e.g., [34] and [48]).

Let $W_1$ and $W_2$ be spectral factors of the same $\Phi \in S_+^{m \times m}(\mathbb{T})$ with $W_1$ square. Then, trivially $U := W_1^{-1}W_2$ is an $m \times p$ all-pass function, i.e.

$$U(e^{i\theta})U(e^{i\theta})^* = I \quad \forall e^{i\theta} \in \mathbb{T}.$$ 

For $\Phi, \Psi \in S_+^{m \times m}(\mathbb{T})$, consider the following function

$$\tilde{d}_H(\Phi, \Psi) = \inf \left\{ \|W_\Psi - W_\phi\|_2^2 : W_\phi, W_\Psi \in L_2^{m \times m} \right\},$$

where $W_\phi W_\phi^* = \Psi, W_\phi W_\phi^* = \Phi$. Theorem 6.1: The following facts hold true:

1) For any square spectral factor $W_\Phi$ of $\Phi$, we have

$$\tilde{d}_H(\Phi, \Psi) = \inf \left\{ \|W_\Psi - W_\phi\|_2^2 : W_\phi \in L_2^{m \times m} \right\},$$

where $W_\phi W_\phi^* = \Phi$. Indeed, the unique spectral factor of $\Phi$ minimizing (51) is given by

$$W_\phi := \Phi^{1/2} \left( \Phi^{1/2} \Psi \Phi^{1/2} \right)^{-1/2} \Phi^{1/2} W_\Psi.$$ 

3) $\tilde{d}_H$ is a bona fide distance function.

4) $\tilde{d}_H$ coincides with the Hellinger distance in the scalar case.

Proof:

1) First of all, observe that, once fixed the spectral factor $W_\Psi$, any square spectral factor $W_\phi$ of $\Psi$ can be written as $W_\phi = W_\Phi U$, where $U$ is an $m \times m$ all-pass. Hence

$$\int \text{tr}(W_\Psi - W_\phi)(W_\Psi - W_\phi)^* d\theta$$

$$= \int \text{tr}(W_\Psi - W_\phi U^*)(W_\Psi - W_\phi U^*)^* d\theta.$$ 

Observe, moreover, that $W_\Phi U^*$ is a square spectral factor of $\Phi$, so that (51) holds.

2) To show that the infimum in (51) is a minimum, notice that (51) may be rewritten in the form

$$\tilde{d}_H(\Phi, \Psi)^2 = \inf \left\{ \int \text{tr}(W_\Psi - \Phi^{1/2} V)(W_\Psi - \Phi^{1/2} V)^* d\theta : V \in L_\infty^{m \times m}, VV^* = I \right\},$$

where $\Delta = \Delta^* > 0$. The first variation of the Lagrangian (at $V$ in direction $\delta V \in L_\infty^{m \times m}$) is

$$\delta L(V; \delta V) = \int \text{tr}[\Delta V - \Phi^{1/2} W_\Psi] \delta V^* + \delta V(\Delta V - \Phi^{1/2} W_\Psi)^*].$$

The second variation of the Lagrangian is

$$\delta^2 L(V; \delta v) = 2 \int \text{tr}[\Phi^{1/2} \delta V \delta V^* \Phi^{1/2} + \Delta(\delta V \delta V^* \Delta - 1)].$$

Hence, $L$ is strictly convex, and therefore, $V$ is a minimizer of the unconstrained minimization problem if and only if

$$\delta L(V; \delta V) = 0, \quad \forall \delta V.$$ 

Condition (53) is clearly equivalent to $\Delta V - \Phi^{1/2} W_\Psi = 0$ or to

$$V = \Delta^{-1} \Phi^{1/2} W_\Psi.$$ 

Thus, if there exists $\Delta = \Delta^* > 0$ such that

$$V V^* = \Delta^{-1} \Phi^{1/2} \Psi \Phi^{1/2} \Delta^{-1} = I$$

then $V$ minimizes (52). Such a $\Delta$ is readily seen to be given by

$$\Delta = \left[ \Phi^{1/2} \Psi \Phi^{1/2} \right]^2.$$ 

In conclusion, the infimum in (51) is a minimum and

$$W_\Psi = \Phi^{1/2} V = \Phi^{1/2} \left[ \Phi^{1/2} \Psi \Phi^{1/2} \right]^{-1/2} \Phi^{1/2} W_\Psi$$

is the unique minimizer.

3) The distance properties of $\tilde{d}_H$ are easy to check: (i) Symmetry is an immediate consequence of the definition of $\tilde{d}_H$. (ii) It is clear that $\tilde{d}_H(\Phi, \Psi) = 0$. Conversely, if $\tilde{d}_H(\Phi, \Psi) = 0$, then $\Phi$ and $\Psi$ share a.e. a common spectral factor, and are therefore, a.e. the same spectral density. (iii) The triangular inequality is inherited by the definition of $\tilde{d}_H$ as the infimum of the $L_2$ distance among spectral factors. Thus, given $\Phi, \Psi$, and $\mathbb{T}$ and choosing an arbitrary square spectral factor $W_\mathbb{T}$ of $\mathbb{T}$, we have

$$\tilde{d}_H(\Phi, \Psi) = \inf_{W_\Phi, W_\Psi} \|W_\Phi - W_\Psi\|_2$$

$$\leq \inf_{W_\Phi, W_\Psi} \|W_\Phi - W_\Psi\|_2 + \|W_\Psi - W_\mathbb{T}\|_2.$$
We establish the existence of a unique spectral factor of the optimal solution. Equations (49) and (59) give the same form for the optimal solution. The multivariable version of Problem 3.4 is the following.

