COMPLEX ANALYSIS - SELECTED PROBLEMS

PIETRO POLESELLO

Exercises from the final examinations
Master degree in Mathematics (2013-2019)

Problem 1. *Let* $\log$ *denote the principal branch of the logarithm and* $[z_1, z_2]$ *the oriented line segment joining* z_1 *with* z_2*.*

(a) *Prove that* $|\log(1-z)| \leq \frac{\pi}{2} - \log(1-\cos\epsilon)$ *for any* $z \in [e^{-i\epsilon}, e^{i\epsilon}]$ *with* $0 < \epsilon < \frac{\pi}{2}$*. (*Hint: *note that* $|1-z| < 1$*.)*

(b) *Show that* $\int_{[e^{-i\epsilon}, e^{i\epsilon}]} \frac{\log(1-z)}{z}\, dz \to 0$ *as* $\epsilon \to 0+$*.*

(c) *Prove that* $\int_{\gamma_\epsilon} \frac{\log(1-z)}{z}\, dz \to 0$ *as* $\epsilon \to 0+$*, for* γ_ϵ *the arc of the unit circle joining* $e^{i\epsilon}$ *with* $e^{-i\epsilon}$*. (*Hint: *use the Cauchy formula.)*

(d) *Show that* $\int_0^{2\pi} \log|1-e^{i\theta}|\, d\theta = 0$*.*

(e) *Set* $g_a(z) = \begin{cases} \frac{a-z}{1-\bar{a}z} & \text{if } |a| < 1 \\ \frac{a-z}{a} & \text{if } |a| = 1 \end{cases}$*. Prove that* $\int_0^{2\pi} \log|g_a(e^{i\theta})|\, d\theta = 0$*.*

(f) *Set* $\mathbb{E} = \{|z| < 1\}$ *and let* $h \in \mathcal{O}(\overline{\mathbb{E}})$ *be never vanishing. Show that* $\int_0^{2\pi} \log|h(e^{i\theta})|\, d\theta = 2\pi \log|h(0)|$*.*

(g) *Let* $f \in \mathcal{O}(\overline{\mathbb{E}})$ *satisfying* $f(0) \neq 0$*. Prove that* f *has a finite number of zeros in* $\overline{\mathbb{E}}$*. Let* $a_1, \dots, a_m$ *those zeros in* $\mathbb{E}$*, counting multiplicities. Prove that*

$$\int_0^{2\pi} \log|f(e^{i\theta})|\, d\theta = 2\pi \sum_{i=1}^m \log\left|\frac{f(0)}{a_i}\right|.$$

*(*Hint: *use the* g_a*'s defined in (e) and (f).)*

Solution. Recall that $\log z = \log|z| + i\arg z$, where $\arg z \in (-\pi, \pi)$ denotes the principal argument.

(a) For $z \in [e^{-i\epsilon}, e^{i\epsilon}]$, with $0 < \epsilon < \frac{\pi}{2}$, we have $0 < |1-z| < 1$, hence $|\arg(1-z)| < \frac{\pi}{2}$ and $\log|1-z| < 0$, and we get

$$|\log(1-z)| \leq |\mathrm{Re}\log(1-z)| + |\mathrm{Im}\log(1-z)| = |\log|1-z|| + |\arg(1-z)| < -\log|1-z| + \frac{\pi}{2}.$$

The result then follows from $|1-z| \geq \mathrm{Re}(1-z) = 1-\cos\epsilon$ for $z \in [e^{-i\epsilon}, e^{i\epsilon}]$.

(b) Since $|z| \geq \mathrm{Re} z = \cos\epsilon$ for any $z \in [e^{-i\epsilon}, e^{i\epsilon}]$, by (a) we get

$$\left|\frac{\log(1-z)}{z}\right| \leq \frac{\frac{\pi}{2} - \log(1-\cos\epsilon)}{\cos\epsilon},$$

hence by the standard estimate

$$\left|\int_{[e^{-i\epsilon}, e^{i\epsilon}]} \frac{\log(1-z)}{z}\, dz\right| \leq 2\sin\epsilon \frac{\frac{\pi}{2} - \log(1-\cos\epsilon)}{\cos\epsilon} \to 0 \quad \text{as } \epsilon \to 0+.$$

Please communicate possible errors to pietro@math.unipd.it.

(c) Since $\log(1-z)$ is holomorphic in $\mathbb{C} \setminus \mathbb{R}_{\geq 1}$, by the Cauchy formula we have

$$0 = 2\pi i \log 1 = \int_{\partial K} \frac{\log(1-z)}{z} \, dz$$

where ∂K is the boundary of the compact delimited by γ_ϵ and by the line segment $[e^{-i\epsilon}, e^{i\epsilon}]$, oriented counterclockwise. By taking $\lim_{\epsilon \longrightarrow 0+}$, we get the result by (b).

(d) By (c), we get

$$0 = \lim_{\epsilon \longrightarrow 0+} \operatorname{Im} \int_{\gamma_\epsilon} \frac{\log(1-z)}{z} \, dz = \lim_{\epsilon \longrightarrow 0+} \operatorname{Im} \int_{\epsilon}^{2\pi - \epsilon} \frac{\log(1-e^{i\theta})}{e^{i\theta}} \, de^{i\theta} = \int_0^{2\pi} \log |1 - e^{i\theta}| \, d\theta.$$

(e) Let $|a| = 1$. Then $a = e^{i\alpha}$ for $\alpha \in [0, 2\pi]$ and by (d) we get

$$\int_0^{2\pi} \log |g_a(e^{i\theta})| \, d\theta = \int_0^{2\pi} \log \left|1 - e^{i(\theta - \alpha)}\right| \, d\theta = 0.$$

If $|a| < 1$, the result follows from $|g_a(e^{i\theta})| = \left| \frac{a - e^{i\theta}}{1 - \bar{a} e^{i\theta}} \right| = \left| \frac{a - e^{i\theta}}{e^{-i\theta} - \bar{a}} \right| = 1$.

(f) Since $\mathbb{E}$ is simply connected and h is never vanishing on $\overline{\mathbb{E}}$, there exists a logarithm l of h on a neighborhood of $\mathbb{E}$. In particular, $\operatorname{Re} l(z) = \log |h(z)|$. The result then follows by taking the real part of the Cauchy formula $\frac{1}{2\pi i} \int_{\partial \mathbb{E}} \frac{l(z)}{z} \, dz = l(0)$.

(g) Since f is not identically zero, its set of zeros is locally finite, hence finite on the compact $\overline{\mathbb{E}}$. Let $b_1, \ldots, b_n$ denotes the zeros in $\partial \mathbb{E}$, counting multiplicities. Then $f g_{a_1}^{-1} \ldots g_{a_m}^{-1} g_{b_1}^{-1} \ldots g_{b_n}^{-1}$ extends to a never vanishing function h on $\overline{\mathbb{E}}$. By (f), thanks to $g_{b_1}(0) = \cdots = g_{b_n}(0) = 1$, we get

$$\int_0^{2\pi} \log |h(e^{i\theta})| \, d\theta = 2\pi \log |h(0)| = \sum_{i=1}^{m} \log \left| \frac{f(0)}{a_i} \right|.$$

$\square$

Problem 2. *Let g be an entire function.*

(a) *Assume that the equation*

$$2g(z) = g\left(\frac{z}{2}\right) + g\left(\frac{z+1}{2}\right)$$

*holds for all $z \in \mathbb{C}$. Prove that g is constant. (*Hint*: apply the Maximum Modulus principle on $\overline{B_r(0)} = \{|z| \leq r\}$ for $r > 1$)*

(b) *Assume that g has no zeros and*

$$cg(z) = g\left(\frac{z}{2}\right) g\left(\frac{z+1}{2}\right) \tag{0.1}$$

*holds for all $z \in \mathbb{C}$ and some $c \in \mathbb{C}^\times$. Prove that $g(z) = ce^{-\frac{b}{2}} e^{bz}$ for some $b \in \mathbb{C}$. (*Hint*: take $\partial \log g$)*

(c) *Assume that g is odd, it has zeros exactly at the points $z \in \mathbb{Z}$, all with multiplicity 1, and*

$$cg(2z) = g(z)g\left(z+\frac{1}{2}\right)$$

*holds for all $z \in \mathbb{C}$ and some $c \in \mathbb{C}^\times$. Prove that $g(z) = 2c\sin(\pi z)$. (*Hint*: consider $\frac{g(z)}{\sin(\pi z)}$ and use the duplication formula for $\sin(\pi z)$)*

Solution. 23 (a) Set $M = \|g\|_{\overline{B_r(0)}}$ for some $r > 1$. By the Maximum Modulus principle (see [7, Lecture 6]), g is either constant on $\overline{B_r(0)}$ (hence on $\mathbb{C}$ by the Identity principle) or there exists $z_0 \in \partial B_r(0)$ satisfying $|g(z_0)| = M$ and $|g(z)| < M$ for any $z \in B_r(0)$. In the latter case, since both $\frac{z_0}{2}$ and $\frac{z_0+1}{2}$ lie in $B_r(0)$, we get

$$2M = |2g(z_0)| \leq \left|g\left(\frac{z_0}{2}\right)\right| + \left|g\left(\frac{z_0+1}{2}\right)\right| < 2M$$

Contradiction.

(b) Since g never vanishes, we may consider the entire function $h = \partial \log g$. By taking the logarithmic derivative of the functional equation in (b), we get that h satisfies the functional equation in (a), hence it is constant. It follows that $g' = bg$ for some $b \in \mathbb{C}$, hence $g(z) = ae^{bz}$ for some $a \in \mathbb{C}$. By using again the functional equation in (b), we get $c = ae^{\frac{b}{2}}$.

(c) Since both $g(z)$ and $\sin(\pi z)$ have zeros exactly at the points $z \in \mathbb{Z}$, all with multiplicity 1, the meromorphic function $h(z) = \frac{g(z)}{\sin(\pi z)}$ extends to a never vanishing entire function, still denoted by h. From the duplication formula $\sin(2\pi z) = 2\sin(\pi z)\cos(\pi z) = 2\sin(\pi z)\sin\left(\pi\left(z+\frac{1}{2}\right)\right)$ follows that h satisfies the functional equation in (b), with c replaced by $2c$, hence $h(z) = 2ce^{-\frac{b}{2}}e^{bz}$ for some $b \in \mathbb{C}$. As $g(z)$ and $\sin(\pi z)$ are odd, h is even hence $b = 0$. □

Problem 3. *Let f be an holomorphic function on the unit disk $\mathbb{E} = \{|z| < 1\}$, with bounded derivative and satisfying $f(0) = 0$, $f'(0) = 1$.*

(a) *Show that there exists $r \in]0,1[$ such that*

$$|f'(z) - 1| \leq \frac{|z|}{r} \quad \text{for any } z \in \mathbb{E}.$$

*(*Hint*: use Schwarz' lemma.)*

(b) *Show that*

$$|f(z) - z| \leq \frac{|z|^2}{2r} \quad \text{for any } z \in \mathbb{E}.$$

*(*Hint*: consider $\int_\gamma (f'(z) - 1)\,dz$ with γ a line segment.)*

(c) *Deduce that, for any $|z| = r$ and $|w| < \frac{r}{2}$, one has*

$$|f(z) - z| < |z - w|.$$

(d) *Show that $B_{\frac{r}{2}}(0) \subset f(B_r(0))$ (recall that $B_r(0) = \{|z| < r\}$). (*Hint*: apply the argument principle to $B_r(0) \ni z \mapsto \frac{f(z)-w}{z-w}$ for $w \in B_{\frac{r}{2}}(0)$.)*

(e) *Conclude that f maps conformally $B_r(0)$ onto an open neighbourhood of $B_{\frac{r}{2}}(0)$.*

Solution. (a) Let us fix a constant $C > 0$ such that $|f'(z)| < C$ for any $z \in \mathbb{E}$.

As $|f'(z) - 1| < C + 1$, the function

$$g(z) = \frac{f'(z) - 1}{C + 1}$$

satisfies $g(0) = 0$ and $|g(z)| < 1$ for any $z \in \mathbb{E}$. We may thus apply Schwarz' lemma and get $|g(z)| \leq |z|$ for any $z \in \mathbb{E}$. One then sets $r = \frac{1}{C+1} \in]0, 1[$.

We may also proceed as follows: as $\lim_{z \to 0} \frac{f'(z)-1}{z} = f''(0)$, the function $\frac{f'(z)-1}{z}$ extends to an holomorphic function $g(z)$ on $\mathbb{E}$. One has

$$|g(z)| = \frac{|f'(z) - 1|}{|z|} < \frac{C + 1}{\rho} \quad \text{for any } |z| = \rho \in]0, 1[.$$

Hence, by the Maximum Modulus theorem, $|g(z)| < \frac{C+1}{\rho}$ for any $|z| \leq \rho$ and, by letting $\rho \to 1-$, one gets $|g(z)| \leq C + 1$ for any $z \in \mathbb{E}$. Again, one sets $r = \frac{1}{C+1}$.

(b) For $z \in \mathbb{E}$ consider the path $\gamma(t) = tz$ with $t \in [0, 1]$. Then

$$|f(z) - z| = \left| \int_\gamma (f'(z) - 1)\, dz \right| \leq |z| \int_0^1 |f'(tz) - 1|\, dt \leq |z| \int_0^1 \frac{|tz|}{r}\, dt = \frac{|z|^2}{2r},$$

where the last inequality follows from (a).

(c) From (b), it follows that

$$|f(z) - z| \leq \frac{r^2}{2r} = \frac{r}{2} \quad \text{for any } |z| = r.$$

For $|w| < \frac{r}{2}$, we thus have $|z - w| > \frac{r}{2} \geq |f(z) - z|$ for any $|z| = r$.

(d) Fix $w \in B_{\frac{r}{2}}(0)$ and consider the function

$$h(z) = \frac{f(z) - w}{z - w}.$$

From (c) it follows that $|h(z) - 1| < 1$ for any $|z| = r$, hence the function $\log h(z)$ is well defined and holomorphic on a neighbourhood of the circle $\{|z| = r\}$. Therefore

$$\int_{|z|=r} \partial \log h(z)\, dz = 0,$$

so that, by the argument principle, $h(z)$ has the same number of zeros and poles on $B_r(0)$, i.e. the function $f(z) - w$ has the same number of zeros on $B_r(0)$ of the function $z - w$. It follows that for any $w \in B_{\frac{r}{2}}(0)$ there exists (only one) $z \in B_r(0)$ such that $f(z) = w$.

