ESERCIZI SUGLI INTEGRALI IMPROPRI

1) $\int_1^{+\infty} \operatorname{sen}(x^2)\,dx$ non è ass. convergente

$$\int_1^{+\infty} |\operatorname{sen}(x^2)|\,dx = \lim_{c\to+\infty} \int_1^c |\operatorname{sen}(x^2)|\,dx \overset{t=x^2}{=} \lim_{c\to+\infty} \int_1^{c^2} \left|\operatorname{sen} t \cdot \frac{1}{2\sqrt{t}}\right| dt$$

$$= \frac{1}{2}\int_1^{+\infty} \frac{|\operatorname{sen} t|}{\sqrt{t}}\,dt$$ divergente, perché se $t \geq 1$

$\Rightarrow \frac{|\operatorname{sen} t|}{\sqrt{t}} \geq \frac{|\operatorname{sen} t|}{t}$ ($\sqrt{t} \leq t$ se $t \geq 1$) e $\int_1^{+\infty} \frac{|\operatorname{sen} t|}{t}\,dt$

è divergente.

2) $\int_0^1 \log x\,dx$ integrale improprio, perché $\lim_{x\to 0+} \log x = -\infty$

Si ha $$\int_0^1 \log x\,dx := \lim_{\varepsilon\to 0+} \int_\varepsilon^1 \log x\,dx \overset{P.P}{=} \lim_{\varepsilon\to 0+} \underbrace{\left(x\log x\Big|_\varepsilon^1 - \int_\varepsilon^1 dx \right)}_{-\varepsilon\log\varepsilon - (1-\varepsilon)} = -1$$

$\Rightarrow$ convergente.

Più in generale: $\int_0^b \log x\,dx = \ldots = b\log b - b \quad \forall\, b>0$

3) $I_n := \int_0^{+\infty} t^n e^{-t}\,dt \quad n\in\mathbb{N}$ è convergente. Infatti:

$$\begin{cases} t^n e^{-t} = o\left(\frac{1}{t^2}\right) \text{ per } t\to+\infty \quad \forall\, n\geq 0 \quad \left(\text{cioè } \lim_{t\to+\infty} \frac{t^n e^{-t}}{\frac{1}{t^2}} = 0\right) \\ t^n e^{-t} \geq 0 \quad \forall\, t\geq 0 \end{cases}$$

Poiché $\int_1^{+\infty} \frac{1}{t^2}\,dt$ converge $\Rightarrow \int_1^{+\infty} t^n e^{-t}\,dt$ converge

$$\Rightarrow \int_0^{+\infty} t^n e^{-t}\,dt = \underbrace{\int_0^1 -}_{\text{integrale "normale"}} + \int_1^{+\infty} - \quad \text{converge}$$

$$I_0 = \int_0^{+\infty} e^{-t}\,dt = \lim_{c\to+\infty}\left(-e^{-c}+1\right) = 1$$

$$I_{n+1} = \int^{+\infty} t^{n+1} e^{-t}\,dt \overset{P.P.}{=} -e^{-t}t^{n+1}\Big|^{+\infty} + \int^{+\infty} (n+1)\, t^n e^{-t}\,dt = (n+1)I_n$$

$$I_{n+1} = \int_0^{+\infty} t^{n+1} e^{-t}\, dt \overset{P.P.}{=} \underbrace{-e^{-t} t^{n+1} \Big|_0^{+\infty}}_{:= \lim\limits_{c \to +\infty} \Big(\underbrace{-e^{-t} t^{n+1} \Big|_0^{c}}_{-e^{-c} c^{n+1}} \Big) = 0} + \int_0^{+\infty} (n+1)\, t^n e^{-t}\, dt = (n+1) I_n$$

Per induzione: $I_n = n \cdot I_{n-1} = n \cdot (n-1)\, I_{n-2} = \dots = n!\, I_0 = n!$

4) $\int_2^{+\infty} \frac{1}{x^\beta} \log^\alpha x\, dx$ al variare di $\alpha, \beta \in \mathbb{R}$

I caso: $\beta = 1 \Rightarrow \int_2^{+\infty} \frac{dx}{x \log^{-\alpha} x}$ convergente $\overset{\text{già visto}}{\Leftrightarrow} -\alpha > 1 \Leftrightarrow \alpha < -1$

II caso: $\beta > 1 \Rightarrow \frac{1}{x^\beta} \log^\alpha x = o\left(\frac{1}{x^\gamma}\right)$ per $x \to +\infty$ per $1 < \gamma < \beta$

