A stochastic control problem connected to the measurement process in stochastic mechanics

Michele Pavon

Abstract—Continuing the research initiated in M. Pavon, J. Math. Physics, vol. 40 (1999), 5565-5577, we seek to model the wave function collapse within Nelson’s stochastic mechanics through a variational mechanism. The variational principle, starting from a reference quantum evolution, produces the correct wavepacket reduction. The new Nelson process is associated to a solution of another Schrödinger equation whose potential function accounts for the interaction of the microscopic system with the measuring apparatus.

Keywords—Stochastic differential game, Measurement in quantum mechanics

I. INTRODUCTION

In quantum mechanics, the evolution is linear and causal as long as no measurement is performed. A measurement changes dramatically the picture: The state changes non-linearly and non causally. Nevertheless, if one models the measuring apparatus together with the microscopic system, then the evolution of the compound system remains linear and causal. This is one of the most striking features of quantum mechanics, origin of an endless discussion. In this paper, continuing the research initiated in [1], we consider the situation in the frame of Nelson’s stochastic mechanics [2], [3], [4]. The novelty is that we model the influence of the measuring device on the microscopic system through a time varying potential. A variational principle in the spirit of [5] produces the correct reduction in the wavepacket. Other recent references on measurement in stochastic mechanics are [6], [7], [8], [9].

Nelson’s stochastic mechanics is a quantization procedure based on stochastic diffusion processes. The Schrödinger equation was originally derived from a continuity type equation plus a Newton type law. The Newton-Nelson law was later shown to follow, in analogy to classical mechanics, from a Hamilton-like stochastic variational principle [10], [5]. Other versions of the variational principle have been proposed in [11], [12], [13], [14], [15]. One of the most striking differences between Nelson’s and other versions of stochastic mechanics such as Bohmian mechanics [16], [17], [18] or the Levy-Krener mechanics [19], [20] is that the former features a kinematics for finite-energy diffusions with two velocities.

Consider the case of a nonrelativistic particle of mass $m$. Let $\{\psi(x,t); t_0 \leq t \leq t_1\}$, satisfying the Schrödinger equation

$$\frac{\partial \psi}{\partial t} = \frac{i\hbar}{2m} \Delta \psi - \frac{i}{\hbar} V(x)\psi,$$

be such that

$$||\nabla \psi||^2 \in L^1_{loc}[t_0, +\infty).$$

This is Carlen’s finite action condition. Under these hypotheses, the Nelson measure $P$ may be constructed on path space, [21],[22], [12, Chapter IV], and references therein. Namely, letting $\Omega := C([t_0, t_1], R^n)$ the $n$-dimensional continuous functions on $[t_0, t_1]$, under the probability measure $P$, the canonical coordinate process $x(t, \omega) = \omega(t)$ is an $n$-dimensional, Markov, finite-energy diffusion process $\{x(t); t_0 \leq t \leq t_1\}$, called Nelson’s process, having (forward) Ito differential

$$dx = \left(\frac{\hbar}{m} \nabla (\Re \log \psi + \Im \log \psi)\right)(x(t), t) dt + \sqrt{\frac{\hbar}{m}} dw(t),$$

where $w$ is a standard, $n$-dimensional Wiener process. Moreover, the probability density $\rho(\cdot,t)$ of $x(t)$ satisfies Born’s relation

$$\rho(x, t) = |\psi(x,t)|^2, \quad \forall t \in [t_0, t_1].$$

Let $V_{r}(x)$ denote the internal or ambient potential function. Let $x_r = \{x_r(t); t_0 \leq t \leq t_1\}$ be the $n$-dimensional Nelson reference process that has been constructed through a variational principle. This process has an associated solution $\{\psi_r(x,t); t_0 \leq t \leq t_1\}$ of the Schrödinger equation

$$\frac{\partial \psi_r}{\partial t} = \frac{i\hbar}{2m} \Delta \psi_r - \frac{i}{\hbar} V_{r}(x)\psi_r,$$

in the sense that the forward drift is

$$b_{r+}(x,t) = \left(\frac{\hbar}{m} \nabla (\Re \log \psi_r(x,t) + \Im \log \psi_r(x,t))\right)$$

and the one-time density of $x_r$ satisfies $\rho_r(x,t) = |\psi_r(x,t)|^2$. As usual, we write

$$\psi_r(x,t) = \exp[R_r(x,t) + \frac{i}{\hbar} S_r(x,t)].$$

II. A STOCHASTIC CONTROL PROBLEM

Suppose that we perform a position measurement during $[t_0, t_1]$. Let $V_{e}(x,t)$ denote the interaction potential, describing the interaction between the microscopic system and the measuring apparatus during $[t_0, t_1]$. Suppose also that at the end of the measurement process at time $t_1$, we get the probability density $\rho_1(x) \neq |\psi_r(x,t_1)|^2$, where
\{ \psi_s(x, t) : t_0 \leq t \leq t_1 \} is the reference quantum evolution (see previous section).