(e) As f is non-constant, from (d) and by the open mapping theorem it follows that f maps $B_r(0)$ onto an open neighbourhood of $B_{\frac{r}{2}}(0)$. It remains to prove that f is also injective on $B_r(0)$.

We may proceed as in (b): given $z_0, z_1 \in B_r(0)$, let $\gamma(t) = (1-t)z_0 + tz_1$ with $t \in [0, 1]$ be the line segment joining them. Then

$$|(f(z_1) - z_1) - (f(z_0) - z_0)| = \left| \int_\gamma (f'(z) - 1)\, dz \right| \leq |z_1 - z_0| \int_0^1 \frac{|\gamma(t)|}{r}\, dt.$$

As $\gamma(t) \in B_r(0)$ for any t, the last integral is < 1. Then $f(z_1) = f(z_0)$ implies $|z_1 - z_0| = 0$, i.e. $z_1 = z_0$.

Note that, from (a) we get immediately that $f'(z) \neq 0$ for any $z \in B_r(0)$, hence f is locally injective on $B_r(0)$. However, this does not prove that f is injective on $B_r(0)$. $\square$

Problem 4. *Let $\mathbb{E}$ denote the unit disk $\{|z| < 1\}$. For $z_1, z_2 \in \mathbb{E}$, set*

$$d(z_1, z_2) = \frac{|z_1 - z_2|}{|1 - \bar{z}_2 z_1|}.$$

(a) *Show that d is a metric on $\mathbb{E}$.*

(b) *Let f be a bounded holomorphic function on $\mathbb{E}$. Show that, if $\|f\|_{\mathbb{E}} \leq 1$, then*

$$d(f(z_1), f(z_2)) \leq d(z_1, z_2) \qquad \text{for any } z_1, z_2 \in \mathbb{E}. \tag{0.2}$$

*(*Hint*: use the Schwarz lemma for a suitable function.)*

(c) *Prove that, in the above situation, f is either a contraction w.r.t. d,* i.e. *strict inequality holds in* (0.2) *for $z_1 \neq z_2$, or an isometry w.r.t. d,* i.e. *equality holds in* (0.2).

(d) *Show that, in the above situation, for any $z \in \mathbb{E}$*

$$|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}.$$

(e) *For $z_1, z_2 \in \mathbb{E}$, characterise the bounded functions $f \in \mathcal{O}(\mathbb{E})$ with $\|f\|_{\mathbb{E}} \leq 1$ and $f(z_1) = z_2$, which maximize $|f'(z_1)|$.*

Solution. (a) By definition, for any $z_1, z_2 \in \mathbb{E}$ we have $d(z_1, z_2) \geq 0$, $d(z_1, z_2) = 0$ iff $z_1 = z_2$, and

$$d(z_2, z_1) = \frac{|z_2 - z_1|}{|\overline{1 - \bar{z}_1 z_2}|} = d(z_1, z_2).$$

As usual, set $g_a(z) = \frac{z-a}{\bar{a}z-1}$ for $a \in \mathbb{E}$. Then

$$d(z_1, z_2) = |g_{z_1}(z_2)| = d(0, g_{z_1}(z_2)).$$

Hence it is enough to prove that $d(z_1, z_2) \leq d(z_1, 0) + d(0, z_2)$, that is,

$$|g_{z_1}(z_2)| \leq |z_1| + |z_2|.$$

This follows from (see [7, Lecture 6] for the first equality)

$$|g_{z_1}(z_2)|^2 = 1 - \frac{(1-|z_1|^2)(1-|z_2|^2)}{|1 - \bar{z}_1 z_2|^2} \leq 1 - \frac{(1-|z_1|^2)(1-|z_2|^2)}{(1+|z_1||z_2|)^2} = \frac{(|z_1|+|z_2|)^2}{(1+|z_1||z_2|)^2} \leq (|z_1|+|z_2|)^2.$$

(b) Let f be a bounded holomorphic function on $\mathbb{E}$ with $\|f\|_{\mathbb{E}} \leq 1$. If $|f(z_0)| = 1$ for some $z_0 \in \mathbb{E}$, then z_0 is a maximum for $|f|$. By the Maximum Modulus theorem, f is constant and the inequality is trivially satisfied.

Suppose that $|f(z)| < 1$ for any $z \in \mathbb{E}$. We have to prove that

$$|g_{f(z_1)} \circ f(z_2)| \leq |g_{z_1}(z_2)| \qquad \text{for any } z_1, z_2 \in \mathbb{E}.$$

Set $F = g_{f(z_1)} \circ f \circ g_{z_1}$. Then F defines an holomorphic map $\mathbb{E} \to \mathbb{E}$ satisfying $F(0) = 0$. By the Schwarz lemma, $|F(z)| \leq |z|$ for any $z \in \mathbb{E}$. It follows that

$$|g_{f(z_1)} \circ f(z_2)| = |g_{f(z_1)} \circ f \circ g_{z_1} \circ g_{z_1}(z_2)| = |F(g_{z_1}(z_2))| \leq |g_{z_1}(z_2)|.$$

(c) If f is constant, then f is trivially a contraction w.r.t. d.

Suppose that $|f(z)| < 1$ for any $z \in \mathbb{E}$. If the equality holds in (0.2) for some $z_1, z_2 \in \mathbb{E}$, then $|F(z_0)| = |z_0|$ for $z_0 = g_{z_1}(z_2) \in E$. Again by the Schwarz lemma, F is a rotation, hence $|F(z)| = |z|$ for all $z \in \mathbb{E}$ and it follows that f is an isometry w.r.t. d. Otherwise f is a contraction w.r.t. d.

(d) If f is constant, then the inequality is trivially satisfied (equality if $|f(z)| = 1$ for any $z \in \mathbb{E}$).

Suppose that $|f(z)| < 1$ for any $z \in \mathbb{E}$. Again by the Schwarz lemma, $|F'(0)| \leq 1$. Since $|g_a'(0)| = 1 - |a|^2$ and $|g_a'(a)| = \frac{1}{1-|a|^2}$, we get

$$|F'(0)| = |g'_{f(z_1)}(f(z_1))f'(z_1)g'_{z_1}(0)| = \frac{1}{1-|f(z_1)|^2}|f'(z_1)|(1-|z_1|^2)$$

for any $z_1 \in \mathbb{E}$ and the result follows.

(e) By (d), for any $z_1, z_2 \in \mathbb{E}$ and any bounded function $f \in \mathcal{O}(\mathbb{E})$ with $\|f\|_{\mathbb{E}} \leq 1$ and $f(z_1) = z_2$, we have $|f'(z_1)| \leq \frac{1-|z_2|^2}{1-|z_1|^2}$. Since $|f(z_1)| = |z_2| < 1$, again by the Schwarz lemma, the equality holds for some $z_1, z_2 \in \mathbb{E}$ iff F is a rotation. It follows that $f(z)$ is an automorphism of $\mathbb{E}$ interchanging z_1 with z_2. □

Problem 5. *Let $f \in \mathcal{O}(\mathbb{E})$ satisfies $f(0) = 1$, where $\mathbb{E}$ denotes the unit disk $\{|z| < 1\}$.*

(a) *Show that, if f extends analytically to a neighbourhood of $\mathbb{E}$ in such a way that $|f(z)| > 1$ for $|z| = 1$, then f has a zero in $\mathbb{E}$.*

(b) *Assume that Re$f \geq 0$ in $\mathbb{E}$. Show that Ref never vanishes in $\mathbb{E}$. (*Hint: *consider $h(z) = e^{-f(z)}$.)*

(c) *Set $g(z) = \frac{f(z)-1}{f(z)+1}$. In the situation of (b), show that $|g(z)| \leq |z|$ for any $z \in \mathbb{E}$.*

(d) *In the situation of (b), prove that $\frac{1-|z|}{1+|z|} \leq |f(z)| \leq \frac{1+|z|}{1-|z|}$ for any $z \in \mathbb{E}$.*

(e) *In the situation of (b), prove that there exists $z_0 \in \mathbb{E} \setminus \{0\}$ making one of the two inequalities in (d) an equality if and only if $f(z) = \frac{1+\lambda z}{1-\lambda z}$ for $|\lambda| = 1$.*

Solution. (a) Assume that f never vanishes in $\mathbb{E}$. Then, by hypothesis, f is holomorphic and never vanishes in a neighbourhood U of $\mathbb{E}$. It follows that $g = \frac{1}{f}$ is holomorphic in U and satisfies $|g(z)| < \|g\|_{\partial\mathbb{E}} < 1 = g(0)$ for any $z \in \mathbb{E}$ by the Maximum Modulus theorem. Contradiction.

(b) By hypothesis, the function $h(z) = e^{-f(z)}$ is holomorphic in $\mathbb{E}$ and satisfies $h(0) = \frac{1}{e}$ and $|h(z)| = e^{-\mathrm{Re}f(z)} \leq 1$ for any $z \in \mathbb{E}$. Assume that Re$f(z_0) = 0$ for $z_0 \in \mathbb{E}$. Then $|h(z)|$ attains its maximum in z_0, hence it has constant value 1 by the Maximum Modulus theorem. Contradiction.

(c) First note that $t(z) = \frac{z-1}{z+1}$ defines a conformal map $\{\mathrm{Re}z > 0\} \to \mathbb{E}$. This follows by a direct computation, or by noticing (see [7, Lecture ?]) that t may be decomposed into

$$\{\mathrm{Re}z > 0\} \underset{a}{\to} \{\mathrm{Re}z > 1\} \underset{b}{\to} \{|z - \frac{1}{2}| < \frac{1}{2}\} \underset{c}{\to} \mathbb{E}$$

where $b(z) = \frac{1}{z}$, $a(z) = z + 1$ and $c(z) = 1 - 2z$ are surjective affine transformations, hence conformal maps; equivalently, $t = h \circ d$ where $d(z) = iz$ and $h(z) = \frac{z-i}{z+i}$ is the Cayley map.

By (b), f defines a map $\mathbb{E} \to \{\mathrm{Re}z > 0\}$, hence $g = t \circ f$ defines a map $\mathbb{E} \to \mathbb{E}$ such that $g(0) = 0$. By the Schwarz's lemma, $|g(z)| \leq |z|$ for any $z \in \mathbb{E}$.

(d) By triangular inequality and (c), for any $z \in \mathbb{E}$

$$\left|\frac{|f(z)|-1}{|f(z)|+1}\right| = \frac{||f(z)|-1|}{|f(z)|+1} \leq \left|\frac{f(z)-1}{f(z)+1}\right| \leq |z|$$

that is,

$$\frac{1-|z|}{1+|z|} \leq |f(z)| \leq \frac{1+|z|}{1-|z|}.$$

(e) By (d), there exists $z_0 \in \mathbb{E} \setminus \{0\}$ making one of the two inequalities in (d) an equality if and only if $|g(z_0)| = |z_0|$. Again by the Schwarz's lemma, this happens if and only if there exists $|\lambda| = 1$ such that $g(z) = \lambda z$, that is, $f(z) = \frac{1+\lambda z}{1-\lambda z}$. $\square$

Problem 6. *Let $D \subset \mathbb{C}$ be an unbounded domain (*i.e. *non-empty, open and connected) and $f \in \mathcal{O}(D)$ be continuous on $\overline{D}$ and bounded on ∂D.*

(a) *Assume $\lim_{z \to \infty, z \in \overline{D}} f(z) = 0$. Prove that the set $K_c = \{z \in \overline{D};\, |f(z)| \geq c\}$ is compact for any $c > 0$.*
(b) *In the situation of (a), prove that $K_c = \varnothing$ for any $c > \|f\|_{\partial D}$. (*Hint: *use the Maximum Modulus theorem.) Deduce that $\|f\|_{\overline{D}} = \|f\|_{\partial D}$ and $|f(z)| < \|f\|_{\partial D}$ for any $z \in D$ if f is non-constant.*
(c) *Assume that $|f(z)| \leq \log(a + |z|)$ for any $z \in D$ and some $a > 0$, and set $h(z) = \frac{f(z)-f(z_0)}{z-z_0}$ for some $z_0 \in D$. Prove that h extends to an holomorphic function on D which is continuos on $\overline{D}$, bounded on ∂D and satisfies $|f^n(z)h(z)| \leq \|f\|_{\partial D}^n \|h\|_{\partial D}$ for any $z \in \overline{D}$ and any $n \geq 0$.*
(d) *In the situation of (c), prove that $\|f\|_{\overline{D}} = \|f\|_{\partial D}$ and $|f(z)| < \|f\|_{\partial D}$ for any $z \in D$ if f is non-constant.*

Solution. (a) Fix $c > 0$. As f is continuous in $\overline{D}$, a closed subset of $\mathbb{C}$, the subset $K_c = \{z \in \overline{D};\, |f(z)| \geq c\} = f^{-1}\{[c, +\infty[\}$ is closed in $\overline{D}$, hence in $\mathbb{C}$.

As $\lim_{z \to \infty, z \in \overline{D}} f(z) = 0$, there exists $R > 0$ such that $|f(z)| < c$ for any $z \in \overline{D} \setminus \overline{B}_R(0)$, therefor $K_c \subset B_R(0)$, *i.e.* it is bounded. It follows that K_c is compact in $\mathbb{C}$, hence in $\overline{D}$.

(b) Fix $c > \|f\|_{\partial D}$ and assume $K_c \neq \varnothing$. As $|f|$ is continuous in the compact subset K_c, by Weierstrass' theorem there exists $z_0 \in K_c$ such that

$$|f(z_0)| \geq |f(z)| \geq c > \|f\|_{\partial D} \qquad \text{for any } z \in K_c \subset \overline{D},$$

hence $z_0 \notin \partial D$, *i.e.* $z_0 \in D$. Moreover, $|f(z_0)| \geq c > |f(z)|$ for any $z \in D \setminus K_c$, therefor z_0 is a (global) maximum point for $|f|$ in D. By the Maximum Modulus theorem, f needs to be constant in D, hence in $\overline{D}$. Contradiction, as $|f(z_0)| > \|f\|_{\partial D}$.