$\Rightarrow \int_2^{+\infty} \frac{1}{x^\beta} \log^\alpha x\, dx$ converge

III caso: $\beta < 1 \Rightarrow \frac{1}{t^\gamma} = o\left(\frac{1}{x^\beta} \log^\alpha x\right)$ per $x \to +\infty$ per $\beta < \gamma < 1$

$\Rightarrow \int_2^{+\infty} \frac{1}{x^\beta} \log^\alpha x\, dx$ divergente

5) $\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}}$ è un integrale improprio perché $\lim\limits_{x \to \pm 1} \frac{1}{\sqrt{1-x^2}} = +\infty$

$$= \underbrace{\int_{-1}^{0} \frac{dx}{\sqrt{1-x^2}}}_{\lim\limits_{c \to -1+} \int_c^0 \frac{dx}{\sqrt{1-x^2}}} + \underbrace{\int_0^1 \frac{dx}{\sqrt{1-x^2}}}_{\lim\limits_{d \to 1+} \int_0^d \frac{dx}{\sqrt{1-x^2}}}$$

I metodo (confronto asintotico): $\frac{1}{\sqrt{1-x^2}} \sim \frac{1}{(1+x)^{1/2}}$ per $x \to -1+$

$$\frac{1}{(1+x)^{1/2}} = \frac{1}{(x - \underbrace{(-1)}_{a})^{1/2}}$$

perché $\lim\limits_{x \to -1+} \frac{\frac{1}{\sqrt{1-x^2}}}{\frac{1}{\sqrt{1+x}}} = \lim\limits_{x \to -1+} \frac{1}{\sqrt{1-x}} = \frac{1}{2}$

$\int_{-1}^{0} \frac{1}{(x-(-1))^{1/2}}\, dx$ converge $\Rightarrow \int_{-1}^{0} \frac{dx}{\sqrt{1-x^2}}$ converge

Poiché $\frac{1}{\sqrt{1-x^2}}$ è pari, si ha anche $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \int_{-1}^{0} \frac{dx}{\sqrt{1-x^2}}$ converge.

II metodo (calcolo diretto): $\int^0 \frac{dx}{\quad} = \lim \int^0 1$

$$\begin{array}{cc} _0\sqrt{1-x^2} & _{-1}\sqrt{1-x^2} \end{array}$$

II metodo (calcolo diretto): $\displaystyle\int_{-1}^{0} \frac{dx}{\sqrt{1-x^2}} = \lim_{c\to -1+} \underbrace{\int_c^0 \frac{1}{\sqrt{1-x^2}}\,dx}_{\operatorname{arcsen} x \,\big|_c^0}$

$$= \lim_{c\to -1+} (\operatorname{arcsen} 0 - \operatorname{arcsen} c) = \frac{\pi}{2}$$

$$\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \underbrace{\int_{-1}^{0} \frac{dx}{\sqrt{1-x^2}}}_{\frac{\pi}{2}} + \underbrace{\int_{0}^{1} \frac{dx}{\sqrt{1-x^2}}}_{\frac{\pi}{2}} = \pi$$

perché $\frac{1}{\sqrt{1-x^2}}$ è pari.

c) $\displaystyle\int_0^{+\infty} \frac{\operatorname{sen} x}{x^2}\,dx$ è integrale improprio perché $[0,+\infty[$ è illimitato

e $\displaystyle\lim_{x\to 0+} \frac{\operatorname{sen} x}{x^2} = \lim_{x\to 0+} \frac{1}{x}\underbrace{\left(\frac{\operatorname{sen} x}{x}\right)}_{\to 1} = +\infty$

(cioè $\frac{\operatorname{sen} x}{x^2}$ è illimitata su $[0,+\infty[$)

$$\int_0^{+\infty} \frac{\operatorname{sen} x}{x^2}\,dx := \underbrace{\int_0^{1} \frac{\operatorname{sen} x}{x^2}\,dx}_{\text{divergente}} + \underbrace{\int_1^{+\infty} \frac{\operatorname{sen} x}{x^2}\,dx}_{\text{ass. convergente}} \quad \text{divergente}$$

ass. convergente, perché $\frac{|\operatorname{sen} x|}{x^2} \le \frac{1}{x^2}$ e $\displaystyle\int_1^{+\infty} \frac{1}{x^2}\,dx$ è convergente