Let \( \mathcal{X}_{\rho_t} \) denote the family of all finite-energy, \( \mathbb{R}^n \)-valued diffusions on \([t_0, t_1]\) with diffusion coefficient \( I_{\eta, \beta} \), and having marginal probability density \( \rho_t \) at time \( t_1 \). We recall \([23], [24]\), that a finite-energy diffusion possesses both a forward and a backward drift \( \beta_t \) and \( \gamma_t \). The sum-semi of these two drifts is called current drift. Let \( \mathcal{V} \) denote the family of finite-energy, \( \mathbb{R}^n \)-valued stochastic processes on \([t_0, t_1]\). For \((x, v, u') \in (\mathcal{X}_{\rho_t}, \mathcal{V}, \mathcal{V})\), define the functional

\[
I(x, v, u) = E \left\{ \int_{t_0}^{t_1} \left[ \frac{1}{2} m \|v(t) - \frac{1}{m} \nabla S_r(x(t), t)\|^2 - \frac{h^2}{8m} \|u'(t) - \nabla \log \rho_r(x(t), t)\|^2 - V_c(x(t), t) \right] dt \right\}
\]

This functional, featuring an "Onsager type Lagrangian", is the natural generalization of the one employed by Guerra and Morato in \([5]\) in view of the "reference" evolution. Consider the stochastic differential game

\[
\min_{x \in \mathcal{X}_{\rho_t}} \max_{v \in \mathcal{V}} \min_{u \in \mathcal{V}} I(x, v, u')
\]

subject to

\[
v(t) \text{ is the current drift of } x, \\
u(t) = \nabla \log \rho_r(x(t), t) \forall t,
\]

where \( \rho_r(t) \) is the probability density of \( x(t) \). We say that \((x^*, v^*, u^*)\) is a saddle-point equilibrium solution of the game if for all \((x_1, v^*, u')\) and \((x_2, v, u')^*\) in \( \mathcal{X}_{\rho_t} \times \mathcal{V} \times \mathcal{V} \) satisfying the constraints we have

\[
I(x_1, v^*, u') \leq I(x^*, v^*, u^*') \leq I(x_2, v, u^*)
\]

**III. VARIATIONAL ANALYSIS**

We introduce a suitable class of Lagrange functionals \([25]\) for the problem. We recall that such a functional must be finite and converge over the set determined by the constraints. Let \( F : (\mathbb{R}^n \times [t_0, t_1]) \rightarrow \mathbb{R} \) be of class \( C^1 \) with compact support. Let \( \lambda : (\mathbb{R}^n \times [t_0, t_1]) \rightarrow \mathbb{R}^n \) be of class \( C^1 \). For such a pair \((F, \lambda)\), define

\[
\Lambda^{F,\lambda}(x, v, u') = E \left\{ F(x(t_1), t_1) - F(x(t_0), t_0) \right\} - \int_{t_0}^{t_1} \left\{ \frac{\partial F}{\partial t} + v(t) \cdot \nabla F \right\} (x(t), t) - \lambda(x(t), t) \cdot \[u'(t) - \nabla \log \rho_r(x(t), t)\] dt \right\}
\]

One can argue as in \([1]\) that Ito's rule for the forward and backward differential of \( x \) imply that for all triples \((x, v, u') \) \( \mathcal{X}_{\rho_t} \times \mathcal{V} \times \mathcal{V} \) satisfying the constraints, we have \( \Lambda^{F,\lambda}(x, v, u') = 0 \). It is possible to rewrite \( \Lambda^{F,\lambda} \) in a form more suited for our purposes along the lines of Guerra-Morato 1983. Observe that, for any \( x \in \mathcal{X}_{\rho_t} \), we have

\[
E \left\{ \lambda(x(t), t) \cdot \nabla \log \rho_r(x(t), t) \right\} = \\
\int_{\mathbb{R}^n} \lambda(x, t) \cdot \nabla \rho_r(x, t) dx = - \int_{\mathbb{R}^n} \nabla \cdot \lambda(x, t) \rho_r(x, t) dx \\
= - E \left\{ \nabla \cdot \lambda(x(t), t) \right\}
\]

where, in the integration by parts, we have used the natural boundary condition at infinity for \( \rho(x(t), t) \). Thus, our Lagrange functionals have now the form

\[
\Lambda^{F,\lambda}(x, v, u') = E \left\{ F(x(t_1), t_1) - F(x(t_0), t_0) \right\} - \int_{t_0}^{t_1} \left\{ \frac{\partial F}{\partial t} + v(t) \cdot \nabla F \right\} (x(t), t) - \lambda(x(t), t) \cdot \[u'(t) - \nabla \cdot \lambda(x(t), t)\] dt \right\}
\]