As $K_c = \varnothing$ for any $c > \|f\|_{\partial D}$, we get $|f(z)| \leq \inf\{c > \|f\|_{\partial D}\} = \|f\|_{\partial D}$ for any $z \in \overline{D}$, hence $\|f\|_{\overline{D}} \leq \|f\|_{\partial D} \leq \|f\|_{\overline{D}}$, *i.e.* $\|f\|_{\overline{D}} = \|f\|_{\partial D}$. If f is non-constant, then $|f(z)| < \|f\|_{\partial D}$ for any $z \in D$, otherwise, if $|f(z_0)| = \|f\|_{\partial D}$ for $z_0 \in D$, then z_0 is a (global) maximum point for $|f|$ in D and f needs to be constant in D.

(c) Clearly $h(z) = \frac{f(z)-f(z_0)}{z-z_0}$ extends to an holomorphic function on D, which is continuos on $\overline{D}$, by setting $h(z_0) = f'(z_0)$. Moreover, for any $z \in \partial D$ we have

$$|h(z)| = \frac{|f(z) - f(z_0)|}{|z - z_0|} \leq \frac{|f(z)| + |f(z_0)|}{|z - z_0|} \leq \frac{\|f\|_{\partial D} + |f(z_0)|}{\operatorname{dist}(z_0, D)},$$

where $\operatorname{dist}(z_0, D) = \min_{z \in \partial D} |z - z_0| > 0$, hence $h(z)$ is bounded on ∂D.

Let's prove that $\lim_{z \to \infty, z \in \overline{D}} f^n(z)h(z) = 0$ for any $n \geq 0$. It will follows from (b) that $|f^n(z)h(z)| \leq \|f^n h\|_{\partial D} \leq \|f\|_{\partial D}^n \|h\|_{\partial D}$ for any $z \in \overline{D}$ and any $n \geq 0$.

For $|z| >> |z_0|$, we have

$$|f^n(z)h(z)| \le |f(z)|^n \frac{|f(z)| + |f(z_0)|}{|z| - |z_0|}$$
$$\le \begin{cases} \|f\|_{\partial D}^n \frac{\|f\|_{\partial D} + |f(z_0)|}{|z|-|z_0|} & \text{if } z \in \partial D, \\ |\log(a+|z|)|^n \frac{|\log(a+|z|)| + |\log(a+|z_0|)|}{|z|-|z_0|} \le 2\frac{\log(a+|z|)|^{n+1}}{|z|-|z_0|} & \text{if } z \in D, \end{cases}$$

where both terms in the right-hand side tend to 0 as $z \to \infty$.

(d) If f is constant, then $\|f\|_{\overline{D}} = \|f\|_{\partial D}$ trivially. If f is non-constant, then $h(z)$ is non identically zero and for any $z \in \overline{D}$ and any $n \ge 0$, we have $|f(z)||h(z)|^{\frac{1}{n}} \le \|f\|_{\partial D} \|h\|_{\partial D}^{\frac{1}{n}}$. It follows that

$$|f(z)| \le \inf \left\{ \left(\frac{\|h\|_{\partial D}}{|h(z)|} \right)^{\frac{1}{n}} \|f\|_{\partial D} \right\} = \|f\|_{\partial D} \qquad \text{for any } z \in \overline{D} \setminus \{z;\, h(z) = 0\},$$

hence $|f(z)| \le \|f\|_{\partial D}$ for any $z \in \overline{D}$, as $\overline{D} \setminus \{z;\, h(z) = 0\}$ is dense in $\overline{D}$. As before, we get $\|f\|_{\overline{D}} = \|f\|_{\partial D}$ and $|f(z)| < \|f\|_{\partial D}$ for any $z \in D$. □

Problem 7. *Let $D \subset \mathbb{C}$ be a domain (*i.e.* non-empty, open and connected) and $\mathbb{E}$ the unit disk. Suppose that there exists a non-constant bounded holomorphic function on D.*

(a) *Show that for any given $z_0 \in D$, there exists $g\colon D \to \mathbb{E}$ holomorphic such that $g(z_0) = 0 \ne g'(z_0)$. (*Hint: *consider $\frac{h(z)-h(z_0)}{(z-z_0)^n}$ for suitable $h(z)$ and n.)*
(b) *For any given $z_0 \in D$, find a holomorphic function $G\colon D \to \mathbb{E}$ satisfying $|G'(z_0)| \ge |g'(z_0)|$ for any holomorphic function $g\colon D \to \mathbb{E}$.*
(c) *For $G(z)$ as in (b), show that $G'(z_0) \ne 0 = G(z_0)$ (*Hint: *consider $g = g_{G(z_0)} \circ G$, for $g_{G(z_0)}$ the Möbius transformation ...)*
(d) *For $G(z)$ as in (b) with $D = \mathbb{E}$ and $z_0 = 0$, prove that G is a rotation.*
(e) *For $G(z)$ as in (b) with D proper simply connected, prove that G is conformal.*
(f) *Show that there exists a holomorphic function $G\colon \mathbb{C}\setminus[-2,2] \to \mathbb{E}$ satisfying $G(i) = 0$, $G'(i) > 0$ and $G'(i) \ge |g'(i)|$ for any holomorphic function $g\colon \mathbb{C}\setminus[-2,2] \to \mathbb{E}$. Show that G cannot be conformal.*

Solution. (a) Let $h \in \mathcal{O}(D)$ be non-constant and bounded and $m \ge 1$ be the multiplicity of h at z_0. Then $(h(z) - h(z_0))(z - z_0)^{-(m-1)}$ extends to a holomorphic function $\tilde{g}(z)$ on D satisfying $\tilde{g}(z_0) = 0 \ne \tilde{g}'(z_0)$. Let $\overline{B_r(z_0)} = \{|z - z_0| \le r\} \subset D$ for some $r > 0$. Then $\tilde{g}(z)$ is bounded on $\overline{B_r(z_0)}$, compact, and also on $D \setminus \overline{B_r(z_0)}$, since on such subset h is bounded and $\left|(z - z_0)^{-(m-1)}\right| < r^{-(m-1)}$. Set $g(z) = \frac{1}{\|\tilde{g}\|_D}\tilde{g}(z)$. As $g(z)$ is non-constant and D connected, then $g(D) \subset \mathbb{E}$ by the Open Mapping theorem.

(b) The set $\mathcal{F} = \{g \in \mathcal{O}(D);\, \|g\|_D \le 1\} \subset \mathcal{O}(D)$ is closed and bounded , hence compact by Montel's theorem. It follows that the continuous function $\mathcal{F} \to \mathbb{R}_{\ge 0}, \quad g \mapsto |g'(z_0)|$ admits a maximum $G \in \mathcal{F}$, *i.e.* $|G'(z_0)| \ge |g'(z_0)|$ for any $g\colon D \to \mathbb{E}$ holomorphic . By (a), there is a function $g \in \mathcal{F}$ with $g'(z_0) \ne 0$, hence $G'(z_0) \ne 0$, so that G is non-constant, therefore $G(D) \subset \mathbb{E}$ by the Open Mapping theorem.

(c) It remains to prove that $G(z_0) = 0$. Recall that for any $a \in \mathbb{E}$ the Möbius transformation $g_a(z) = \frac{z-a}{\bar{a}z-1}$ defines a conformal map of $\mathbb{E}$ interchanging a with 0 and satisfying

$|g'_a(a)| = \frac{1}{1-|a|^2}$. Then

$$g(z) = g_{G(z_0)} \circ G(z) = \frac{G(z) - G(z_0)}{\overline{G(z_0)}G(z) - 1} \in \mathcal{F},$$

hence

$$|G'(z_0)| \geq |g'(z_0)| = |g'_{G(z_0)}(G(z_0))G'(z_0)| = \frac{|G'(z_0)|}{1 - |G(z_0)|^2}$$

Since $G'(z_0) \neq 0$, we get $G(z_0) = 0$.

(d) By (c), $G\colon \mathbb{E} \to \mathbb{E}$ satisfies $G(0) = 0$, hence $|G'(0)| \leq 1$ by the Schwarz lemma. As the identity function $z \mapsto z$ belongs to $\mathcal{F}$, we also have $|G'(0)| \geq 1$, hence equality holds. Again by the Schwarz lemma, we get that G is a rotation, *i.e.* $G(z) = \alpha z$ with $|\alpha| = 1$.

(e) By the Riemann Mapping theorem, there exists a conformal map $\varphi\colon D \to \mathbb{E}$ satisfying $\varphi(z_0) = 0$. Set $F = G \circ \varphi^{-1}\colon \mathbb{E} \to \mathbb{E}$. Then $|F'(0)| = |G'(z_0)\frac{1}{\varphi'(z_0)}| \geq |f'(0)|$ for any holomorphic function $f\colon \mathbb{E} \to \mathbb{E}$, as $f = g \circ \varphi^{-1}$ for a unique $g\colon D \to \mathbb{E}$. By (d), F is a rotation, *i.e.* $G(z) = \alpha\varphi(z)$ with $|\alpha| = 1$. Hence G is conformal.

(f) Recall that $f(z) = z + \frac{1}{z}$ defines a conformal map $\mathbb{E} \setminus \{0\} \to \mathbb{C} \setminus [-2, 2]$, hence its inverse is a non-constant bounded function on $\mathbb{C} \setminus [-2, 2]$. By (b) and (c), there exists a holomorphic function $\tilde{G}\colon \mathbb{C} \setminus [-2, 2] \to \mathbb{E}$ satisfying $\tilde{G}(i) = 0 \neq |\tilde{G}'(i)| \geq |g'(i)|$ for any holomorphic function $g\colon \mathbb{C} \setminus [-2, 2] \to \mathbb{E}$. Then $G(z) = \frac{|\tilde{G}'(z_0)|}{\tilde{G}'(z_0)}\tilde{G}(z)$ satisfies the required properties. Clearly G cannot be conformal, as $\mathbb{C} \setminus [-2, 2]$ is not simply connected. $\square$

Problem 8. *21 Let $\mathbb{H}$ denote the upper half plane $\{Im z > 0\}$, $\mathbb{E}$ the unit disk $\{|z| < 1\}$ and $\overline{\mathbb{C}}$ the extended complex plane $\mathbb{C} \cup \{\infty\}$.*

(a) *Show that $h_\alpha(z) = \frac{z-\alpha}{z-\bar{\alpha}}$ defines a conformal map $\mathbb{H} \to \mathbb{E}$ for any $\alpha \in \mathbb{H}$, and compute its inverse.*
(b) *Show that any conformal map $f\colon \mathbb{H} \to \mathbb{E}$ may be written uniquely as $f(z) = \mu h_\alpha(z)$ with $|\mu| = 1$ and $\alpha \in \mathbb{H}$ satisfying $f(\alpha) = 0$.*
(c) *Show that a Möbius transformation $z \mapsto \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$ defines a conformal map $\mathbb{H} \to \mathbb{H}$ if and only if $ad - bc > 0$. Show that any conformal map $g\colon \mathbb{H} \to \mathbb{H}$ may be written (not uniquely) as $g(z) = \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$ satisfying $ad - bc > 0$. (*Hint*: use that $g = h_i^{-1} \circ \mu h_\alpha$ for some μ and α as in (b).)*
(d) *Let $\varnothing \neq D \subset \overline{\mathbb{C}}$ be a simply connected open subset. Show that one, and only one, of the following is true: 1) $D = \overline{\mathbb{C}}$; 2) D is conformally equivalent to $\mathbb{C}$; 3) D is conformally equivalent to $\mathbb{E}$. Here, if $\infty \in D \cup D'$, a conformal map $D \to D'$ is by definition a Möbius transformation. (*Hint*: if $\infty \in D \subsetneq \overline{\mathbb{C}}$, find a Möbius transformation $D \to D'$ with $\infty \notin D'$.)*
(e) *Let D as in (d). Show that the group* Aut $(D) = \{$*conformal maps* $D \to D\}$ *is isomorphic to one of the following:*

$$G_1 = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \mathbb{C}, \det = 1\}/\{\pm 1\};$$

$$G_2 = \{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}; a, b, d \in \mathbb{C}, \det = 1\}/\{\pm 1\};$$

$$G_3 = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \mathbb{R}, \det = 1\}/\{\pm 1\}.$$

Solution. (a) Since $\alpha \in \mathbb{H}$, one has $-\bar{\alpha}+\alpha = 2i\mathrm{Im}\alpha \neq 0$, hence h_α is a Möbius transformation. It thus defines a conformal map $\mathbb{C}\setminus\{\bar{\alpha}\} \to \mathbb{C}\setminus\{1\}$ with inverse $w \mapsto \frac{\bar{\alpha}w-\alpha}{w-1}$. It remains to show that $h_\alpha(\mathbb{H}) = \mathbb{E}$. As $\partial h_\alpha(\mathbb{H}) = h_\alpha(\partial\mathbb{H})$ and $h_\alpha(\alpha) = 0$, it is enough to check that h_α maps $\partial\mathbb{H}$ (= real axis) onto $\partial\mathbb{E}$ (= unit circle), *i.e.* $|z-\alpha| = |z-\bar{\alpha}|$ for any $z \in \mathbb{R}$. This follows from $z-\bar{\alpha} = \overline{z-\alpha}$.

(b) Clearly $z \mapsto \mu h_\alpha(z)$ defines a conformal map $\mathbb{H} \to \mathbb{E}$, since $z \mapsto \mu z$ is a rotation. Conversely, let $f\colon \mathbb{H} \to \mathbb{E}$ be a conformal map. Set $\alpha = f^{-1}(0) \in \mathbb{H}$. Then $f \circ h_\alpha^{-1}$ is a conformal map $\mathbb{E} \to \mathbb{E}$ which fixes 0, hence $f \circ h_\alpha^{-1}(z) = \mu z$ for some $|\mu| = 1$ as a consequence of the Schwarz lemma. Therefore, $f(z) = \mu h_\alpha(z)$.