divergente perché $\frac{\operatorname{sen} x}{x^2} \sim \frac{1}{x}$ per $x\to 0$, e $\displaystyle\int_0^1 \frac{1}{x}\,dx$ divergente

$$\int_\alpha^\beta \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}} \quad \text{con } \alpha<\beta \text{ è convergente}$$

$$\|$$

$$\underbrace{\int_\alpha^c \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}}_{\text{convergente}} + \underbrace{\int_c^\beta \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}}_{\text{convergente}} \quad \text{per } \alpha<c<\beta$$

perché $\begin{cases} \dfrac{1}{\sqrt{(x-\alpha)(\beta-x)}} \sim \dfrac{1}{(x-\alpha)^{1/2}} \text{ per } x\to\alpha+ \text{ e } \displaystyle\int_\alpha^c \frac{dx}{(x-\alpha)^{1/2}} \text{ convergente} \\ \dfrac{1}{\sqrt{(x-\alpha)(\beta-x)}} \sim \dfrac{1}{(\beta-x)^{1/2}} \text{ per } x\to\beta- \text{ e } \displaystyle\int_c^\beta \frac{dx}{(\beta-x)^{1/2}} \text{ convergente} \end{cases}$

Mostriamo che $\displaystyle\int^\beta \frac{dx}{\ldots} = \pi$

Mostriamo che $\int_\alpha^\beta \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}} = \pi$

$]0,+\infty[\longrightarrow]\alpha,\beta[$

$t \longmapsto \frac{\alpha+t^2\beta}{1+t^2} = x$

$$\int_0^{+\infty} \frac{2\,dt}{1+t^2} = 2 \operatorname{arctg} t \Big|_0^{+\infty} = 2\cdot\frac{\pi}{2} = \pi$$

$t = \sqrt{\frac{(x-\alpha)}{(\beta-x)}} \Rightarrow x = \frac{\alpha+t^2\beta}{1+t^2}$

$\Rightarrow \sqrt{(x-\alpha)(\beta-x)} = (\beta-x)\underbrace{\sqrt{\frac{(x-\alpha)}{(\beta-x)}}}_{t} = \left(\beta - \frac{\alpha+t^2\beta}{1+t^2}\right)t$

$= (\beta-\alpha)\frac{t}{1+t^2}$

$\Rightarrow \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}} = \frac{2\,dt}{1+t^2}$

CALCOLI DI LIMITI CON GLI SVILUPPI DI TAYLOR

Nota: 1) $x^n = o(x^m)$ $\begin{cases} \text{per } x \to 0 \iff n > m \\ \text{per } x \to \pm\infty \iff n < m \end{cases}$ $\left(x^n + o(x^m) = o(x^m) \text{ per } x \to 0 \right)$

2) $\begin{cases} f = o(g) & \text{per } x \to c \\ g = o(h) & \text{"} \quad \text{"} \end{cases} \Rightarrow f = o(h) \text{ per } x \to c$

(Se pensiamo $o(g) = \{ f \text{ definita in } I \setminus \{c\} \mid f = o(g) \text{ per } x \to c \}$ allora 2) si legge $g = o(h) \Rightarrow o(g) \subseteq o(h)$)

3) $f = o(g)$ per $x \to c$ $\Rightarrow fh = o(hg)$ per $x \to c$ $\forall h$ definita in $I \setminus \{c\}$

Se h è limitata in $I \setminus \{c\} \Rightarrow fh = o(g)$ per $x \to c$

$\left(h\,o(g) \subseteq o(hg) \overset{h \text{ limitata}}{\subseteq} o(g) \right)$

4) $\begin{cases} f_1 = o(g_1) & \text{per } x \to c \\ f_2 = o(g_2) & \text{"} \end{cases} \Rightarrow f_1 f_2 = o(g_1 g_2)$ per $x \to c$

$\left(o(g_1)\,o(g_2) \subseteq o(g_1 g_2) \right)$

5) $f_1, f_2 = o(g)$ per $x \to c$ $\Rightarrow f_1 \pm f_2 = o(g)$ per $x \to c$

$\left(o(g) \pm o(g) \subseteq o(g) \right)$

Es: $x^\alpha = o(e^x)$ per $x \to +\infty$ $\quad \forall \alpha \in \mathbb{R}$