(VII) \( \hat{d}_H \) -OPTIMAL MULTIVARIABLE SPECTRUM APPROXIMATION

Theorem 6.1 shows that \( \hat{d}_H \) is a natural extension to the multivariable case of the Hellinger distance. The corresponding multivariable version of Problem 3.4 is the following.

Problem 7.1: Given \( \Psi \in S_+^{m \times m} (T) \), find \( \Phi \in S_+^{m \times m} (T) \) that solves

\[
\begin{align*}
\text{minimize} & \quad \Phi_H (\Phi, \Psi) \\
\text{subject to} & \quad \int G \Phi G^* = I.
\end{align*}
\]

It is in this form that the optimization problem is amenable to the variational analysis even in multivariable version. Let

\[ \mathcal{L}^H := \{ \Lambda \in \mathbb{C}^{n \times n} | \Lambda = \Lambda^*, I + G^* \Lambda G > 0 \forall e^{i\theta} \in T \} \]

and

\[ \mathcal{L}^H \Gamma := \mathcal{L}^H \cap \text{Range}(\Gamma) \]

where \( \Gamma \) was defined in (28). The following is our main result.

Theorem 7.2: Assume condition (6) [or, equivalently, condition (8)] is satisfied. Then, there exists a unique \( \Lambda \in \mathcal{L}^H \Gamma \) such that

\[ \int G(I + G^* \hat{\Lambda} G)^{-1} \Psi(I + G^* \hat{\Lambda} G)^{-1} G^* = I. \]

The unique solution of the constrained approximation Problem 7 is then given by

\[ \hat{\Phi}_H := (I + G^* \hat{\Lambda} G)^{-1} \Psi(I + G^* \hat{\Lambda} G)^{-1}. \]

Remark 7.3: Let \( \Psi_0 \in S_+ (T) \) and suppose \( \Psi = \Psi_0 I \) has the form of a scalar matrix. Then, a simple calculation shows that (49) and (59) give the same form for the optimal solution \( \hat{\Phi} \).

We break the proof of Theorem 7.2 into two parts: First, by unconstrained minimization of the Lagrangian function, we obtain an expression for a spectral factor of the optimal \( \Phi \) depending on the Lagrange multiplier matrix \( \Lambda \) (Lemma 7.4). Second, we establish the existence of a unique \( \Lambda \in \mathcal{L}^H \Gamma \) satisfying (58) (Theorem 7.7).

For \( \Lambda \in \mathcal{L}^H \), \( \Psi \) a spectral factor of \( \Psi \), and \( W, W^{-1} \in L_{\infty}^{m \times m} (T) \), form the Lagrangian function

\[ L(W, \Lambda) = \text{tr} \left( (I + G^* \Lambda G)^{-1} W \Psi (I + G^* \Lambda G)^{-1} W^* \right) + \text{tr} \left( \int G W W^* G^* - I \right). \]

Consider the unconstrained minimization problem

\[ \min_W \{ L(W, \Lambda) | W, W^{-1} \in L_{\infty}^{m \times m} (T) \}. \]

Lemma 7.4: The unique solution to problem (61) is given by

\[ \hat{W} = (I + G^* \Lambda G)^{-1} W \Psi. \]

Proof: Let \( \delta W \in L_{\infty}^{m \times m} (T) \). The first variation of the Lagrangian is:

\[ \delta L(W, \Lambda; \delta W) = \text{tr} \left( [\delta W (W - W \Psi)^* + (W - W \Psi) \delta W^* + \Lambda (G \delta W W^* G^* + GW \delta W^* G^* + GW^* \delta W^* G^*)] \right) \]

\[ = \text{tr} \left( (W - W \Psi + G^* \Lambda G W) \delta W^* \right) \]

\[ + \text{tr} \left( \int (W - W \Psi + G^* \Lambda G W) \delta W^* \right). \]

By taking into account the cyclic property of the trace operator, the second variation of the Lagrangian is easily seen to be given by

\[ \delta^2 L(W, \Lambda; \delta W) = 2 \text{tr} \int \delta W^* (I + G^* \Lambda G) \delta W \]

which is clearly positive for any \( \Lambda \in \mathcal{L}^H \) and \( \delta W \neq 0 \). Hence \( L \) is strictly convex with respect to \( W \). Moreover, the set \( \mathcal{L}^H \) is open and convex. To find the minimum point of \( L \), we impose \( \delta L(W, \Lambda; \delta W) = 0 \) in each direction \( \delta W \). This yields (62). We now consider the question of existence of a matrix \( \Lambda \in \mathcal{L}^H \) satisfying (58). To this end, we introduce the dual functional

\[ L(\hat{W}, \Lambda) \]

\[ = \text{tr} \left( (I + G^* \Lambda G)^{-1} W \Psi - W \Psi \right) \]

\[ \times ((I + G^* \Lambda G)^{-1} W \Psi - W \Psi)^* \]

\[ + \text{tr} \left[ \Lambda \left( \int G(I + G^* \Lambda G)^{-1} W \Psi \times W \Psi (I + G^* \Lambda G)^{-1} G^* - I \right) \right] \]

\[ = \text{tr} \left( \Psi - (I + G^* \Lambda G)^{-1} \Psi \right) - \text{tr} \Lambda, \quad \Lambda \in \mathcal{L}^H. \]

Instead of maximizing (64), we consider the equivalent problem of minimizing the functional

\[ J_\Psi(\Lambda) := -L(\hat{W}, \Lambda) + \text{tr} \int \Psi \]

\[ = \text{tr} \left( (I + G^* \Lambda G)^{-1} \Psi \right) + \text{tr} \Lambda, \quad \Lambda \in \mathcal{L}^H. \]

Lemma 7.5: The functional (65) is convex and its restriction to \( \mathcal{L}^H \Gamma \) [defined in (57)] is strictly convex.