(c) We know (see for example [7, Lecture 5]) that for $a, b, c, d \in \mathbb{R}$ satisfying $ad-bc \neq 0$, the Möbius transformation $z \mapsto \frac{az+b}{cz+d}$ defines a conformal map $f\colon \mathbb{C}\setminus\{-\frac{d}{c}\} \to \mathbb{C}\setminus\{\frac{a}{c}\}$ with inverse $w \mapsto \frac{dw-b}{-cw+a}$. Since

$$\frac{az+b}{cz+d} = \frac{ac|z|^2+bd+adz+bc\bar{z}}{|c\bar{z}+d|^2},$$

we get that $\mathrm{Im}(\frac{az+b}{cz+d}) > 0$ if and only if $\mathrm{Im}z$ and $ad-bc$ have the same sign. Hence f restricts to a conformal map $\mathbb{H} \to \mathbb{H}$ if and only if $ad-bc > 0$.

Conversely, let $g\colon \mathbb{H} \to \mathbb{H}$ be a conformal map. By (b), $h_i \circ g = \mu h_\alpha$ for some $|\mu| = 1$ and $\alpha \in \mathbb{H}$. Then

$$g(z) = h_i^{-1}(\mu h_\alpha(z)) = i\frac{z(1+\mu)-(\bar{\alpha}+\mu\alpha)}{z(1-\mu)-(\bar{\alpha}-\mu\alpha)} = \frac{zi(\mu-\bar{\mu})-[i(\bar{\alpha}-\alpha)-i(\bar{\mu}\bar{\alpha}-\mu\alpha)]}{z|1-\mu|^2-[\bar{\alpha}+\alpha-(\bar{\mu}\bar{\alpha}+\mu\alpha)]}.$$

This is a Möbius transformation $\mathbb{H} \to \mathbb{H}$ with real coefficients, hence they need to be positive.

(d) Let $\varnothing \neq D \subset \overline{\mathbb{C}}$ be a simply connected open subset. If $\infty \notin D$, then $D \subset \mathbb{C}$ is either $\mathbb{C}$ or conformally equivalent to $\mathbb{E}$ by the Riemann Mapping theorem. If $\infty \in D$, then D is either $\overline{\mathbb{C}}$ or $\overline{\mathbb{C}}\setminus D \neq \varnothing$. In the latter case, the Möbius transformation $z \mapsto \frac{1}{z-z_0}$ for $z_0 \in \overline{\mathbb{C}}\setminus D$ extends to an homomorphism $D \to D'$ with $\infty \notin D'$. Since D' is simply connected, it is either $\mathbb{C}$ or conformally equivalent to $\mathbb{E}$.

Finally, there cannot be conformal maps $\overline{\mathbb{C}} \to \mathbb{C}$, $\overline{\mathbb{C}} \to \mathbb{E}$ or $\mathbb{C} \to \mathbb{E}$, since $\overline{\mathbb{C}}$ is compact and any holomorphic map $\mathbb{C} \to \mathbb{E}$ is constant by Liouville theorem.

(e) Let $D = \overline{\mathbb{C}}$. Then one easily checks that the assignment $\frac{az+b}{cz+d} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ defines a group isomorphism

$$\mathrm{Aut}\,(\overline{\mathbb{C}}) = \{\text{Möbius transformations}\} \simeq \{\begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \mathbb{C},\ \det \neq 0\}/\{\mathbb{C}^\times\}.$$

By sending a matrix A to $\frac{1}{\sqrt{\det A}}A$ for a chosen square root of $\det A$, we get that the right-hand side is isomorphic to G_1.

Let $\varnothing \neq D \subset \overline{\mathbb{C}}$ be conformally equivalent to $\mathbb{C}$. Then the group isomorphism $\mathrm{Aut}\,(\overline{\mathbb{C}}) \simeq G_1$ restricts to

$$\mathrm{Aut}\,(D) \simeq \mathrm{Aut}\,(\mathbb{C}) = \{\text{affine transformations } z \mapsto az+b\} \simeq G_2.$$

Finally, if $\varnothing \neq D \subset \overline{\mathbb{C}}$ is conformally equivalent to $\mathbb{E}$, it is also conformally equivalent to $\mathbb{H}$ by (a). Hence by (c), the group isomorphism $\mathrm{Aut}\,(\overline{\mathbb{C}}) \simeq G_1$ restricts to

$$\mathrm{Aut}\,(D) \simeq \mathrm{Aut}\,(\mathbb{H}) \simeq G_3.$$

(Note that one uses the fact that $ad-bc > 0$ has real square roots.) $\square$

Problem 9. *For a domain $D \subset \mathbb{C}$ (*i.e. *non-empty, open and connected) and $a \in D$, set*

$$\operatorname{Aut}_a(D) = \{f : D \to D \text{ biholomorphism};\ f(a) = a\}$$

and let $\sigma_a : \operatorname{Aut}_a(D) \to \mathbb{C}$ be the map $f \mapsto f'(a)$.

(a) *Show that $\sigma_a : \operatorname{Aut}_a(D) \to \mathbb{C}$ defines an homomorphism of the group $\operatorname{Aut}_a(D)$ into the group $\mathbb{C}^\times$.*

(b) *Describe $\operatorname{Aut}_a(D)$ and σ_a for $D = \mathbb{E}$, the unit disc, and $a = 0$, and for $D = \mathbb{C}^\times, \mathbb{C}$ and any a.*

(c) *Prove that $\sigma_a : \operatorname{Aut}_a(D) \to \mathbb{C}^\times$ is injective with image $\mathbb{U} = \{z;\ |z| = 1\}$ for any proper simply connected domain $D \subset \mathbb{C}$.*

(d) *Let $D \subset \mathbb{C}$ be a proper domain. Prove that, if D is simply connected or biholomorphic to $\mathbb{C}^\times$, then any $f \in \operatorname{Aut}_a(D)$ satisfying $f'(a) > 0$ is the identity of D.*

Solution. (a) As any biholomorphism is locally injective, $\sigma_a(f) = f'(a) \in \mathbb{C}^\times$ for any $f \in \operatorname{Aut}_a(D)$. Moreover, $\sigma_a(\mathrm{id}_D) = 1$ and

$$\sigma_a(g \circ f) = (g \circ f)'(a) = g'(f(a))f'(a) = \sigma_a(g)\sigma_a(f)$$

for any $f, g \in \operatorname{Aut}_a(D)$.

(b) By the Schwarz lemma, any biholomorphism f of the unit disc $\mathbb{E}$ satisfying $f(0) = 0$ is a rotation, *i.e.* $f(z) = \lambda z$ for some $\lambda \in \mathbb{U} = \{z;\ |z| = 1\}$ (see [7, Lecture 6]). It follows that $\operatorname{Aut}_0(\mathbb{E})$ is isomorphic through σ_0 to $\mathbb{U}$.

The biholomorphisms of $D = \mathbb{C}$ are the affine transformations $cz + b$ with $c \in \mathbb{C}^\times$ (see [7, Lecture 5]). Hence any $f \in \operatorname{Aut}_a(\mathbb{C})$ has the form $c(z - a) + a$ with $c \in \mathbb{C}^\times$ and $\sigma_a : \operatorname{Aut}_a(\mathbb{C}) \to \mathbb{C}^\times$ is an isomorphism.

Similarly, the biholomorphisms of $D = \mathbb{C}$ have the form cz^ϵ with $c \in \mathbb{C}^\times$ and $\epsilon = \pm 1$. Solving the equation $ca^\epsilon = a$ gives $\epsilon = c = 1$ or $\epsilon = -1$ and $c = a^2$. It follows that $\operatorname{Aut}_a(\mathbb{C}^\times)$ is isomorphic through σ_a to the group $\{\pm 1\}$.

(c) By the Riemann Mapping Theorem, given a proper simply connected domain $D \subset \mathbb{C}$ and $a \in D$ there exists a biholomorphism $g : D \to \mathbb{E}$ satisfying $g(a) = 0$. The map $\mathrm{Ad}_g : f \mapsto g \circ f \circ g^{-1}$ defines a group isomorphism $\operatorname{Aut}_a(D) \to \operatorname{Aut}_0(\mathbb{E})$ satisfying

$$\mathrm{Ad}_g(f)'(0) = g'(f(a))f'(a)(g^{-1})'(0) = g'(a)f'(a)\frac{1}{g'(a)} = f'(a)$$

Hence $\sigma_a = \sigma_0 \circ \mathrm{Ad}_g$ and the result then follows from (b).

(d) For a proper simply connected domain $D \subset \mathbb{C}$, it follows from (c) that σ_a is injective with image $\mathbb{U} = \{z;\ |z| = 1\}$. Then $|f'(a)| = |\sigma_a(f)| = 1$ for any $f \in \operatorname{Aut}_a(D)$ and $f'(a) > 0$ implies $f'(a) = 1$, hence $f = \mathrm{id}_D$, as σ_a is injective.

If D is biholomorphic to $\mathbb{C}^\times$, one may replace D by $\mathbb{C}^\times$ along the same lines of (c) and the result follows from (b). $\square$

Problem 10. *Let $D \subset \mathbb{C}$ be a domain (*i.e. *non-empty, open and connected), and $S \subset D$ a closed subset. Set*

$$\operatorname{Aut}_S(D) = \{f : D \to D \text{ conformal map};\ f(S) = S\}.$$

(a) *Show that the restriction map $f \mapsto f|_{D \setminus S}$ defines an injective group morphism $\rho : \operatorname{Aut}_S(D) \to \operatorname{Aut}(D \setminus S) = \{\text{conformal maps } D \setminus S \to D \setminus S\}$.*

(b) *Show that ρ is never an isomorphism for $D = \mathbb{C}$ and $S = \{c\}$.*

(c) *Show that ρ is an isomorphism for $D = \mathbb{E} = \{|z| < 1\}$, the unit disk, and $S = \{c\}$.*

(d) *Show that ρ is an isomorphism for D proper and simply connected and $S = \{c\}$. Compute explicitly the group* $\mathrm{Aut}\,(D \setminus \{c\})$*.*

(e) *Show that ρ is an isomorphism for D bounded with ∂D not containing isolated points and S discrete. (*Hint: *use that for any open subset $\Omega \subset \mathbb{C}$, if $\Omega \cup \{c\}$ is open then either $c \in \Omega$, or c is an isolated point of $\partial\Omega$).*

Solution. (a) Clearly, $\mathrm{Aut}_S(D)$ is a group. For $f \in \mathrm{Aut}_S(D)$, we have $f(D \setminus S) = D \setminus S$, as f is bijective and $f(S) = S$, hence $f|_{D\setminus S} \colon D \setminus S \to D \setminus S$ is bijective and holomorphic, *i.e.* a conformal map. Therefore ρ is well-defined and it is clearly a group morphism.

Given conformal maps $f, g \colon D \to D$, if $f|_{D\setminus S} = g|_{D\setminus S}$, we get $f = g$ by the identity principle (as $D \setminus S$ is non-empty and open in D, connected), hence ρ is injective.

(b) The conformal map $\tau(z) = z - c$ sends c to 0, hence $f \mapsto \tau \circ f \circ \tau^{-1}$ defines group isomorphisms $\mathrm{Aut}_{\{c\}}(\mathbb{C}) \xrightarrow{\sim} \mathrm{Aut}_{\{0\}}(\mathbb{C})$ and $\mathrm{Aut}\,(\mathbb{C} \setminus \{c\}) \xrightarrow{\sim} \mathrm{Aut}\,(\mathbb{C} \setminus \{0\})$ compatible with ρ. It follows that we may suppose $c = 0$. A conformal map $\mathbb{C} \to \mathbb{C}$ is an affine map $z \mapsto \alpha z + \beta$ with $\alpha \neq 0$, and it fixes 0 iff $\beta = 0$, whereas a conformal map $\mathbb{C}^\times \to \mathbb{C}^\times$ is of the form $z \mapsto \alpha z^{\pm 1}$ with $\alpha \neq 0$. Hence ρ is not surjective.

(c) Let $f \colon \mathbb{E} \setminus \{c\} \to \mathbb{E} \setminus \{c\}$ be a conformal map. Then c is an isolated singularity for f and, f being bounded, it must be removable. Let $\tilde{f}$ be the analytic continuation of f at $\mathbb{E}$. Then $\tilde{f}$ is continuous and injective, since $\mathbb{E} \setminus \{c\}$ is dense in $\mathbb{E}$ (see [7, Lecture 5]), hence $\tilde{f}(c) \in \overline{\tilde{f}(\mathbb{E} \setminus \{c\})} \setminus \tilde{f}(\mathbb{E} \setminus \{c\}) = \partial(\mathbb{E} \setminus \{c\}) = \{c\} \cup \partial\mathbb{E}$. Since $|\tilde{f}(z)| < 1$ for any $z \neq c$, then $\tilde{f}(c) = c$, otherwise it would be constant by the Maximum Modulus theorem. It follows that ρ is surjective.

(Note that, as in (e), one may use the Open Mapping theorem to get $\tilde{f}(\mathbb{E}) \subset \mathbb{E}$ and to deduce $\tilde{f}(c) = c$. Note also that, as for (b) with τ the Möbius transformation $z \mapsto \frac{z-c}{1-\bar{c}z}$, one might have supposed $c = 0$.)

(d) By the Riemann Mapping theorem, there exists a conformal map $\varphi \colon D \to \mathbb{E}$ sending c to 0, hence $f \mapsto \varphi \circ f \circ \varphi^{-1}$ and $f \mapsto \varphi|_{D\setminus\{c\}} \circ f \circ \varphi|_{D\setminus\{c\}}^{-1}$ define the vertical group isomorphisms in the following commutative diagram

$$\begin{tikzcd}
\mathrm{Aut}_{\{c\}}(D) \arrow[r, "\rho"] \arrow[d, "\sim"] & \mathrm{Aut}\,(D \setminus \{c\}) \arrow[d, "\sim"] \\
\mathrm{Aut}_{\{0\}}(\mathbb{E}) \arrow[r, "\rho"] & \mathrm{Aut}\,(\mathbb{E}^\times)
\end{tikzcd}$$

(where $\mathbb{E}^\times = \mathbb{E} \setminus \{0\}$). By (c), we get that the upper horizontal arrow is an isomorphism.

By the Schwarz lemma, a conformal map $f \colon \mathbb{E} \to \mathbb{E}$ satisfying $f(0) = 0$ is a rotation, *i.e.* $f(z) = \alpha z$ for some $\alpha \in \mathbb{U} = \{|z| = 1\}$, hence $\mathrm{Aut}\,(D \setminus \{c\}) \xrightarrow{\sim} \mathrm{Aut}\,(\mathbb{E}^\times) \xleftarrow{\sim} \mathrm{Aut}_{\{0\}}(\mathbb{E}) \simeq \mathbb{U}$ (the last arrow being a group isomorphism).