Consider now the unconstrained problem

\[
\min_{x \in \mathcal{X}_{\rho_t}} \max_{v \in \mathcal{V}} \min_{u \in \mathcal{V}} (I + \Lambda^{F,\lambda})(x, v, u').
\]

Obviously, if \((x, v, u') \in (\mathcal{X}_{\rho_t} \times \mathcal{V} \times \mathcal{V})\) satisfying the constraints is a saddle-point solution for \((I + \Lambda^{F,\lambda})\), then it also solves the original problem with cost function \( I \). For each fixed \( x \in \mathcal{X}_{\rho_t} \), and each \( t \in [t_0, t_1] \), we study the finite-dimensional problem

\[
\min_{v \in \mathcal{V}} \max_{u \in \mathcal{V}} \left\{ \frac{1}{2} m \|v - \frac{1}{m} \nabla S_r(x(t), t)\|^2 - \frac{h^2}{8m} \|u' - \nabla \log \rho_r(x(t), t)\|^2 - V_c(x(t), t) \cdot v - \nabla \cdot \lambda(x(t), t) \right\}.
\]

We get the optimality conditions

\[
v^*_c(x(t)) = \frac{1}{m} \nabla S_r(x(t), t) + \frac{1}{m} \nabla F(x(t), t), \\
u^*_c(x(t)) = \nabla \log \rho_r(x(t), t) + \frac{4m}{h^2} \lambda(x(t), t).
\]

**Remark I:** If a stochastic process with the prescribed \( v^*_c(x(t)) \) and \( u^*_c(x(t)) \) does exist, then the first optimality condition implies that it is a Markov process and the second that \( \lambda \) is the gradient of some scalar function. More precisely, the second optimality condition implies that

\[
\lambda(x, t) = \frac{h^2}{4m} \nabla \log \left( \frac{\rho_r^*}{\rho_r} \right)(x, t)
\]

where \( \rho_r^*(x, t) \) denotes the probability density of \( x^*(t) \). Notice that \( v^* \) and \( u^* \) belong to \( \mathcal{V} \). Consider next the minimization of

\[
\left. (I + \Lambda^{F,\lambda})(x, v_*, u^*_*) = \\
E \{ F(x(t_1), t_1) - F(x(t_0), t_0) \} + \int_{t_0}^{t_1} \left[ - \frac{\partial F}{\partial t}(x(t), t) - \frac{1}{m} \|\nabla F(x(t), t)\|^2 - V_c(x(t), t) \cdot \nabla F(x(t), t) \right. \\
+ \frac{2m}{h^2} \|\lambda(x(t), t)\|^2 + \nabla \log \rho_r(x(t), t) \cdot \lambda(x(t), t) \\
+ \nabla \cdot \lambda(x(t), t) \right] dt \right|_{x \in \mathcal{X}_{\rho_t}}
\]

on the space \( \mathcal{X}_{\rho_t} \). We wish to choose \( F \) and \( \lambda \) such that the functional becomes constant with respect to the process.
\( x \in \mathcal{X}_\rho \). Suppose that the pair \((F, \lambda)\) satisfies on \(\mathbb{R}^n \times [t_0, t_1]\) the equation
\[
\frac{\partial F}{\partial t} + \frac{1}{2m} \nabla F \cdot \nabla F + \frac{1}{m} \nabla S_r \cdot \nabla F + V(x, t) \\
- \frac{2m}{\hbar^2} \lambda \cdot \lambda - \nabla \log \rho_r \cdot \lambda - \nabla \cdot \lambda = 0,  
\]
and the boundary condition \(F(x, t_0) = 0\). With this choice of \((F, \lambda), (1 + \Delta \psi_\lambda)(x, v^*_r, u^*_r) \equiv E\{F(x(t_1), t_1)\}\) which is constant on \(\mathcal{X}_\rho\). Hence, any \(x\) in \(\mathcal{X}_\rho\) solves with \((v^*, u^*)\), the unconstrained problem.