Proof: First of all, observe that \( \mathcal{L}^H \) is an open, convex subset of the Hermitian matrices in \( C^{n \times n} \). For \( \delta \Lambda \in C^{n \times n} \) Hermitian, we compute the directional derivative

\[ \delta J_\Psi(\Lambda; \delta \Lambda) \]

\[ = -\text{tr} \left( (I + G^* \Lambda G)^{-1} G^* \delta \Lambda G(I + G^* \Lambda G)^{-1} \Psi \right) + \text{tr} \delta \Lambda \]

\[ = \text{tr} \left[ \left( I - \int G(I + G^* \Lambda G)^{-1} \Psi(I + G^* \Lambda G)^{-1} G^* \right) \delta \Lambda \right]. \]
The second variation is then given by
\[
\delta^2 J_{\Psi}(\Lambda; \delta \Lambda) = 2 \text{tr} \int W^*_\Psi (I + G^* G)^{-1} G^* \delta G \times \\
(I + G^* G)^{-1} G^* \delta G (I + G^* G)^{-1} W_{\Psi}
\]
(67)
which is clearly a nonnegative quantity. Hence, \( J_{\Psi} \) is convex on \( \mathcal{L}_I^H \). In view of (31), we have that \( \delta^2 J_{\Psi}(\Lambda; \delta \Lambda) \) is strictly positive for any nonzero \( \delta \Lambda \in \text{Range}(\Gamma) \), and consequently, \( J_{\Psi} \) is strictly convex on \( \mathcal{L}_I^H \). As an immediate consequence of the aforesaid lemma, we have the following corollary.

**Corollary 7.6:** The dual problem

Find \( \Lambda \in \mathcal{L}_I^H \) minimizing \( J_{\Psi}(\Lambda) \)

admits at most one solution. Moreover, (58) is necessary and sufficient for \( \Lambda \) to solve the dual problem (68).

We now tackle the existence issue for the dual problem. Although this is a finite-dimensional, convex optimization problem, the existence question is quite delicate since the set \( \mathcal{L}_I^H \) is open and unbounded. The proof of the following theorem is partially inspired by the proof of the corresponding result for the scalar, Kullback–Leibler case in [Sec. 2] [11].

**Theorem 7.7:** If Problem 7.1 is feasible, i.e., (6) [or, equivalently, condition (8)] is satisfied, then the dual functional (65) has a unique minimum point in \( \mathcal{L}_I^H \).

**Proof:** In view of Corollary 7.6, we only need to show that \( J_{\Psi} \) takes a minimum value on \( \mathcal{L}_I^H \). First, we observe that \( J_{\Psi} \) is continuous on its domain. Second, we show that \( J_{\Psi} \) is bounded below on \( \mathcal{L}_I^H \). Indeed, by feasibility, there exists a \( \bar{\Phi} \in S_{n \times m}^+ (\mathbb{T}) \) such that \( \int \bar{\Phi} G^* = I \). Hence, for all \( M \in \mathbb{C}^{n \times n}, \int \bar{\Phi} G^* M = M \), which implies

\[
\text{tr} M = \text{tr} \int \bar{\Phi}^{1/2} G^* M G \bar{\Phi}^{1/2}. \tag{69}
\]
Recalling that, for \( \Lambda \in \mathcal{L}_I^H \), \( I + G^* (e^{i \theta}) A G (e^{i \theta}) \) is positive definite for all \( \theta \in [0, 2\pi] \), and using the monotonicity property of the trace, we get

\[
\text{tr} \Lambda = \text{tr} \int \bar{\Phi}^{1/2} G^* A G \bar{\Phi}^{1/2} > \text{tr} \int \bar{\Phi}, \quad \forall \Lambda \in \mathcal{L}_I^H. \tag{70}
\]
Define \( \bar{f} := \text{tr} \int \bar{\Phi} < 0 \). We get

\[
J_{\Psi}(\Lambda) := \text{tr} \int (I + G^* \Lambda G)^{-1} \Psi + \text{tr} \Lambda \\
= \text{tr} \int \Psi^{1/2} (I + G^* \Lambda G)^{-1} \Psi^{1/2} + \text{tr} \Lambda \\
> \text{tr} \bar{f}, \quad \forall \Lambda \in \mathcal{L}_I^H. \tag{71}
\]
where we have used \( \text{tr} \int \Psi^{1/2} (I + G^* \Lambda G)^{-1} \Psi^{1/2} > 0 \) on \( \mathcal{L}_I^H \).

Finally, we show that \( J_{\Psi} \) is inf-compact, i.e., the sublevel sets \( J_{\Psi}^{-1}(-\infty, r) \) are compact. This implies existence of a minimum point. Indeed, observing that \( J_{\Psi}(0) = \text{tr} \int \Psi \), we can then restrict the search for a minimum point to the compact set \( J_{\Psi}^{-1}(-\infty, \text{tr} \int \Psi) \). Existence for the latter problem then follows from a version of Weierstrass Theorem since an inf-compact function has closed level sets, and is therefore, lower semicontinuous [37, p. 56]. To prove inf-compactness of \( J_{\Psi} \), we proceed to show that

1) \[
\lim_{\Lambda \to \partial \mathcal{L}_I^H} J_{\Psi}(\Lambda) = +\infty
\]
where \( \partial \mathcal{L}_I^H \) denotes the boundary of \( \mathcal{L}_I^H \).