(e) Let $f \colon D \setminus S \to D \setminus S$ be a conformal map. Then any $c \in S$ is a removable singularity for f, as f is bounded and S discrete. Let $\tilde{f}$ be the analytic continuation of f at D. Since $\tilde{f}$ is continuous and injective (as in (c), $D \setminus S$ being dense in D) and $\tilde{f}(D) = \tilde{f}(D \setminus S) \cup \tilde{f}(S) = (D \setminus S) \cup \tilde{f}(S)$ is open by the Open Mapping theorem, it follows from the *Hint* (by induction) that $\tilde{f}(S)$ consists of isolated points of $\partial(D \setminus S) = S \cup \partial D$. Since ∂D has no isolated points, it follows that $\tilde{f}(S) \subset S$.

By replacing f with f^{-1}, we get an holomorphic map $\widetilde{f^{-1}}\colon D \to D$ satisfying $\widetilde{f^{-1}} \circ \tilde{f} = \mathrm{id} = \tilde{f} \circ \widetilde{f^{-1}}$ on $D \setminus S$, hence on D by the identity principle. It follows that $\tilde{f}$ is conformal, hence ρ is surjective. $\square$

Problem 11. *Let f be a meromorphic function on $\mathbb{C}$.*

(a) *Show that f is locally injective at a pole $a \in \mathbb{C} \cup \infty$ iff $\operatorname{ord}_a f = -1$.*
(b) *Show that f is injective iff it is a Möbius transformation. (*Hint*: show that if $z_1, z_2 \in \mathbb{C}$, $z_1 \neq z_2$, are poles for f, then f cannot be injective.)*

Solution. (a) First, recall that the zeros and poles of f form a closed and discrete set. Let $a \in \mathbb{C}$ be a pole of order m of f. As $\lim_{z \to a} f(z) = \infty$, locally at a we have $f(z) \neq 0$, hence $\frac{1}{f}$ is well-defined and (extends to an holomorphic function which) has a zero of order m at a. Clearly, f is locally injective at a iff so does $\frac{1}{f}$, iff $(\frac{1}{f})'(a) \neq 0$, that is, iff $1 = \operatorname{ord}_a \frac{1}{f} = -\operatorname{ord}_a f$.

If $a = \infty$, replace $f(z)$ by $f(\frac{1}{z})$ and a by 0.

(b) Suppose that $z_1, z_2 \in \mathbb{C}$, $z_1 \neq z_2$, are poles for f. As $\lim_{z \to z_i} f(z) = \infty$ for $i = 1, 2$, we may find two disjoint open neighbourhoods V_1 and V_2 of z_1, z_2 such that $f(V_1)$ and $f(V_1)$ are open neighbourhoods of ∞. Then $f(V_1) \cap f(V_2)$ is also an open neighbourhood of ∞, hence non-empty. Contradiction.

It follows that f has at most one pole, which must be simple by (a). To conclude one then follows the lines of [7, Lecture 5] and prove that f is a Möbius transformation either of the form $a\frac{1}{z-c} + b = \frac{bz+a-bc}{z-c}$ if it has a pole, or $a(z-c) + b = az + b - ac$ for $a \in \mathbb{C}^\times$ and $b \in \mathbb{C}$. $\square$

Problem 12. *Set $\mathbb{E} = \{|z| < 1\}$ and let $f\colon \mathbb{E} \to \mathbb{E}$ be holomorphic and proper (*i.e.* the pre-image of any compact subset is compact).*

(a) *Show that $\partial f(S) \subset f(\partial S)$ for any subset $S \subset \mathbb{E}$ with $\partial S \subset \mathbb{E}$.*
(b) *Prove that $\lim_{z \to \partial \mathbb{E}} |f(z)| = 1$.*
(c) *Given $a_1, \ldots, a_n \in \mathbb{E}$, not necessarily distinct, show that*

$$B_{a_1,\ldots,a_n}(z) = \prod_{i=1}^{n} \frac{z - a_i}{1 - \bar{a}_i z}$$

defines a proper holomorphic map $\mathbb{E} \to \mathbb{E}$, which is conformal if and only if $n = 1$.
(d) *Prove that there exist $\lambda \in \partial \mathbb{E}$ and $a_1, \ldots, a_n \in \mathbb{E}$ such that $f(z) = \lambda B_{a_1,\ldots,a_n}(z)$. (*Hint*: use the Maximum Modulus theorem.)*
(e) *Suppose that f extends analytically on $\mathbb{C}$. Show that $f(z) = \lambda z^n$ for some $n \geq 1$.*

Solution. (a) Since f is proper and $\mathbb{E} = f^{-1}(f(\mathbb{E}))$ is non-compact, f cannot be constant, hence it is open by the Open Mapping theorem. It follows that $f(\operatorname{Int}(S)) \subset \operatorname{Int}(f(S))$ for any subset $S \subset \mathbb{E}$. Let us prove that $\overline{f(S)} \subset f(\overline{S})$ for any subset $S \subset \mathbb{E}$ with $\partial S \subset \mathbb{E}$. It will follow that

$$\partial f(S) = \overline{f(S)} \setminus \operatorname{Int}(f(S)) \subset f(\overline{S}) \setminus f(\operatorname{Int}(S)) = f(\partial S).$$

Take $w = \lim_{n \to +\infty} f(z_n) \in \overline{f(S)}$ for a sequence $\{z_n\}_n \subset S$. Since $\overline{S}$ is compact, $\{z_n\}_n$ has a convergent subsequence to some $z \in \overline{S} \subset \mathbb{E}$, hence $f(z) = w$.

(b) We have to prove that $\lim_{z\to\partial\mathbb{E}} f(z) \in \partial\mathbb{E}$. Let $\{z_n\}_n \subset \mathbb{E}$ be a sequence convergent to $\partial\mathbb{E}$ and suppose that $\{f(z_n)\}_n$ does not converge to $\partial\mathbb{E}$. Then there exists a subsequence $\{f(z_{n_k})\}_k$ lying in the compact subset $\overline{B_r(0)}$ for some $0 < r < 1$. Since f is proper, $f^{-1}\left(\overline{B_r(0)}\right)$ is compact. Hence $\{z_{n_k}\}_k \subset f^{-1}\left(\overline{B_r(0)}\right) \subset \mathbb{E}$ has a convergent subsequence to some $z \in \mathbb{E}$. Contradiction. It follows that $\lim_{n\to+\infty} f(z_n) \in \partial\mathbb{E}$.

(c) Since $|a_i| < 1$ for any $i = 1, \dots, n$, each $\frac{z-a_i}{1-\bar{a}_i z}$ is a conformal self-map of $\mathbb{E}$ (see [7, Lecture 6]), which is in particular proper and sends a_i to 0. Hence their product $B_{a_1,\dots,a_n}$ defines a proper holomorphic map $\mathbb{E} \to \mathbb{E}$, which cannot be conformal if $n > 1$, as $B^{-1}_{a_1,\dots,a_n}(\{0\}) = \{a_1, \dots, a_n\} \neq \{0\}$ if there are $a_i \neq a_j$, otherwise $B_{a_1,\dots,a_n} = \left(\frac{z-a_1}{1-\bar{a}_1 z}\right)^n$.

(d) By (b), we get $\partial f(\mathbb{E}) \subset \partial\mathbb{E}$, hence $0 \in f(\mathbb{E})$, otherwise the line segment joining 0 and $f(0)$ would contain a point in $\partial f(\mathbb{E})$, contradiction. Since f is proper, the non-empty discrete subset $f^{-1}(\{0\}) \subset \mathbb{E}$ is compact, hence finite.

Let $a_1, \dots, a_n \in \mathbb{E}$ be the zeros of f, counted according to their multiplicities. Then the function $\frac{f(z)}{B_{a_1,\dots,a_n}(z)}$ can be analytically extended to a never vanishing function $h \in \mathcal{O}(\mathbb{E})$. Since $|B_{a_1,\dots,a_n}(z)| = 1$ for $|z| = 1$ and $\lim_{|z|\to 1^-} |f(z)| = 1$ by (b), we get $\lim_{|z|\to 1^-} |h(z)| = 1$. By the Maximum Modulus theorem, we have both $|h(z)| \leq 1$ and $\frac{1}{|h(z)|} \leq 1$ for any $z \in \overline{\mathbb{E}}$. It follows that $h(z) = \lambda$ with $|\lambda| = 1$.

(e) Supposing f non-constant, by (d) we have $f(z) = \lambda B_{a_1,\dots,a_n}(z)$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. If f extends analytically on $\mathbb{C}$, then $a_1 = \dots = a_n = 0$ and $f(z) = \lambda z^n$. □

Problem 13. *Denote by* $\overline{\mathbb{E}}$ *the closure of the unit disk* $\mathbb{E} = \{|z| < 1\}$ *and set* $\partial\mathbb{E} = \overline{\mathbb{E}} \setminus \mathbb{E}$. *Let* $f \in \mathcal{O}(\overline{\mathbb{E}})$ *satisfy* $f(\partial\mathbb{E}) \subset \partial\mathbb{E}$.

(a) *Show that* $f(\overline{\mathbb{E}}) \subset \overline{\mathbb{E}}$.
(b) *Prove that either* $f(\overline{\mathbb{E}}) = \overline{\mathbb{E}}$, *or* f *is constant. (*Hint*: use the Open Mapping theorem)*
(c) *Prove that if* $Re f' > 0$ *in* $\mathbb{E}$, *then* f *restricts to a conformal map* $\mathbb{E} \to \mathbb{E}$. *(*Hint*: compute* $\int_\gamma f'(z)dz$ *for an appropriate path* γ*)*
(d) *Prove that, if* f *is non-constant, then there exists* $\lambda \in \partial\mathbb{E}$ *such that* $f(z) = \lambda B(z)$, *where* B *is a finite Blaschke product.*
(e) *Suppose that* f *extends analytically on* $\mathbb{C}$. *Show that* $f(z) = \lambda z^n$ *for some* $n \in \mathbb{N}$.
(f) *Prove that any* $g \in \mathcal{O}(\mathbb{C})$ *mapping a circle to a circle has the form* $g(z) = a(z - z_0)^n + w_0$ *for some* $a, z_0, w_0 \in \mathbb{C}$ *and* $n \in \mathbb{N}$.

Solution. (a) Since $|f(z)| = 1$ for $|z| = 1$, it follows by the Maximum Modulus theorem that $|f(z)| \leq 1$ for any $|z| \leq 1$, *i.e.* $f(\overline{\mathbb{E}}) \subset \overline{\mathbb{E}}$.

(b) If f is non-constant, then f is open by the Open Mapping theorem, hence $\partial f(\overline{\mathbb{E}}) \subset f(\partial\mathbb{E}) \subset \partial\mathbb{E}$. It follows that $f(\overline{\mathbb{E}}) \supset \overline{\mathbb{E}}$, as it contains any line segment $S \subset \overline{\mathbb{E}}$ starting from $f(0)$. Indeed, if it is not the case, there would exist a point in $S \setminus \partial\mathbb{E}$ belonging to $\partial f(\overline{\mathbb{E}})$, a contradiction.

(c) As f cannot be constant, we have $|f(z)| < 1$ for any $z \in \mathbb{E}$ by the Maximum Modulus theorem, *i.e.* f restricts to an holomorphic map $\mathbb{E} \to \mathbb{E}$. From $f(\overline{\mathbb{E}}) = \overline{\mathbb{E}}$ and $f(\partial\mathbb{E}) \subset \partial\mathbb{E}$ it follows that $f(\mathbb{E}) = \mathbb{E}$. It remains to prove that f is injective.

Let $[z_1, z_2]$ be the line segment joining two points $z_1, z_2 \in \mathbb{E}$. Then

$$f(z_2) - f(z_1) = \int_{[z_1,z_2]} f'(z)dz = (z_2 - z_1) \int_0^1 f'(z_1 + t(z_2 - z_1))dt.$$

Since $\mathrm{Re} f' > 0$ in $\mathbb{E}$, we have $\mathrm{Re} \int_0^1 f'(z_1+t(z_2-z_1))dt > 0$, hence $\int_0^1 f'(z_1+t(z_2-z_1))dt \neq 0$ and the result follows.

(d) As $\overline{\mathbb{E}}$ is compact, f has a finite number of zeros $a_1, \dots, a_n$ in $\overline{\mathbb{E}}$, counted according to their multiplicities. As $f(\partial\mathbb{E}) \subset \partial\mathbb{E}$, none of the a_i's belongs to $\partial\mathbb{E}$, hence $a_1, \dots, a_n \in \mathbb{E}$. It follows that the finite Blaschke product

$$B(z) = \prod_{i=1}^{n} \frac{z - a_i}{\bar{a}_i z - 1} \frac{|a_i|}{a_i}$$

(where we set $\frac{|a|}{a} = 1$ if $a = 0$) belongs to $\mathcal{O}(\overline{\mathbb{E}})$ and $\frac{f(z)}{B(z)}$ can be analytically extended to a never vanishing function $h \in \mathcal{O}(\overline{\mathbb{E}})$ satisfying $|h(z)| = \frac{|f(z)|}{|B(z)|} = 1$ for $|z| = 1$. By the Maximum Modulus theorem, we have both $|h(z)| \leq 1$ and $\frac{1}{|h(z)|} \leq 1$ for any $z \in \overline{\mathbb{E}}$. It follow that $h(z) = \lambda$ with $|\lambda| = 1$.

(e) Supposing f non-constant, by (d) we have $f(z) = \lambda B(z)$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. If f extends analytically on $\mathbb{C}$, then $a_1 = \dots = a_n = 0$ and $f(z) = \lambda z^n$.

