Rational approximations of fractional powers of matrices

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Fractional powers of matrices can be used to construct numerical methods for the solution of problems involving fractional derivatives. For instance, denoting by A the approximation of the standard Laplacian with homogeneous Dirichlet boundary conditions obtained by using any finite difference method, the so-called *matrix transfer technique* introduced by Ilić et al. in [4, 5] approximates the fractional Laplacian operator of order 2α, $\alpha \in (1/2, 1]$, by A^{α}. The idea can be extended to other kind of fractional derivatives whenever A represents the discretization of the corresponding integer order one [1].

In this view, any numerical scheme able to compute the matrix fractional powers can be potentially used to define a method for fractional equations. Nevertheless, when working with fractional powers, it must be kept in mind that raising to a fractional number destroys the sparsity structure of the underlying integer order approximation. As a consequence, the corresponding solver may be extremely expensive for large size matrices. In order to tackle this problem, in [2, 3] we have studied a rational approximation to A^{α}, that is,

$$A^{\alpha} \approx [q_k(A)]^{-1} p_k(A), \tag{1}$$

where $p_k, q_k \in \Pi_k$, the set of polynomials of degree k and smaller. Considering that good accuracy is attainable for values of the bandwidth k much less than the size of A^{α}, the action of A^{α} is then approximated through the action of sparse matrices. However, the condition number of $p_k(A)$ and $q_k(A)$ becomes (with respect to k) quickly very large.

In this talk, we first recall the basic features about this rational approximation, that is essentially a scaled Padé form, and then we present a simple but reliable strategy that allows to keep the conditioning under control.

References

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