2) \[
\lim_{\|\Lambda\| \to \infty} J_{\Psi}(\Lambda) = +\infty.
\]
To prove property 1), notice that \( \partial \mathcal{L}_I^H \) is the set of \( \Lambda \) in \( \text{Range}(\Gamma) \) for which: i) \( I + G^* \Lambda G \) is positive semidefinite on \( \mathbb{T} \) and ii) \( \exists \theta \) s.t. \( I + G^* (e^{i \theta}) A G (e^{i \theta}) \) is singular. Thus, for \( \Lambda \to \partial \mathcal{L}_I^H \), all the eigenvalues of \( [I + G^* \Lambda G]^{-1} \) are positive on \( \mathbb{T} \) and at least one of them has a pole tending to the unit circle \( \{I + G^* \Lambda G\}^{-1} \) has all eigenvalues positive on \( \mathbb{T} \) and at least one of them with a pole tending to the unit circle as \( \Lambda \to \partial \mathcal{L}_I^H \). Rewrite now \( J_{\Psi} \), as in (71), in the form \( J_{\Psi} = \text{tr} \int \Psi^{1/2} (I + G^* \Lambda G)^{-1} \Psi^{1/2} + \text{tr} \Lambda \). Since \( \text{tr} \Lambda \) is bounded below in view of (70), we get the conclusion.

Point 2) is more delicate. Let \( \Lambda_k \in \mathcal{L}_I^H \) be a sequence such that \( \lim_{k \to \infty} \|\Lambda_k\| = \infty \). Let \( \Lambda_k^0 := \frac{\Lambda_k}{\|\Lambda_k\|} \). It is easy to see that if \( \Lambda \in \mathcal{L}_I^H \), then \( \alpha \Lambda \in \mathcal{L}_I^H \) for all \( \alpha \in [0, 1] \). Hence, for sufficiently large \( k \), we have \( \Lambda_k^0 \in \mathcal{L}_I^H \).

Let \( \eta := \inf \text{tr} \Lambda_k^0 \). We want to show that \( \eta \) is strictly positive. We first observe that \( \eta \geq 0 \). In fact, \( \text{tr} \Lambda_k^0 = \frac{1}{\|\Lambda_k\|} \text{tr} \Lambda_k > \frac{1}{\|\Lambda_k\|} \int \bar{f} \to 0 \), where we have used (70).

Consider a subsequence of \( \Lambda_k^0 \) such that the limit of its trace is \( \eta \). Since this subsequence remains on the surface of the unit ball \( \partial B := \{ \Lambda = \Lambda^* : \|\Lambda\| = 1 \} \), which is compact, it has a subsubsequence converging in \( \partial B \). Let \( \Lambda_\infty \) be such a subsubsequence, and let \( \Lambda_\infty \in \partial B \) be its limit. Clearly,

\[
\lim_{i \to \infty} \text{tr} \Lambda_k^0 = \text{tr} \Lambda_\infty = \eta. \tag{72}
\]
We now prove that \( \Lambda_\infty \in \mathcal{L}_I^H \). To this aim, notice that \( \Lambda_\infty \) is the limit of a sequence in the finite-dimensional linear space \( \text{Range}(\Gamma) \), and hence, it belongs to the same space \( \text{Range}(\Gamma) \). It remains to show that \( (I + G^* \Lambda_\infty G) \) is positive definite on \( \mathbb{T} \). Indeed, since (the unnormalized subsequence) \( \Lambda_k \) belongs to \( \mathcal{L}_I^H \), we have that \( (I + G^* \Lambda_k G) \) is positive definite on \( \mathbb{T} \) so that \( \Lambda_k^0 = \frac{\Lambda_k}{\|\Lambda_k\|} \) is also positive definite on \( \mathbb{T} \) for each \( i \). Taking the limit for \( i \to \infty \), we get that \( G^* \Lambda_\infty G \) is positive semidefinite on \( \mathbb{T} \) so that \( (I + G^* \Lambda_\infty G) \) is strictly positive definite on \( \mathbb{T} \). This proves that \( \Lambda_\infty \in \mathcal{L}_I^H \). The latter, together with (69) yields

\[
\text{tr} \Lambda_\infty = \text{tr} \int \bar{\Phi}^{1/2} G^* \Lambda_\infty G \bar{\Phi}^{1/2}. \tag{73}
\]
As seen before, \( G^* \Lambda_\infty G \) is positive semidefinite on \( \mathbb{T} \). Moreover, \( G^* \Lambda_\infty G \) is not identically zero since \( \Lambda_\infty \in \text{Range}(\Gamma) \) [see (31)], and \( \Lambda_\infty \neq 0 \) (it is not the zero matrix) since \( \Lambda_\infty \in \partial B \).

We conclude, in view of (72) and (73), that \( \eta = \text{tr} \Lambda_\infty > 0 \).
Finally, we have
\[ J_\Psi(\Lambda_k) = \text{tr}\left[ \Phi^{1/2}(I + G^*\Lambda_k G)^{-1}\Phi^{1/2} + \text{tr}\Lambda_k \right] \geq \|\Lambda_k\| \text{tr}\Lambda_k^0. \] (74)
Since \( \|\Lambda_k\| \to \infty \) and \( \lim \inf \text{tr}\Lambda_k^0 > 0 \), we get
\[ \lim_{k \to \infty} J_\Psi(\Lambda_k) = +\infty. \] (75)

Let \( \hat{\Lambda} \in L_H^H \) be the unique solution of the dual problem whose existence has just been proven in Theorem 7.7. We show below that it also provides via (59) the unique solution to the primal problem 7.1.