(f) Let S be the circle of radius $r > 0$ centred in $z_0 \in \mathbb{C}$, and suppose that $g(S)$ is the circle of radius $\rho > 0$ centred in w_0. Consider the conformal maps of the plane $h_1(z) = \frac{1}{r}(z - z_0)$ and $h_2(w) = \frac{1}{\rho}(w - w_0)$. Then $h_1(S) = \partial\mathbb{E} = h_2(g(S))$, hence $\tilde{g} = h_2 \circ g \circ h_1^{-1} \in \mathcal{O}(\mathbb{C})$ maps the unit circle to the unit circle. It follows by (e) that $\tilde{g}(z) = \lambda z^n$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, hence

$$g(z) = h_2^{-1} \circ \tilde{g} \circ h_1(z) = \frac{\rho\lambda}{r^n}(z - z_0)^n + w_0.$$

□

Problem 14. *Let $D \subset \mathbb{C}$ be a domain (*i.e.* non-empty, open and connected).*

(a) *Let $g, h \colon D \to D$ be holomorphic maps, with g non-constant and satisfying $g = h \circ g$. Prove that $h = \mathrm{id}_D$.*

(b) *Let $\{h_k\}_{k\geq 0}$ be a sequence which converges in $\mathcal{O}(D)$ to a non-constant function h. Prove that if $h_k(D) \subset E$ for any k then $h(D) \subset E$. (*Hint*: use Hurwitz's theorem.)*

(c) *Let $f \colon D \to D$ be an holomorphic map and set $f^{\circ 1} = f$, $f^{\circ(n+1)} = f \circ f^{\circ n}$ for any $n \geq 1$. Assume that there exists a subsequence $\{f^{\circ n_k}\}_{k\geq 0}$ which converges in $\mathcal{O}(D)$ to a non-constant function g and set $h_k = f^{\circ(n_{k+1}-n_k)}$ for any $k \geq 0$.*

   (i) *Prove that every convergent subsequence of $\{h_k\}_{k\geq 0}$ converges in $\mathcal{O}(D)$ to id_D. (*Hint*: note that $f^{\circ n_{k+1}} = h_k \circ f^{\circ n_k}$ and use that $\cdot \circ \cdot$ is continuous.)*

   (ii) *Suppose that D is bounded. Show that $\{h_k\}_{k\geq 0}$ converges in $\mathcal{O}(D)$ to id_D. (*Hint*: use Montel's convergence criterion.) Deduce that f is conformal.*

(d) *Suppose that D is bounded and let $f \colon D \to D$ be an holomorphic map such that $f(z_0) = z_0$ for some $z_0 \in D$. Assume that either $|f'(z_0)| = 1$, or $f(z_1) = z_1$ for some $z_1 \in D \setminus \{z_0\}$. Prove that f is conformal.*

Solution. (a) h is the identity on the (non-empty) subset $g(D) \subset D$, which is open by the open mapping theorem, since g is non-constant. As D is connected, $h = \mathrm{id}_D$ by the identity principle.

(b) Take $w \notin E$. Then $h_k(z) - w$ never vanishes in D for any k and by Hurwitz's theorem so does $h(z) - w$, as it is non-constant. Hence $w \notin h(D)$, that is $h(D) \subset E$.

(c) (i) Let $\{h_{k_l}\}_{l\geq 0}$ be a subsequence convergent to some h in $\mathcal{O}(D)$. As $\cdot\circ\cdot$ is continuous, taking the limit of $f^{\circ n_{k+1}} = h_k \circ f^{\circ n_k}$, we get $g = h \circ g$. Since g is non-constant, we can apply (a) and get $h = \mathrm{id}_D$.

(ii) Since $f(D) \subset D$, by induction we get $h_k(D) \subset D$ for any k. As D is bounded, the sequence $\{h_k\}_{k\geq 0}$ is bounded, and we can use Montel's convergence criterion: by (i) every convergent subsequence converges to id_D in $\mathcal{O}(D)$, hence so does $\{h_k\}_{k\geq 0}$.
By induction $f^{\circ n}(D) \subset f(D)$ for any n, hence $h_k(D) \subset f(D)$ for any k. By (b) we get $D \subset f(D) \subset D$, that is $f(D) = D$.
Given $f(z_1) = f(z_2)$, we have $f_n(z_1) = f_n(z_2)$ for any n, hence $h_k(z_1) = h_k(z_2)$ for any k. Taking the limit we get $z_1 = z_2$. Therefore f is conformal.

(d) By Montel's theorem, the bounded sequence $\{f^{\circ n}\}_{n\geq 1}$ has a subsequence $\{f^{\circ n_k}\}_{k\geq 0}$ which converges in $\mathcal{O}(D)$ to some g. In both cases, g is non-constant and we may apply (c) (ii). Indeed, in one case $|g'(z_0)| = \lim_k |f^{\circ n_k\prime}(z_0)| = \lim_k |f'(z_0)|^{n_k} = 1$, whereas in the other case g has two distinct fixed points, since so does any $f^{\circ n}$. $\square$

Problem 15. *Let Z be a connected component of $\mathbb{C} \setminus K$ for a compact $K \subset \mathbb{C}$.*

(a) *Show that $\partial Z \subset \partial K$.* (Hint: *If $B_r(z_0) \subset \mathbb{C} \setminus K$ for $z_0 \in \partial Z$ and $r > 0$, then ...)*
(b) *Suppose that Z is bounded and $K \cup Z \subset U$ for an open subset $U \subset \mathbb{C}$. Prove that $\|f\|_{\overline{Z}} \leq \|f\|_K$ for any $f \in \mathcal{O}(U)$.*
(c) *In the situation of (b), prove that for any $z_0 \in Z$ there exists $C_{K,z_0} > 0$ such that $\left\| \frac{1}{z-z_0} - f(z) \right\|_K \geq C_{K,z_0}$ for any $f \in \mathcal{O}(U)$.*
(d) *Assume $\mathbb{C} \setminus K$ not connected. Show that there exists $f \in \mathcal{O}(K)$ which cannot be approximated on K by polynomials.*
(e) *Assume $\mathbb{C} \setminus K$ connected. Show that for any $z_0 \in \mathbb{C} \setminus K$ there exist a polynomial p and $f \in \mathcal{O}(K \cup \{z_0\})$ satisfying $f(z_0) = 1$, $f|_K = 0$ and $\|f - p\|_{K\cup\{z_0\}} < \frac{1}{2}$.*
(f) *Prove that $\mathbb{C} \setminus K$ is connected if and only if for any $z_0 \in \mathbb{C} \setminus K$ there exists a polynomial p such that $\|p\|_K < |p(z_0)|$.*

Solution. (a) Let $z_0 \in \partial Z = \overline{Z} \setminus Z$. It is enough to prove that z_0 is a limit point for K, *i.e.* for any $r > 0$ the open disc $B_r(z_0)$ intersects K. As K is closed and $z_0 \notin \mathrm{Int}K$, it will follow that $z_0 \in \partial K = K \setminus \mathrm{Int}K$.
Assume $B_r(z_0) \subset \mathbb{C} \setminus K$. Then $Z \subset Z \cup B_r(z_0) \subset \mathbb{C} \setminus K$ and $Z \cup B_r(z_0)$ is connected, as $B_r(z_0)$ and Z are connected and $B_r(z_0) \cap Z \neq \varnothing$. Since Z is maximal, we have $Z \cup B_r(z_0) = Z$, that is, $B_r(z_0) \subset Z$. Contradiction.

(b) By (a), we have $\partial Z \subset \partial K \subset K$, hence $\overline{Z} \subset K \cup Z \subset U$. As $\overline{Z}$ is bounded, by the Maximum Modulus theorem, we get

$$\|f\|_{\overline{Z}} = \|f\|_{\partial Z} \leq \|f\|_K \quad \text{for any } f \in \mathcal{O}(U).$$

(c) By (b), for $z_0 \in Z$ and any $f \in \mathcal{O}(U)$ we have

$$\left\| \frac{1}{z - z_0} - f(z) \right\|_K \geq \frac{\|1-(z-z_0)f(z)\|_K}{\|z - z_0\|_K} \geq \frac{\|1-(z-z_0)f(z)\|_{\overline{Z}}}{\|z - z_0\|_K}$$
$$\geq \frac{\|1-(z-z_0)f(z)\|_{z_0}}{\|z - z_0\|_K} = \frac{1}{\|z - z_0\|_K} = C_{K,z_0} > 0$$

(d) If $\mathbb{C} \setminus K$ is not connected, it has a bounded component Z (see [7, Lecture 16]) and $f(z) = \frac{1}{z-z_0} \in \mathcal{O}(K)$ for $z_0 \in Z$. By (c), there exists $C_{K,z_0} > 0$ such that

$$\|f - p\|_K \geq C_{K,z_0} \quad \text{for any } p \in \mathbb{C}[z] \subset \mathcal{O}(\mathbb{C}),$$

hence f cannot be approximated on K by polynomials.

(e) For $z_0 \in \mathbb{C} \setminus K$, take $V, W \subset \mathbb{C}$ disjoint open subsets satisfying $z_0 \in V$ and $K \subset U$, and f the function which has value 1 on V and 0 on W. Then f is locally constant, hence holomorphic, in $V \cup W \supset K \cup \{z_0\}$, that is, $f \in \mathcal{O}(K \cup \{z_0\})$.

If $\mathbb{C} \setminus K$ is connected, then $\mathbb{C} \setminus \{K \cup \{z_0\}\} = \{\mathbb{C} \setminus K\} \setminus \{z_0\}$ is connected as well, hence by Runge's theorem there exists a polynomial p such that $\|f - p\|_{K \cup \{z_0\}} < \frac{1}{2}$.

(f) Assume $\mathbb{C} \setminus K$ connected and take $z_0 \in \mathbb{C} \setminus K$. By (e), there exist $f \in \mathcal{O}(K \cup \{z_0\})$ and a polynomial p satisfying $f(z_0) = 1$, $f|_K = 0$ and $\|f - p\|_{K \cup \{z_0\}} < \frac{1}{2}$. It follows that

$$\|p\|_K = \|f - p\|_K \leq \|f - p\|_{K \cup \{z_0\}} < \frac{1}{2}$$

and

$$|1 - |p(z_0)|| \leq |1 - p(z_0)| = \|f - p\|_{\{z_0\}} \leq \|f - p\|_{K \cup \{z_0\}} < \frac{1}{2},$$

hence $\|p\|_K < \frac{1}{2} < |p(z_0)|$.

Conversely, if $\mathbb{C} \setminus K$ is not connected, it has a bounded component Z. Take $z_0 \in Z$. It follows by (b) that

$$|p(z_0)| \leq \|p\|_{\overline{Z}} \leq \|p\|_K \quad \text{for any } p \in \mathbb{C}[z].$$

$\square$

Problem 16. *For a domain (*i.e. *non-empty, open and connected)* $D \subset \mathbb{C}$ *and* $a \in D$, *set*

$$\operatorname{Aut}(D) = \{f \colon D \to D \text{ biholomorphism}\}, \quad \operatorname{Aut}_a(D) = \{f \in \operatorname{Aut}(D);\ f(a) = a\}.$$

(a) *Suppose that* $\mathbb{C} \setminus D$ *is non-empty and without bounded components. Prove that any* $f \in \operatorname{Aut}_a(D)$ *satisfies* $|f'(a)| = 1$, *and* $f'(a) = 1$ *if and only if* f *is the identity.*
(b) *In the situation of (a), suppose moreover that* D *is symmetric with respect to the real axis and* $a \in D \cap \mathbb{R}$. *Given* $f \in \operatorname{Aut}_a(D)$, *prove that* $f(\overline{z}) = \overline{f(z)}$ *on* D *if and only if* $f'(a) = \pm 1$. *(*Hint: *consider* $\tilde{f}(z) = \overline{f(\overline{z})}$.*)*
(c) *Let* D *be bounded. Prove that any* $f \in \operatorname{Aut}_a(D)$ *satisfies* $|f'(a)| = 1$. *(*Hint: *consider the sequence* $\{f^{\circ n} = f \circ f \circ \cdots \circ f\}_{n \geq 1}$ *and compute* $(f^{\circ n})'(a)$.*)*
(d) *In the situation of (c), prove that any* $f \in \operatorname{Aut}_a(D)$ *satisfying* $f'(a) = 1$ *is the identity. (*Hint: *use that, if* $z + c_m z^m + \ldots$ $(m \geq 2)$ *is the Taylor series at* 0 *of an holomorphic function* f, *then* $z + nc_m z^m + \ldots$ *is the Taylor series at* 0 *of* $f^{\circ n}$. *Prove this fact by induction, if time allows.)*

(e) *In the situation of (c), suppose moreover that $D = \mathrm{Int}\,\overline{D}$. Prove that a convergent sequence $\{f_n\}_{n\geq 0} \in \mathrm{Aut}\,(D)$ converges either to some $f \in \mathrm{Aut}\,(D)$ or to a constant $\notin D$. (*Hint: *use the sequence $\{f_n^{-1}\}_{n\geq 0} \in \mathrm{Aut}\,(D)$.) Deduce that $\mathrm{Aut}_a(D) \subset \mathcal{O}(D)$ is compact.*

Solution. (a) As a consequence of Runge's little theorem (see [7, Lecture 16]), $D \subset \mathbb{C}$ is a proper simply connected domain, hence by the Riemann Mapping Theorem, for any $a \in D$ there exists a biholomorphism $g\colon D \to \mathbb{E}$ satisfying $g(a) = 0$. Then $\mathrm{Ad}_g(f) = g \circ f \circ g^{-1} \in \mathrm{Aut}_0(\mathbb{E})$ is a rotation by the Schwarz lemma, *i.e.* it has the form λz for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. It follows that $f'(a) = (\mathrm{Ad}_g(f))'(0) = \lambda$, hence $|f'(a)| = 1$, and $f'(a) = 1$ if and only if $\mathrm{Ad}_g(f) = \mathrm{id}_{\mathbb{E}}$ if and only if $f = \mathrm{id}_D$.

(b) Set $\tilde{f}(z) = \overline{f(\overline{z})}$. Since $\overline{D} = D$ and $\overline{a} = a$, we get a bijection $\tilde{f}\colon D \to D$ satisfying $\tilde{f}(a) = a$. Moreover, since

$$\tilde{f}'(z) = \lim_{h\to 0} \overline{\left(\frac{f(\overline{z}+\overline{h}) - f(\overline{h})}{\overline{h}}\right)} = \overline{f'(\overline{z})},$$

it follows that $\tilde{f}$ is holomorphic, hence $\tilde{f} \in \mathrm{Aut}_a(D)$, and $\tilde{f}'(a) = \overline{f'(a)}$.