$e^x = o(|x|^\alpha)$ per $x \to 0$ $\quad \forall \in \mathbb{R}$

$\log x = o(x^\alpha)$ per $x \to +\infty$ $\quad \forall \alpha > 0$

$x^\alpha = o(\log x)$ $\quad$ " $\quad \forall \alpha \leq 0$

$\log x = o(x^\alpha)$ per $x \to 0^+$ $\quad \forall \alpha < 0$

$x^\alpha = o(\log x)$ per $x \to 0^+$ $\quad \forall \alpha \geq 0$

teorema (FORMULA DI TAYLOR): se f è derivabile $(n-1)$-volte in I: intorno di x_0 e $f^{(n)}(x_0)$ esiste, allora

$$f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k}_{\text{polinomio di Taylor di } f \text{ in } x_0 \text{ di ordine } n} + o((x-x_0)^n) \quad \text{per } x \to x_0$$

(= somma parziale n-esima della serie di Taylor di f centrata in x_0)

Es: $(1+x)^\alpha = \sum_{k=0}^{n} \binom{\alpha}{k} x^k + o(x^n)$ per $x \to 0$ con $\begin{cases} \binom{\alpha}{k} := \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!} & k \geq 1 \\ \binom{\alpha}{0} := 1 \end{cases}$

$\forall \alpha \in \mathbb{R}$

OPERAZIONI CON GLI SVILUPPI DI TAYLOR

Nota: 6) $f_1, f_2 = o(x^n)$ per $x \to 0$ $\Rightarrow$ $f_1 \pm f_2 = o(x^n)$ per $x \to 0$

7) $\begin{cases} f_1 = o(x^n) & \text{per } x \to 0 \\ f_2 = o(x^m) & " \end{cases}$ $\Rightarrow$ $f_1 f_2 = o(x^{n+m})$ per $x \to 0$

In particolare: $f = o(x^n)$ per $x \to 0$ $\Rightarrow$ $f^p = o(x^{pn})$ per $x \to 0$

8) $\begin{cases} f_1 = o(x^n) & \text{per } x \to 0 \\ f_2 = o(x^m) & " \end{cases}$ $\Rightarrow$ $f_1(f_2(x)) = o(f_2^n) = o((x^m)^n) \overset{\text{perché } m \cdot n \geq n}{=} o(x^n)$

per $x \to 0$

Prop: $f_i(x) = P_n^i(x) + o(x^n)$ per $x \to 0$ $\quad i=1,2$ $\quad$ ($P_n^i(x)$: polinomio di Taylor di f in $x=0$ di ordine n)

1) $f_1(x) \pm f_2(x) = P_n^1(x) + P_n^2(x) + o(x^n)$ per $x \to 0$

$$f_1(x) f_2(x) = \underbrace{\left(P_n^1(x) \cdot P_n^2(x)\right)_{\leq n}}_{:= \text{ termini di grado } \leq n} + o(x^n) \quad \text{per } x \to 0$$

2) se inoltre $f_2(x) \neq 0$ in I: intorno di 0, allora $\exists\, Q_n(x)$ t.c.

$$P_n^1(x) = P_n^2(x)\, Q_n(x) + o(x^n) \quad \text{per } x \to 0$$

$\uparrow$ polinomio

$$\Rightarrow \frac{f_1(x)}{f_2(x)} = Q_n(x) + o(x^n) \quad \text{per } x \to 0$$

ALGORITMO DI EUCLIDE

3) se $\begin{cases} P_n^1(x) = a_0 + a_1 x + \dots + a_n x^n \\ \lim_{x \to 0} f_2(x) = 0 \iff P_n^2(0) = 0 \end{cases}$

$$\Rightarrow f_1(f_2(x)) = a_0 + a_1 P_n^2(x) + a_2 \left[\left(P_n^2(x)\right)^2\right]_{\leq n} + a_3 \left[\left(P_n^2(x)\right)^3\right]_{\leq n} + \dots$$

$$\dots + a_n \left[\left(P_n^2(x)\right)^n\right]_{\leq n} + o(x^n) \quad \text{per } x \to 0$$

Es: 1) Trovare lo sviluppo di Taylor di $e^{\operatorname{sen} x}$ in $x=0$ fino all'ordine 4.