**Proof of Theorem 7.2:** Let \( W_\Psi \) be any spectral factor of \( \Psi \). Let \( \hat{W} = (I + G^*AG)^{-1}W_\Psi \) as in (62). Let \( W \), belonging to \( L_{\infty}^{m \times m}(T) \) together with its inverse, satisfy the constraint
\[ \int GWW^*G^* = I. \] (76)

By Lemma (7.4), and by the strict convexity of the functional \( L(\cdot, \hat{\Lambda}) \), we get
\[ \| \hat{W} - W_\Psi \|^2_2 = L(\hat{W}, \hat{\Lambda}) < L(W, \hat{\Lambda}) = \| W - W_\Psi \|^2_2. \]

Thus, \( \hat{W} \) minimizes the \( L_2 \) distance to \( W_\Psi \) among \( W \) belonging to \( L_{\infty}^{m \times m}(T) \) together with their inverse and satisfying constraint (76). Theorem 6.1 now shows that \( \hat{\Phi}_H = \hat{W}\hat{W}^* \) [coinciding with \( \Phi_H \) in (59)], is the unique solution to the multivariable approximation Problem 7.1.

**Remark 7.8** Consider the important covariance extension problem when, as it is often the case, the process \( y \) is real-valued. Then \( A \) and \( B \) are real matrices and \( \Psi \) is a real spectral density, i.e., \( \Psi(z) \) is real (and symmetric) for all \( z \in \mathbb{T} \). In this case, \( \Lambda \) is a real symmetric matrix.

**VIII. Numerical Solution of the Dual Problem**

**A. Matricial Newton-Type Algorithm**

We now show how to efficiently implement a modified Newton algorithm with backtracking (see, e.g., [4, Ch. 9]) for the computation of \( \Lambda \) (convergence of the algorithm, however, will be discussed elsewhere). This task requires some care because we are working in a matricial space and vectorization does not appear to be convenient. The road map is the following. We have to find the minimum of the functional (65) which is strictly convex on \( L_H^H \). This is then equivalent to finding a matrix \( \hat{\Lambda} \in L_H^H \) that annihilates the derivative of \( J_\Psi(\Lambda) \), i.e., such that (58) is satisfied. According to the abstract version of the Newton algorithm, \( \hat{\Lambda} \) may be found as the limit of the sequence obtained by iterating the following steps:

1. Choose an initial estimate \( \Lambda_0 \in L_H^H \) of \( \hat{\Lambda} \) (the simplest choice being \( \Lambda_0 = 0 \)).
2. Let \( \Lambda_k \) be the current estimate of \( \hat{\Lambda} \). Compute the directional derivative \( J_\Psi(\Lambda_k; \delta \Lambda) \) at the point \( \Lambda_k \) in direction \( \delta \Lambda \), as in (66).
3. Compute the “Hessian” (second directional derivative) \( \delta^2 J_\Psi(\Lambda_k; \delta \Lambda_1, \delta \Lambda_2) \) at the point \( \Lambda_k \) in direction \( \delta \Lambda_1 \) and \( \delta \Lambda_2 \). This may be done in the same way in which we computed \( \delta^2 J_\Psi(\Lambda; \delta \Lambda) \) in (67). Indeed, the latter may be viewed as the “diagonal” part of the Hessian, i.e.,
\[ \delta^2 J_\Psi(\Lambda_k; \delta \Lambda) = \delta^2 J_\Psi(\Lambda_k; \delta \Lambda, \delta \Lambda) \]
\[ = \text{tr} \left[ \int GQ_i^{-1} \left[ (G^*\delta \Lambda_2 GQ_i^{-1}\Psi) + (G^*\delta \Lambda_1 GQ_i^{-1}\Psi)^* \right] Q_i^{-1}G^* \right] \] (77)

where \( Q_i := (I + G^* \Lambda_i G) \).

4. Solve the following equation for \( X \) such that \( (\Lambda_k + X) \in L_H^H \)
\[ \delta^2 J_\Psi(\Lambda_k; \delta \Lambda, X) = -\delta J_\Psi(\Lambda_k; \delta \Lambda) \quad \forall \delta \Lambda \] (78)
and set the \((i+1)\)th estimate of \( \hat{\Lambda} \) to the value \( \Lambda_{i+1} = \Lambda_k + X \).

5. Let \( \varepsilon \) be a suitably small number. If
\[ \left\| \int G(I + G^* \Lambda_{i+1} G)^{-1} \Psi(I + G^* \Lambda_{i+1} G)^{-1} G^* - I \right\| > \varepsilon \] (79)
then go to Step 2). Otherwise, set \( \hat{\Lambda} = \Lambda_{i+1} \).

There are some very delicate points to be addressed. First of all, we need to worry about the existence of solutions for (78).

**Proposition 8.1:** Assume condition (6) [or, equivalently, condition (8)] is satisfied. There exists a unique \( X \in \text{Range}(\Gamma) \) solving (78).