One has $f(\overline{z}) = \overline{f(z)}$ if and only if $\tilde{f} = f$, if and only if $f^{-1} \circ \tilde{f} \in \mathrm{Aut}_a(D)$ is the identity. By (a), this happens if and only if $(f^{-1} \circ \tilde{f})'(a) = \frac{\overline{f'(a)}}{f'(a)} = 1$, that is, $f'(a) = \pm 1$, since $|f'(a)| = 1$ by (a).

(c) Given $f \in \mathrm{Aut}_a(D)$, one easily checks that $(f^{\circ n})(a) = a$ and $(f^{\circ n})'(a) = f'(a)^n$. Since D is bounded, then $\mathrm{Aut}_a(D)$ is bounded, as $\sup_{f\in \mathrm{Aut}_a(D)} \|f\|_K \leq \sup_{z\in D} |z| < +\infty$ for any compact $K \subset D$. By Montel's theorem, it follows that the sequence $\{f^{\circ n}\}_{n\geq 1} \subset \mathrm{Aut}_a(D)$ has a convergent subsequence $\{f^{\circ n_k}\}_{k\geq 1}$, hence the sequence $\{(f^{\circ n_k})'(a) = f'(a)^{n_k}\}_{k\geq 1}$ is convergent. This can happen only if $|f'(a)| \leq 1$, otherwise $\lim_n |f'(a)|^n = +\infty$ so that it cannot have convergent subsequences. By considering $f^{-1} \in \mathrm{Aut}_a(D)$, one also gets $|f^{-1\prime}(a)| = \frac{1}{|f'(a)|} \leq 1$, hence $|f'(a)| = 1$.

(d) Given $f \in \mathrm{Aut}_a(D)$ satisfying $f'(a) = 1$, one has $(f^{\circ n})(a) = a$, $(f^{\circ n})'(a) = 1$, and $(f^{\circ n})''(a) = nf''(a)$ by the hint. As in (c), by Montel's theorem the sequence $\{f^{\circ n}\}_{n\geq 1} \subset \mathrm{Aut}_a(D)$ has a convergent subsequence $\{f^{\circ n_k}\}_{k\geq 1}$, hence the sequence $\{(f^{\circ n_k})''(a) = n_k f''(a)\}_{k\geq 1}$ is convergent. This can happen if and only if $f''(a) = 0$. By induction, one gets that $f^{(m)}(a) = 0$ for any $m \geq 2$. By the identity principle, it follows that $f(z) = z$ on D.

Let us prove the content of the hint. Given the Taylor series $f(z) = z + c_m z^m + \dots$ ($m \geq 2$) centred at 0, by induction hypothesis and by using $f^{\circ n}(0) = 0$, we get the following Taylor series centred at 0

$$\begin{aligned} f^{\circ n+1}(z) = f(f^{\circ n}(z)) &= f^{\circ n}(z) + c_m f^{\circ n}(z)^m + \dots \\ &= z + nc_m z^m + \dots + c_m(z + nc_m z^m + \dots)^m + \dots \\ &= z + (n+1)c_m z^m + \dots \end{aligned}$$

(e) Let $\{f_n\}_{n\geq 0} \in \mathrm{Aut}\,(D)$ be a sequence convergent to f in $\mathcal{O}(D)$. By Hurwitz's theorem, either f has constant value $\in \overline{D}$, or f is injective, hence open by the Open Mapping theorem, so that $f(D) \subset \mathrm{Int}\overline{D} = D$.

Consider the sequence of the inverses $\{f_n^{-1}\}_{n\geq 0} \in \mathrm{Aut}\,(D)$. Since D is bounded, then $\mathrm{Aut}\,(D)$ is bounded, hence, by Montel's theorem there exists a convergent subsequence $\{f_{n_k}^{-1}\}$ which converges to g in $\mathcal{O}(D)$. It follows that

$$g'(f(z))f'(z) = \lim_k (f_{n_k}^{-1} \circ f_{n_k})'(z) = 1 \quad \text{for any } z \in D \text{ satisfying } f(z) \in D.$$

Therefore, if f has constant value, necessarily $\in \overline{D} \setminus D$, whereas if f is injective, g cannot be constant, hence it is injective and open, with $g(D) \subset \mathrm{Int}\overline{D} = D$. As

$$g \circ f = \lim_k (f_{n_k}^{-1} \circ f_{n_k}) = \mathrm{id}_D = \lim_k (f_{n_k} \circ f_{n_k}^{-1}) = f \circ g$$

it follows that f (and g) is bijective.

If $\{f_n\}_{n\geq 0} \in \mathrm{Aut}_a(D)$ is a sequence converging to f, then $f(a) = \lim_n f_n(a) = a \in D$, hence f cannot have constant value $\notin D$. It follows that $\mathrm{Aut}_a(D)$ is closed in $\mathcal{O}(D)$. Since it is also bounded, it is compact by Montel's theorem. □

Problem 17. *Let $D \subset \mathbb{C}$ be a domain (*i.e.* non-empty, open and connected).*

(a) *For $z_0 \in D$, let $ev_{z_0} \colon \mathcal{O}(D) \to \mathbb{C}$, $f \mapsto f(z_0)$ be the evaluation map. Show that $ev_{z_0}^{-1}(\{0\})$ is a closed and maximal ideal of $\mathcal{O}(D)$ (*i.e.* $\mathcal{O}(D)$ is the only ideal which properly contains it).*

(b) *Let $I \subset \mathcal{O}(D)$ be an ideal. Show that the following are equivalent:*
   (i) *I is closed and maximal;*
   (ii) *$I = \{f \in \mathcal{O}(D);\ f(z_0) = 0\}$ for some $z_0 \in D$;*
   (iii) *$I = \ker\chi$ for a $\mathbb{C}$-linear ring homomorphism $\chi \colon \mathcal{O}(D) \to \mathbb{C}$.*
   *(*Hint*: show that $z_0 = \chi(\mathrm{id}_D) \in D$ and $\chi = ev_{z_0}$.)*

Solution. (a) The evaluation map is continuous for the topology of compact convergence on $\mathcal{O}(D)$, hence $ev_{z_0}^{-1}(\{0\}) \subset D$ is a closed subset, and one easily checks that this is a proper ideal of $\mathcal{O}(D)$. Let $I \subset \mathcal{O}(D)$ be an ideal properly containing $ev_{z_0}^{-1}(\{0\})$ and take $f \in I$ such that $f(z_0) \neq 0$. Then $f(z) = f(z_0) + (z - z_0)g(z)$ for $g \in \mathcal{O}(D)$, hence $1 = f(z_0)^{-1}f(z_0) = f(z_0)^{-1}\left(f(z) - (z - z_0)g(z)\right) \in I$, so that $I = \mathcal{O}(D)$.

(b) (i) $\Rightarrow$ (ii) By the main result of ideal theory for holomorphic functions (see [7, Lectures 20,21]), every closed ideal is principal, hence $I = g\,\mathcal{O}(D)$ for some $g \in \mathcal{O}(D)$. Since I is proper, g needs to have a zero $z_0 \in D$, otherwise $\frac{1}{g} \in \mathcal{O}(D)$ and $I = \frac{1}{g}I = \frac{1}{g}g\,\mathcal{O}(D) = \mathcal{O}(D)$. It follows that $I \subset \{f \in \mathcal{O}(D);\ f(z_0) = 0\}$ and equality holds by maximality of I.

(ii) $\Rightarrow$ (i) It follows from (a).

(ii) $\Rightarrow$ (iii) One easily checks that $ev_{z_0} \colon \mathcal{O}(D) \to \mathbb{C}$ is a $\mathbb{C}$-linear ring homomorphism, hence $\{f \in \mathcal{O}(D);\ f(z_0) = 0\} = ev_{z_0}^{-1}(\{0\}) = \ker ev_{z_0}$.

(iii) $\Rightarrow$ (ii): Set $z_0 = \chi(\mathrm{id}_D) \in \mathbb{C}$. Then $\chi(z - z_0) = \chi(\mathrm{id}_D) - z_0 = 0$, hence $z_0 \in D$, otherwise $(z - z_0)^{-1} \in \mathcal{O}(D)$ and $1 = \chi\left((z - z_0)^{-1}(z - z_0)\right) = \chi(z - z_0)^{-1}\chi(z - z_0) = 0$.

Let $f \in \mathcal{O}(D)$. Then $f(z) = f(z_0) + (z - z_0)g(z)$ for $g \in \mathcal{O}(D)$, hence $\chi(f) = \chi(f(z_0)) + \chi((z - z_0))\chi(g(z)) = f(z_0) = ev_{z_0}(f)$. It follows that $\ker\chi = \{f \in \mathcal{O}(D);\ f(z_0) = 0\}$. □

Problem 18. *Set $\mathbb{E} = \{|z| < 1\}$ and* $b_a(z) = \begin{cases} \frac{|a|}{a}\frac{z-a}{\bar{a}z-1} & \text{if } a \in \mathbb{E} \setminus \{0\}, \\ z & \text{if } a = 0. \end{cases}$

(a) *Show that for any $a \in \mathbb{E}$, the map $b_a(z)$ defines a biholomorphism of $\mathbb{E}$ satisfying $b_a(\partial\mathbb{E}) = \partial\mathbb{E}$.*

(b) *Prove that $|b_a(z) - 1| \leq \frac{(1-|a|)(1+|z|)}{1-|z|}$ for any $a, z \in \mathbb{E}$.*

(c) *Let $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{E}$ be a sequence such that the series $\sum_{n\geq 0}(1 - |a_n|)$ converges. Prove that the infinite product $\prod_{n\geq 0} b_{a_n}(z)$ defines a holomorphic map $f\colon \mathbb{E} \to \mathbb{E}$ having a zero at any a_n.*

Solution. (a) If $a = 0$ there is nothing to prove. If $a \in \mathbb{E} \setminus \{0\}$, then $\left|\frac{|a|}{a}\right| = 1$ hence $b_a(z)$ is a Möbius transformation which restricts to a conformal map $\mathbb{E} \to \mathbb{E}$ (see [7, Lecture 5]). It remains to prove that $b_a(\partial\mathbb{E}) = \partial\mathbb{E}$. We have

$$|b_a(z)| = \frac{|z-a|}{|\bar{a}z - 1|} = 1 \iff |z-a|^2 = |\bar{a}z - 1|^2 \iff (z-a)(\bar{z} - \bar{a}) = (1 - \bar{a}z)(1 - \bar{z}a).$$

This is equivalent to $(1 - |a|^2)(1 - |z|^2) = 0$, which holds iff $|z| = 1$, as $|a| < 1$.

(b) Let $z \in \mathbb{E}$. If $a = 0$, then $|z - 1| \leq 1 + |z| \leq \frac{1+|z|}{1-|z|}$. For $a \in \mathbb{E} \setminus \{0\}$, we have

$$|b_a(z) - 1| = \left|\frac{|a|(z-a) - a(\bar{a}z - 1)}{a(\bar{a}z - 1)}\right| = \frac{(1-|a|)\,|(|a|z + a)|}{|a|\,|(\bar{a}z - 1)|} \leq \frac{(1-|a|)(1+|z|)}{1-|z|}$$

since $|\bar{a}z - 1| \geq 1 - |\bar{a}z| > 1 - |z|$.

(c) For any compact $K \subset \mathbb{E}$, we have

$$\|b_{a_n}(z) - 1\|_K \leq (1 - |a_n|)\frac{1 + \|z\|_K}{1 - \|z\|_K}$$

by (b), as $a_n \in \mathbb{E}$ for any $n \geq 0$. Since $\sum_{n\geq 1}(1 - |a_n|) < +\infty$, the series $\sum_{n\geq 1}(b_{a_n}(z) - 1)$ is normally convergent in $\mathcal{O}(\mathbb{E})$, hence $\prod_{n\geq 0} b_{a_n}(z)$ defines a holomorphic map $f \in \mathcal{O}(\mathbb{E})$ having a zero at any a_n (but not identically zero as none of its factors vanishes). By (a), for any $z \in \mathbb{E}$ we have $|f(z)| = \prod_{n\geq 0} |b_{a_n}(z)| \leq 1$, hence $f(\mathbb{E}) \subset \mathbb{E}$ by the Open Mapping Theorem, f being non-constant. □

Problem 19. *Let $\mathbb{T}$ denote the right half-plane $\{Re\ z > 0\}$ and $\mathbb{E}$ the unit disk $\{|z| < 1\}$.*

(a) *Show that $t(z) = \frac{z-1}{z+1}$ defines a conformal map $t\colon \mathbb{T} \to \mathbb{E}$, and compute its inverse.*

(b) *Let $d\colon \mathbb{T} \to \mathbb{N}$ be a positive divisor and $\{a_n\}_{n\geq 1} \subset \mathbb{C}^\times$ a sequence such that $\operatorname{supp} d = \{a_n; n \geq 1\}$ and $d(a_n) =$ cardinality of $\{a_k;\ k \geq 1, a_k = a_n\}$.*

Prove that the following are equivalent:

(i) *$d = (f)$ for a bounded function $f \in \mathcal{O}(\mathbb{T})$;*

(ii) *the series $\sum_{n\geq 1} \frac{Re\ a_n}{|1+a_n|^2}$ converges.*

*(*Hint*: use the Blaschke condition and the inequality $\frac{1}{2}(1 - |w|^2) \leq 1 - |w| \leq (1 - |w|^2)$ in $\mathbb{E}$.)*

(c) *Let $\{r_n\}_{n\geq 1} \subset \mathbb{R}_{>0}$ be a sequence with $r_n > 1$ for any $n \geq 1$ and such that $\sum_{n\geq 1} \frac{1}{r_n} = +\infty$. Show that, if $f \in \mathcal{O}(\mathbb{T})$ is a bounded function such that $f(r_n) = 0$ for any $n \geq 1$, then f vanishes identically on $\mathbb{T}$.*

Solution. (a) We know (see [7, Lecture 6]) that $z \mapsto \frac{1}{z}$ maps conformally $\{\mathrm{Re} z > 1\}$ onto the open disk $\{|z - \frac{1}{2}| < \frac{1}{2}\}$. It follows that $t(z) = \frac{z-1}{z+1}$ defines a conformal map of $\mathbb{T}$ onto $\mathbb{E}$, as it may be decomposed into

$$\mathbb{T} \underset{a}{\to} \{\mathrm{Re} z > 1\} \underset{b}{\to} \{|z - \frac{1}{2}| < \frac{1}{2}\} \underset{c}{\to} \mathbb{E}$$

where $b(z) = \frac{1}{z}$, and $a(z) = z + 1$ and $c(z) = 1 - 2z$ are surjective affine transformations.