Ho $\begin{cases} e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + o(x^4) & \text{per } x \to 0 \quad (f_1(x)) \\ \operatorname{sen} x = x - \frac{x^3}{3!} + o(x^4) & \text{per } x \to 0 \quad (f_2(x)) \end{cases}$

$$\Rightarrow e^{\operatorname{sen} x} = 1 + \operatorname{sen} x + \frac{\operatorname{sen}^2 x}{2!} + \frac{\operatorname{sen}^3 x}{3!} + \frac{\operatorname{sen}^4 x}{4!} + \underbrace{o(\operatorname{sen}^4 x)}_{= o(x^4)}$$

(perché $\operatorname{sen}^4 x \sim x^4$ per $x \to 0$)

$$= 1 + \left(x - \frac{x^3}{3!}\right) + \frac{1}{2}\left[\left(x - \frac{x^3}{3!}\right)^2\right]_{\leq 4} +$$

$$+ \frac{1}{6}\left[\left(x - \frac{x^3}{3!}\right)^3\right]_{\leq 4} + \frac{1}{24}\left[\left(x - \frac{x^3}{3!}\right)^4\right]_{\leq 4} + o(x^4)$$

$$= 1 + x - \frac{x^3}{6} + \frac{1}{2}\underbrace{\left[x^2 - 2 \cdot x \cdot \frac{x^3}{6} + \left(\frac{x^3}{6}\right)^2\right]_{\leq 4}} +$$

$$+\frac{1}{6}\underbrace{\Big[\Big(x-\frac{x^3}{6}\Big)\overbrace{\Big(x-\frac{x^3}{6}\Big)^2}^{\left(x^2-\frac{1}{3}x^4\right)}\Big]_{\le 4}}_{\underbrace{\left[\left(x-\frac{x^3}{6}\right)\left(x^2-\frac{1}{3}x^4\right)\right]_{\le 4}}_{x^3}} + \frac{1}{24}\underbrace{\Big[\Big(x-\frac{x^3}{6}\Big)x^3\Big]_{\le 4}}_{x^4} + o(x^4)$$

$$= 1 + x - \frac{x^3}{6} + \frac{1}{2}x^2 - \frac{1}{6}x^4 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + o(x^4)$$

$$= 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + o(x^4)$$

Equivalente: $\operatorname{sen} x = x - \frac{x^3}{6} + o(x^4)$

$$\Rightarrow \operatorname{sen}^2 x = \Big(x - \frac{x^3}{6} + o(x^4)\Big)^2 = \Big(x-\frac{x^3}{6}\Big)^2 + o(x^4)$$

$$= x^2 - \frac{x^4}{3} + o(x^4)$$

$$\Rightarrow \operatorname{sen}^3 x = \Big(x^2 - \frac{x^4}{3} + o(x^4)\Big)\Big(x - \frac{x^3}{6} + o(x^4)\Big) = x^3 + o(x^4)$$

$$\Rightarrow \operatorname{sen}^4 x = \big(x^3 + o(x^4)\big)\Big(x - \frac{x^3}{6} + o(x^4)\Big) = x^4 + o(x^4)$$

2) $\log(\cos x)$

Ho
$$\begin{cases} \log(1+x) = x - \frac{x^2}{2} + o(x^2) & \text{per } x \to 0 \\ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + o(x^4) & \text{per } x \to 0 \end{cases}$$

$$\Rightarrow \cos x - 1 \sim x^2 \Rightarrow (\cos x - 1)^2 \sim x^4 \Rightarrow o\big((\cos x - 1)\big) = o(x^4)$$

Inoltre: $(\cos x - 1)^2 = \frac{x^4}{4} + o(x^4)$ per $x \to 0$

$$\Rightarrow \log(\cos x) = \log(1 + \cos x - 1) = (\cos x - 1) - \frac{(\cos x - 1)^2}{2} + o\big((\cos x - 1)^2\big)$$

$$= -\frac{x^2}{2} + \frac{x^4}{24} - \frac{x^4}{8} + o(x^4) \quad \text{per } x \to 0$$

$$= -\frac{x^2}{2} - \frac{1}{12}x^4 + o(x^4)$$

3) $\dfrac{1}{-\sqrt{8}\operatorname{sen} x - 2\cos x}$ all'ordine 3

4) $\lim_{x \to 0} \frac{2 \operatorname{sen} x \cos x - 2e^{x} \log(1+x) + x^2}{\operatorname{tg} x - \operatorname{sen} x}$

3) $(1+x)^x$ all'ordine 4

4) $\lim_{x \to 0} \frac{\operatorname{sen}(\alpha x) - 4x + 1 - \cos(2x)}{\operatorname{tg}(x^2) - x^2 + \sqrt{1+\alpha x} - x - 1}$ $\forall \alpha \in \mathbb{R}$ $\sqrt{1+\alpha x} = (1+\alpha x)^{1/2}$