**Proof:** Equation (78) may be rewritten as
\[ \int GQ_i^{-1} \left[ (G^*XGQ_i^{-1}\Psi) + (G^*XGQ_i^{-1}\Psi)^* \right] Q_i^{-1}G^* = \int GQ_i^{-1}\Psi Q_i^{-1}G^* - I \] (80)
where we have eliminated \( \delta \Lambda \). Notice that the map \( \phi \) associating to \( X \in \text{Range}(\Gamma) \) the matrix
\[ \phi(X) := \int GQ_i^{-1} \left[ (G^*XGQ_i^{-1}\Psi) + (G^*XGQ_i^{-1}\Psi)^* \right] Q_i^{-1}G^* \]
defines a linear transformation of \( \text{Range}(\Gamma) \) into itself. In fact, clearly
\[ Q_i^{-1} \left[ (G^*XGQ_i^{-1}\Psi) + (G^*XGQ_i^{-1}\Psi)^* \right] Q_i^{-1} = \left[ Q_i^{-1} \left[ (G^*XGQ_i^{-1}\Psi) + (G^*XGQ_i^{-1}\Psi)^* \right] Q_i^{-1} \right] \] (81)
so that by definition \( \phi(X) \in \text{Range}(\Gamma) \). The linear map \( \phi \) has trivial kernel. In fact, if in some \( X \in \text{Range}(\Gamma) \), \( \phi(X) = 0 \), then
\[ 0 = \text{tr}[\phi(X)X] = \delta^2 J_\Psi(\Lambda_k; X, X). \]
Taking into account the positive definiteness of \( \delta^2 J_\Psi(\Lambda_k; X, X) \) on \( \text{Range}(\Gamma) \), this implies \( X = 0 \). As a consequence, the image of \( \phi \) is the whole linear space \( \text{Range}(\Gamma) \). It only remains to observe that \( \int GQ_i^{-1}\Psi Q_i^{-1}G^* - I \in \text{Range}(\Gamma) \). Indeed, \( I \in \text{Range}(\Gamma) \) and \( \int GQ_i^{-1}\Psi Q_i^{-1}G^* \in \text{Range}(\Gamma) \) by definition of \( \text{Range}(\Gamma) \).
Proposition 8.1 does not guarantee that the solution \( X \) of (78) satisfies \( (\Lambda_t + X) \in \mathcal{L}_I^H \), as requested in Step 4. To overcome this problem, we design a variant of the Newton method with backtracking. In this variant, the following substeps are employed in place of Step 4:

4.1) Solve equation (78) for \( X \in \text{Range}(\Gamma) \).

4.2) Let \( k = 1 \) and double \( k \) until both conditions

\[
\left( \Lambda_i + \frac{1}{k} X \right) \in \mathcal{L}_I^H \tag{82}
\]

and

\[
\| V_{i,k} \| < \| V_i \| \tag{83}
\]

are satisfied, where

\[
V_i := \int G(I + G^* \Lambda_i G)^{-1} \Psi(I + G^* \Lambda_i G)^{-1} G^* - I
\]

and

\[
V_{i,k} := \int G \left( I + G^* \left( \Lambda_i + \frac{1}{k} X \right) G \right)^{-1} \Psi \times \left( I + G^* \left( \Lambda_i + \frac{1}{k} X \right) G \right)^{-1} G^* - I.
\]

4.3) Set the \((i+1)\)th estimate of \( \hat{\Lambda} \) to the value

\[
\Lambda_{i+1} = \Lambda_i + \frac{1}{k} X.
\]

This procedure guarantees that each \( \Lambda_i \in \mathcal{L}_I^H \) even when \( X \notin \mathcal{L}_I^H \). Notice that, by convexity of the problem, \( X \) is a descent direction so that, for sufficiently large \( k \), (83) is certainly satisfied. Moreover, since \( \mathcal{L}_I^H \) is an open set, (82) is also satisfied for sufficiently large \( k \).

B. Computation of the Solution of (80)

The next point that needs to be addressed is the computation of \( X \) (Step 4.1). In fact, although (80) is a linear equation, it is not obvious how to solve it in a numerically efficient way. To simplify notation, we drop the subscript “\( i \)” in \( \Lambda_i \) and \( Q_i := (I + G^* \Lambda_i G) \). Consider the following equation

\[
\int GQ^{-1} \left[ (G^* X GQ^{-1} \Psi) + (G^* X GQ^{-1} \Psi)^* \right] Q^{-1} G^* = \int GQ^{-1} \Psi Q^{-1} G^* - I. \tag{84}
\]

We propose the following procedure.

1) Choose a set \( \{ X_i \} \) of linearly independent matrices such that \( \text{span}\{ X_i \} = \text{Range}(\Gamma) \).

2) Compute the quantities

\[
Y_i := \int GQ^{-1} \left[ (G^* X GQ^{-1} \Psi) + (G^* X GQ^{-1} \Psi)^* \right] Q^{-1} G^*
\]

\[
Y := \int GQ^{-1} \Psi Q^{-1} G^* - I. \tag{85}
\]

3) Solve for the scalar unknowns \( y_i \), equation

\[
\sum_i y_i Y_i = Y.
\]

4) Set

\[
X = \sum_i y_i X_i.
\]

Steps 3) and 4) do not present any difficulty. Concerning point 1), employing the characterization (29) of \( \text{Range}(\Gamma) \), we simply have to solve the following Lyapunov equations

\[
\Sigma - \Sigma \Delta^* \Delta = BH_{h,k} + H_{h,k} B^* \tag{87}
\]

where \( H_{h,k} \in \mathbb{R}^m \times n \) is the matrix in which the entry in position \((h,k)\) is 1 and all the other entries are zero. As \( k \) and \( h \) vary in their respective range, (87) yields \( n \times m \) equations whose solutions are \( n \times m \) square matrices. Such matrices span \( \text{Range}(\Gamma) \), but are not linearly independent. It is easy, however, to employ the singular value decomposition algorithm (which is very stable and robust) and reduce to a basis \( \{ X_i \} \) of \( \text{Range}(\Gamma) \).

Concerning point 2), in the case when \( \Psi \) is a rational matrix function, we can compute the integrals in (85) and (86) very efficiently and precisely. We first describe in detail the computation of \( Y \). To compute \( \int GQ^{-1} \Psi Q^{-1} G^* \), we observe that \( \chi := GQ^{-1} \Psi Q^{-1} G^* \) is a spectral density. Let \( W_\chi \) be an analytic spectral factor of \( \chi \) (namely a function \( W_\chi \) analytic outside the unit disk and such that \( \chi = W_\chi W_\chi^* \)). Then, \( \int \chi \) is the steady-state covariance of the output of a filter with transfer function \( W_\chi \) fed by normalized white noise. To compute a realization of \( W_\chi \), we implement the following steps:

2a) Compute a coanalytic square spectral factor \( W_\phi^* \) of \( \Psi \) (namely \( \Psi = W_\phi^* W_\phi \), \( W_\phi \) being square and analytic outside the unit disk). This requires (see, e.g., [18]):

- decomposing \( \Psi \) as \( \Psi = Z + Z^* \) with \( Z \) being analytic inside the unit disk: This may be done by partial fraction expansion.