IV. SOLUTION TO THE VARIATIONAL PROBLEM

In view of Remark 1, we now write
\[
\lambda(x, t) = \frac{\hbar^2}{2m} \nabla G(x, t),
\]
for some scalar \(C^1\) function \(G\). Equation (6) then becomes
\[
\frac{\partial F}{\partial t} + \frac{1}{2m} \nabla F \cdot \nabla F + \frac{1}{m} \nabla S_r \cdot \nabla F + V(x, t) \\
- \frac{h^2}{2m} \left[ \nabla G \cdot \nabla G + \Delta G + \nabla \log \rho_r \cdot \nabla G \right] = 0.
\]

In the ideal case where \(V = V_r, \nabla S_r \equiv 0\) and \(\rho_r = \text{const}\), this equation reduces to one of the Madelung equations [4]. If \(F\) and \(G\) may be found satisfying (7) with \(F(x, t_0) = 0\), then any \(x\) in \(\mathcal{X}_\rho\) solves the unconstrained problem. If we can find one \(x^*\) in \(\mathcal{X}_\rho\), that also satisfies the constraints that \(v^*_r(t) = \frac{1}{m} \nabla S_r(x(t), t)\) and \(\frac{1}{m} \nabla F(x(t), t)\) is the current drift of \(x^*\) at time \(t\) and \(\nabla \log \rho_r(x(t), t) + 2\nabla V_r(x, t)\), \(\forall t\), then \(x^*\) solves the original stochastic differential game. In that case, we define
\[
S^*(x, t) := S_r(x(t), t) + F(x, t),
\]
so that \(v^*_r(t) = \frac{1}{m} \nabla S^*(x(t), t)\) and we have
\[
G(x, t) = \frac{1}{2} \log \rho_r^*(x, t). 
\]

Suppose that we have found such a process \(x^*\) with density \(\rho^*\). As observed before, it is a Markov process and so is the Nelson reference process \(x^*\). Then the corresponding Fokker-Planck (continuity) equations read
\[
\frac{\partial \rho^*}{\partial t} + \nabla \cdot \left( \frac{1}{m} \nabla S^* \rho^* \right) = 0, \\
\frac{\partial \rho_r}{\partial t} + \nabla \cdot \left( \frac{1}{m} \nabla S_r \rho_r \right) = 0.
\]
Using these equations, and relations (8)-(9), we get that \(F\) and \(G\) must satisfy another equation
\[
\frac{\partial G}{\partial t} + \frac{1}{m} \nabla (F - S^*) \cdot \nabla G + \frac{1}{2m} \Delta F + \frac{1}{2m} \Delta F \cdot \nabla \log \rho_r = 0. 
\]

Thus, \(F\) and \(G\) also satisfy
\[
\frac{\partial G}{\partial t} + \frac{1}{m} \nabla (F - S^*) \cdot \nabla G + \frac{1}{2m} \Delta F + \frac{1}{2m} \Delta F \cdot \nabla \log \rho_r = 0. 
\]

In the ideal case where \(V = V_r, \nabla S_r \equiv 0\) and \(\rho_r = \text{const}\), this equation reduces to the other Madelung equation. Moreover,
\[
G(x, t_1) = \frac{1}{2} \log \frac{\rho_1}{\rho_r}(x, t).
\]
Define \(\theta = \exp(G + \frac{\hbar^2}{2m} F)\). Then, equations (7)-(10) imply that \(\theta\) satisfies
\[
\frac{\partial \theta}{\partial t} + \left( \frac{1}{m} \nabla S_r + \frac{\hbar^2}{2m} \nabla R_r \right) \cdot \nabla \theta - \frac{i\hbar}{2m} \Delta \theta + \frac{i\hbar}{2m} V_r(x) \theta = 0 
\]

Define now
\[
\psi_n := \left( \rho^*_n \right)^{1/2} \exp \frac{i}{\hbar} S^*_n = \psi_r \theta.
\]

It follows from (5) and (12) that \(\psi_n\) satisfies
\[
\frac{\partial \psi_n}{\partial t} = \frac{i\hbar}{2m} \Delta \psi_n - \frac{i}{\hbar} \left( V(x) + V_r(x, t) \right) \psi_n, 
\]
and \(\psi_n(x, t_1) = (\rho_1(x))^{1/2} \exp(\frac{\hbar^2}{2m} S_r(x, t_1)\).

V. CONCLUSION

Hence, we find that the solution to the stochastic differential game is the Nelson process associated to a solution of the new Schrödinger equation (13) with \(V(x) = V(x) + V_r(x, t)\). Moreover, \(|\psi_n(x, t_1)|^2\) coincides with the observed probability density \(\rho_1\) at time \(t_1\), and the phase of \(\psi_n(x, t_1)\) is the same as for the reference evolution \(S_r(x, t_1)\). All of this is in agreement with the postulates of the orthodox theory of measurement in quantum mechanics.

REFERENCES