- solving an algebraic Riccati equation of dimension equal to the McMillan degree of \( Z \).

2b) Factorize \( Q = (I + G^* \Lambda G) \) as \( Q = \Delta^* \Delta \), with \( \Delta \) being square and analytic together with its inverse outside the unit disk. This can be done by computing the stabilizing solution \( X_\gamma \) of the following algebraic Riccati equation:

\[
X = A^* X A - A^* X B (I + B^* X B)^{-1} B^* X A + \Lambda.
\]

We have the following realization for \( \Delta^{-1} \):

\[
\Delta^{-1} = (I + B^* X B)^{-1/2} \left[ (I + B^* X B)^{-1} \right]^* \times B(I + B^* X B)^{-1/2} \tag{89}
\]

with \( \Gamma_\gamma := A - B(I + B^* X B)^{-1} B^* X A \) having all its eigenvalues inside the unit circle.

2c) Compute a realization of \( H^* := \Delta^{-1} W_\phi^* \). Notice that \( H^* \) is a coanalytic spectral factor of \( \Delta^* \Psi \Delta^{-1} \).

2d) From \( H^* \), compute an analytic spectral factor \( H_1 \) of \( \Delta^{-1} \Psi \Delta^{-1} \) using the procedure detailed in Lemma A.1 in the Appendix.

2e) Compute a minimal realization of \( W_\chi := G \Delta^{-1} H_1 \)

\[
W_\chi = C_\chi (zI - A_\chi)^{-1} B_\chi.
\]
We get
\[ \int GQ^{-1}\Psi Q^{-1}G^* = \int W_\chi W_\chi^* = C_\chi \Sigma_\chi C_\chi^* \]  
with \( \Sigma_\chi \) being the solution of the Lyapunov equation
\[ \Sigma_\chi - A_\chi \Sigma_\chi A_\chi^* = B_\chi B_\chi^*. \]  
(90)

For the computation of the integrals \( Y_i \) in (85), we employ the same technique. The main difference is that the integrand of (85) is not a spectral density. Nevertheless, we observe that, by factoring \( Q \) as \( Q = \Delta^* \Delta \) [exactly as we have done in point 2b] earlier] and by defining the functions \( \Phi_1, \Phi_2 \in C_H(\mathbb{T}) \) as
\[ \Phi_1 := \Delta^{-*}G^* XG\Delta^{-1}, \quad \Phi_2 := \Delta^{-*}\Psi\Delta^{-1} \]  
(92)

we may rewrite such an integrand in the form
\[ G\Delta^{-1}[(\Phi_1 + \Phi_2)(\Phi_1 + \Phi_2)^* - \Phi_1\Phi_1^* - \Phi_2\Phi_2^*]\Delta^{-*}G^*. \]  
(93)

It is therefore clear that the integrand of (85) is a difference of spectral densities. Hence, the integral (85) may be computed by resorting to the same technique detailed earlier for the computation of \( Y_i \).

C. Simulation Results

We have applied the procedure described in Section VIII-A to many different examples and it performed very well even in the case of large values of \( n \) and \( m \) (recall that \( B \in \mathbb{C}^{n \times m} \)). In the scalar case (\( m = 1 \)), some examples are discussed in [19] for the case of the Hellinger distance, and in [45], for the case of the KL pseudodistance. In the following we discuss a simple multivariable example (\( n = 3, \ m = 2 \)). Choose
\[ A = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \]
\[ \Sigma = \begin{bmatrix} 9/4 & 12/5 & 0 \\ 12/5 & 16/3 & 16/7 \\ 0 & 16/7 & 32/15 \end{bmatrix}. \]

\( A, B \) and \( \Sigma \) satisfy the feasibility condition (6). After renormalizing so that \( \Sigma = I \), we get
\[ A \simeq \begin{bmatrix} 0.2309 & -0.3657 & -0.2046 \\ 0.1368 & 0.7935 & 0.2434 \\ -0.0886 & -0.4086 & 0.0590 \end{bmatrix} \]
\[ B \simeq \begin{bmatrix} 0.6191 & -0.0471 \\ 0.2697 & 0.2711 \\ -0.0458 & 0.6393 \end{bmatrix}. \]

Finally, we have chosen the reference spectral density \( \Psi \) to be identically equal to \( I \) (the identity). In this case, the Kullback–Leibler approximation has the interpretation of maximum entropy solution and may be obtained in closed form [25]
\[ \hat{\Psi}_{KL} = [G^* B(B^*B)^{-1} B^* G]^{-1}. \]  
(94)

For the computation of the Hellinger approximation, we have set in (79) \( \varepsilon := \exp(-12) \). Our procedure converged in 12 steps of the Newton algorithm with backtracking (less than 10 s) to the matrix
\[ \hat{A}_H \simeq \begin{bmatrix} -0.4267 & -0.0621 & 0.0005 \\ -0.0621 & -0.1007 & -0.0269 \\ 0.0005 & -0.0269 & -0.4690 \end{bmatrix}. \]

Let \( \hat{\Phi}_H \) be the corresponding Hellinger approximation computed as in (59). Let \( \hat{\Phi}_{KL}^{(i,j)} \) be the entry in row \( i \) and column \( j \) of \( \hat{\Phi}_H \), and similarly define \( \hat{\Phi}_KL \). In Fig. 1, \( \hat{\Phi}_H \), \( \hat{\Phi}_{KL} \), and the real and imaginary parts of \( \hat{\Phi}_{KL} \) are depicted together with the corresponding entries of \( \hat{\Phi}_{KL} \).

It may be worthwhile to observe that, with respect to the \( L_1 \) distance\(^1\), the Hellinger approximation is, in this example, closer to the prior than the Kullback–Leibler approximation. More precisely, we have
\[ \text{tr} \left\{ \left( \hat{\Phi}_H - I \right)^2 \right\}^{1/2} \simeq 1.6982 \]
\[ \text{tr} \left\{ \left( \hat{\Phi}_{KL} - I \right)^2 \right\}^{1/2} \simeq 2.5079. \]

\(^1\)For a matrix function \( M : \mathbb{T} \rightarrow \mathbb{C}^{k \times k} \), the \( L_1 \) norm here considered is \( \text{tr} \left\{ |M^*M|^{1/2} \right\} \).
APPENDIX
SPECTRAL FACTORIZATION RESULT

In this appendix, we collect a side result connecting left- and right-spectral factors of a rational spectral density.

**Lemma A.1.** Let $A$ be a stability matrix and $H(z) = C(zI - A)^{-1}B + D$ be a minimal realization. Let $P$ be the solution of the Lyapunov equation

$$P - A^*PA = C^*C.$$  \hfill (95)

Let $\begin{bmatrix} K \\ J \end{bmatrix}$ be an orthonormal basis of $\ker [A^*P^{1/2}C^*]$, i.e.,

$$\begin{bmatrix} A^*P^{1/2} & C^* \end{bmatrix} \begin{bmatrix} K \\ J \end{bmatrix} = 0, \quad \begin{bmatrix} K^* & J^* \end{bmatrix} \begin{bmatrix} K \\ J \end{bmatrix} = I. \tag{96}$$

Let $G := P^{-1/2}K$ and

$$H_1(z) := (D^*C + B^*PA)(zI - A)^{-1}G + B^*PG + D^*J.$$

Then, $H^*H = H_1^*H_1^\dagger$.

**Proof:** Let $Q := C(zI - A)^{-1}G + J$. We first show that $QQ^* = I$, so that $Q$ is inner. We then prove that $Q^*H = H_1^\dagger$, concluding the proof. We have

$$Q^*Q = G^*(z^{-1}I - A)^{-1}C^*C(zI - A)^{-1}G + G^*(z^{-1}I - A)^{-1}C^*J + J^*C(zI - A)^{-1}G + J^*J.$$ \hfill (98)

Now let $P > 0$ be the solution of the Lyapunov equation (95). Then,

$$C^*C = -(z^{-1}I - A^*)P(zI - A)$$

$$+ (z^{-1}I - A^*)Pz + z^{-1}P(zI - A).$$ \hfill (99)

Substituting (99) into (98) we obtain

$$Q^*Q = -G^*PG + G^*Pz(zI - A)^{-1}G$$

$$+ G^*(z^{-1}I - A)^{-1}C^*J + J^*C(zI - A)^{-1}G + J^*J.$$ \hfill (100)

Moreover,

$$z(zI - A)^{-1} = I + A(zI - A)^{-1}$$

$$+ (z^{-1}I - A^*)^{-1}z^{-1}I + (z^{-1}I - A^*)^{-1}A^*.$$ \hfill (101)

so that

$$Q^*Q = (J^*C + G^*PA)(zI - A)^{-1}G$$

$$+ (J^*C + G^*PA)(zI - A)^{-1}G^*$$

$$+ G^*PG + J^*J.$$ \hfill (102)

Taking (96) into account, it is easy to see that $Q^*Q = I$. Therefore, $H^*H = H_1^*QH$. Recalling (99) and (101), we eventually get

$$Q^*H =$$

$$= (G^*(z^{-1}I - A)^{-1}C^* + J^*)(C(zI - A)^{-1}B + D)$$

$$= -G^*(z^{-1}I - A^*)^{-1}(z^{-1}I - A^*)P(zI - A)G(zI - A)^{-1}B$$

$$+ G^*(z^{-1}I - A)^{-1}z^{-1}P(zI - A)(zI - A)^{-1}B$$

$$+ G^*(z^{-1}I - A)^{-1}z^{-1}P(zI - A)(zI - A)^{-1}B$$

$$+ G^*(z^{-1}I - A^*)^{-1}C^*D + J^*C(zI - A)^{-1}B + J^*D$$

$$= -G^*PB + G^*Pz(zI - A)^{-1}B$$

$$+ G^*(z^{-1}I - A^*)^{-1}z^{-1}PB$$

$$+ G^*(z^{-1}I - A)^{-1}z^{-1}PB$$

$$+ G^*(z^{-1}I - A)^{-1}C^*D + J^*C(zI - A)^{-1}B + J^*D$$

$$= -G^*PB + G^*P(I + A(zI - A)^{-1})B$$

$$+ G^*(I + (z^{-1}I - A)^{-1})A^*PB$$

$$+ G^*(z^{-1}I - A^*)^{-1}C^*D + J^*C(zI - A)^{-1}B + J^*D$$

$$= G^*PB + G^*PA(zI - A)^{-1}B$$

$$+ G^*(z^{-1}I - A^*)^{-1}A^*PB$$

$$+ G^*(z^{-1}I - A^*)^{-1}C^*D + J^*C(zI - A)^{-1}B + J^*D$$

$$= G^*(z^{-1}I - A^*)^{-1}(C^*D + A^*PB)$$

$$+ G^*PB + J^*D$$

$$= H_1^\dagger.$$ \hfill (103)

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