As $t(z)$ is the Möbius transformation associated to the projective matrix $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $t^{-1}\colon \mathbb{E} \to \mathbb{T}$ is given by the Möbius transformation associated to the inverse projective matrix $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, that is, $t^{-1}(z) = \frac{1+z}{1-z}$.

One may also proceed as follows: $t(z)$ is a Möbius transformation, hence it defines a conformal map of $\mathbb{C} \setminus \{-1\}$ onto $\mathbb{C} \setminus \{1\}$, which sends simply connected open subsets to simply connected open subsets. By a direct computation, one gets $t(\partial\mathbb{T}) = \partial\mathbb{E} \setminus \{1\} \subset \partial\mathbb{E} \subset \partial t(\mathbb{T})$. It follows that $\partial t(\mathbb{T}) = \partial\mathbb{E}$, hence $t(\mathbb{T}) = \mathbb{E}$. The determination of the inverse follows also by an easy computation.

(b) The conformal map $t\colon \mathbb{T} \to \mathbb{E}$ induces a ring isomorphsim

$$\mathcal{O}(\mathbb{T}) \to \mathcal{O}(\mathbb{E}), \quad f \mapsto \tilde{f} = f \circ t^{-1},$$

which sends bounded functions on $\mathbb{T}$ to bounded functions on $\mathbb{E}$. Then $(f) = d$ if and only if $(\tilde{f}) = \tilde{d}$, where $\tilde{d} = d \circ t^{-1}\colon \mathbb{E} \to \mathbb{N}$ is a positive divisor. By the Blaschke condition, this happens if and only if the series $\sum_{n\geq 1}(1 - |t(a_n)|)$ converges, as $\tilde{d}(t(a_n)) = d(a_n)$. From the inequality in $\mathbb{E}$

$$\frac{1}{2}(1 - |w|^2) \leq 1 - |w| \leq (1 - |w|^2),$$

it follows that the above series converges if and only if so does the series $\sum_{n\geq 1}(1 - |t(a_n)|^2)$. One then concludes, since for any $z \in \mathbb{T}$

$$1 - |t(z)|^2 = 4\frac{\mathrm{Re}\ z}{|1 + z|^2}.$$

(c) Let $f \in \mathcal{O}(\mathbb{T})$ be a bounded function satifying $f(r_n) = 0$ for any $n \geq 1$. If f was not identically zero, $d = (f)$ would be a positive divisor. By (a), the series $\sum_{n\geq 1} \frac{r_n}{(1+r_n)^2}$ would converge, as a sub-series of $\sum_{n\geq 1} \frac{\mathrm{Re}\ a_n}{|1+a_n|^2}$, with $\{a_n\}_{n\geq 1} \subset \mathbb{C}^\times$ a sequence satisfying $\mathrm{supp}\, d = \{a_n; n \geq 1\}$ and $d(a_n) =$ cardinality of $\{a_k; a_k = a_n,\ k \geq 1\}$. But this is impossible, as

$$\sum_{n\geq 1} \frac{r_n}{(1 + r_n)^2} \geq \frac{1}{4} \sum_{n\geq 1} \frac{1}{r_n} = +\infty.$$

□

Problem 20. *Consider the infinite product*

$$\prod_{n\geq 1}(1 + q^n z) \qquad \textit{for } q \in \mathbb{C}$$

(a) *Find the q's for which the infinite product converges normally in $\mathcal{O}(\mathbb{C})$ to $f_q(z)$.*

(b) *For any q as in (a), show that $\prod_{n\geq 1,\, n\ odd}(1+q^n)$ and $\prod_{n\geq 1,\, n\ odd}(1-q^n)^{-1}$ may be recovered as special values of $f_p(w)$ for some p and w. (*Hint*: for the second use $(1+q^n)(1-q^n)=(1-q^{2n})$.)*

(c) *For any q as in (a), show that $f_q(z)$ never vanishes in the unit disk $\mathbb{E}=\{|z|<1\}$, and compute $\partial \log f_q(0)$.*

(d) *For any q as in (a), compute the Taylor series of $f_q(z)$ at $z=0$. (*Hint*: use $\prod_{n\geq 1}(1+q^n z)=(1+qz)\prod_{n\geq 1}(1+q^n qz)$.)*

Solution. (a) The infinite product $\prod_{n\geq 1}(1+q^n z)$ converges normally in $\mathcal{O}(\mathbb{C})$ iff the series $\sum_{n\geq 1} q^n z$ converges normally in $\mathcal{O}(\mathbb{C})$, and this happens iff $|q|<1$. For any such q, set $f_q(z)$ the entire function defined by the corresponding infinite product.

(b) Since $\prod_{n\geq 1}(1+q^n)$ is convergent, so does $\prod_{n\geq 1,\, n \text{ odd}}(1+q^n)$ and, if $q\neq 0$, this is equal to $\prod_{n\geq 1}(1+q^{2n-1})=f_{q^2}(q^{-1})$ (note that, if $|q|<1$, then $|q^2|<1$ and f_{q^2} is well-defined).

From $(1+q^n)(1-q^n)=(1-q^{2n})$, one has

$$\prod_{n=1}^{k}(1+q^n)=\frac{1-q^2}{1-q}\frac{1-q^4}{1-q^2}\frac{1-q^6}{1-q^3}\frac{1-q^8}{1-q^4}\cdots\frac{1-q^{2k}}{1-q^k}=\prod_{n=1,\, n \text{ odd}}^{k}(1-q^n)^{-1}\prod_{n=k+1,\, n \text{ even}}^{2k}(1-q^n).$$

As

$$\lim_{k\longrightarrow+\infty}\prod_{n=k+1}^{2k}(1-q^n)=\lim_{k\longrightarrow+\infty}\frac{\prod_{n\geq k+1}(1-q^n)}{\prod_{n\geq 2k+1}(1-q^n)}=\frac{1}{1},$$

we get $f_q(1)=\prod_{n\geq 1,\, n \text{ odd}}(1-q^n)^{-1}$.

(c) $f_q(z)=0$ iff one of the factors $1+q^n z=0$, i.e. $z=-q^{-n}$ for some $n\geq 1$ if $q\neq 0$. As $|q|<1$, then $|-q^{-n}|=|q|^{-n}>1$ for any $n\geq 1$, so that $f_q(z)$ never vanishes in $\mathbb{E}$. One has

$$\partial \log f_q(z)=\sum_{n\geq 1}\partial \log(1+q^n z)=\sum_{n\geq 1}\frac{q^n}{1+q^n z},$$

whence

$$\partial \log f_q(0)=\sum_{n\geq 1} q^n=\frac{1}{1-q}-1=\frac{q}{1-q}.$$

(d) From $\prod_{n\geq 1}(1+q^n z)=(1+qz)\prod_{n\geq 1}(1+q^n qz)$, we get the functional equation

$$f_q(z)=(1+qz)f_q(qz).$$

Let $f_q(z)=\sum_{n\geq 0} a_n z^n$ be the Taylor series at $z=0$. We have $a_0=f_q(0)=1$, and the functional equation gives $a_n=a_n q^n+qa_{n-1}q^{n-1}$ for any $n\geq 1$, i.e. $a_n=\frac{q^n}{1-q^n}a_{n-1}$. By induction, one gets $a_n=\frac{q^{1+2+\cdots+n}}{(1-q)(1-q^2)\ldots(1-q^n)}$ for any $n\geq 1$. Whence

$$f_q(z)=1+\sum_{n\geq 1}\frac{q^{\frac{n(n+1)}{2}}}{(1-q)(1-q^2)\ldots(1-q^n)}z^n.$$

□

Problem 21. *Consider the infinite products*

$$\prod_{n\geq 1}(1+(-1)^n z^n),\quad \prod_{n\geq 1}(1+z^{2n-1}).$$

(a) *Show that they converge normally in* $\mathcal{O}(\mathbb{E})$ *to never vanishing functions* $f(z)$ *and* $g(z)$. *(Here* $\mathbb{E} = \{|z| < 1\}$ *denotes the unit disk.)*
(b) *Compute the product* $f(z)g(z)$ *for any* $z \in \mathbb{E}$.

Solution. (a) The infinite products converge normally in $\mathcal{O}(\mathbb{E})$, since the series $\sum_{n\geq 1}(-1)^n z^n$ and $\sum_{n\geq 1} z^{2n-1}$ converge normally in $\mathcal{O}(\mathbb{E})$. They define holomorphic functions $f(z)$ and $g(z)$ on $\mathbb{E}$, which never vanish, since none of their factors vanishes on $\mathbb{E}$.

(b) From $(1+(-1)^n z^n)(1-(-1)^n z^n) = (1 - z^{2n})$, one has

$$\prod_{n=1}^{k}(1+(-1)^n z^n) = \frac{\prod_{n=1}^{k}(1-z^{2n})}{\prod_{n=1}^{k}(1-(-1)^n z^n)}.$$

It follows that

$$\lim_{k\longrightarrow+\infty}\prod_{n=1}^{k}(1+(-1)^n z^n)\prod_{n=1}^{k}(1+z^{2n-1}) = \lim_{k\longrightarrow+\infty}\frac{\prod_{n=1}^{2k}(1-(-1)^n z^n)}{\prod_{n=1}^{k}(1-(-1)^n z^n)},$$

hence $f(z)g(z) = 1$ for any $z \in \mathbb{E}$. □

Problem 22. (a) *Prove that* $\prod_{n\geq 1}\frac{(2n)^2}{(2n-1)(2n+1)}$ *is convergent. Prove that* $\prod_{n\geq 1}\frac{1+\frac{2z}{2n-1}}{1+\frac{2z}{2n+1}} = 1+2z$ *in* $\mathcal{O}(B_1(0))$.
(b) *Prove that* $z\prod_{n\geq 1}\left(1-\frac{z^2}{n^2}\right)$ *is normally convergent to an entire function* $g(z)$, *and compute its zeros with multiplicities.*
(c) *Prove that* $g(z)$ *satisfies the functional equation* (0.1) *in* **Problem** *2. Deduce the sine product* $\frac{\sin(\pi z)}{\pi} = z\prod_{n\geq 1}\left(1-\frac{z^2}{n^2}\right)$. *(*Hint*: use* $\frac{(2n-1)(2n+1)}{(2n)^2}\left(1-\frac{(2z)^2}{(2n-1)^2}\right) = \frac{1+\frac{2z}{2n-1}}{1+\frac{2z}{2n+1}}\left(1-\frac{(z+\frac{1}{2})^2}{n^2}\right)$.*)*

Solution. (a) The first infinite product is convergent, since

$$\frac{(2n)^2}{(2n-1)(2n+1)} - 1 = \frac{1}{4n^2-1} \sim \frac{1}{4n^2} \qquad \text{for } n \to +\infty$$

with $\sum_{n\geq 1}\frac{1}{4n^2} < +\infty$.

For the second, we have

$$\prod_{n\geq 1}^{k}\frac{1+\frac{2z}{2n-1}}{1+\frac{2z}{2n+1}} = \frac{1+2z}{1+\frac{2z}{3}}\frac{1+\frac{2z}{3}}{1+\frac{2z}{5}}\cdots\frac{1+\frac{2z}{2k-1}}{1+\frac{2z}{2k+1}} = \frac{1+2z}{1+\frac{2z}{2k+1}}$$

with $\frac{1+\frac{2z}{2n-1}}{1+\frac{2z}{2n+1}}$ holomorphic in $B_1(0)$ for any $n \geq 1$, hence we get

$$\prod_{n\geq 1}\frac{1+\frac{2z}{2n-1}}{1+\frac{2z}{2n+1}} = \lim_{k\longrightarrow+\infty}\frac{1+2z}{1+\frac{2z}{2k+1}} = 1+2z \text{ in } \mathcal{O}(B_1(0)).$$

(b) For any compact $K \subset \mathbb{C}$ and any $z \in K$, we have

$$\left|1-\frac{z^2}{n^2}-1\right| \leq \frac{\|z\|_K^2}{n^2}$$

with $\sum_{n\geq 1}\frac{1}{n^2}<+\infty$, hence the infinite product is normally convergent in $\mathcal{O}(\mathbb{C})$. Since $\left(1-\frac{z^2}{n^2}\right)=0$ iff $z=\pm n$ for any $n\geq 1$, it follows that $g(z)=z\prod_{n\geq 1}\left(1-\frac{z^2}{n^2}\right)$ has zeros exactly at the points $z\in\mathbb{Z}$, all with multiplicity 1.

(c) Set $\lambda=\prod_{n\geq 1}\frac{(2n)^2}{(2n-1)(2n+1)}$. By using normal convergence, we get

$$\begin{aligned}
g(2z) &= 2z\prod_{n\geq 1}\left(1-\frac{(2z)^2}{(2n)^2}\right)\prod_{n\geq 1}\left(1-\frac{(2z)^2}{(2n-1)^2}\right)\\
&= 2g(z)\prod_{n\geq 1}\left(1-\frac{(2z)^2}{(2n-1)^2}\right)\\
&= 2g(z)\lambda\prod_{n\geq 1}\frac{(2n-1)(2n+1)}{(2n)^2}\left(1-\frac{(2z)^2}{(2n-1)^2}\right)\\
&= 2g(z)\lambda\prod_{n\geq 1}\frac{1+\frac{2z}{2n-1}}{1+\frac{2z}{2n+1}}\left(1-\frac{(z+\frac{1}{2})^2}{n^2}\right)\\
&= 2g(z)\lambda(1+2z)\prod_{n\geq 1}\left(1-\frac{(z+\frac{1}{2})^2}{n^2}\right)=4\lambda g(z)g\left(z+\frac{1}{2}\right)
\end{aligned}$$

Hence g satisfies the functional equation (0.1) in **Problem** 2 with $c=\frac{1}{4\lambda}$.

As g is entire and odd, from (0.1) it follows that $g(z)=\frac{1}{2\lambda}\sin(\pi z)$. Finally, since

$$1=\lim_{z\longrightarrow 0}\frac{\sin(\pi z)}{\pi z}=\lim_{z\longrightarrow 0}\frac{2\lambda g(z)}{\pi z}=\frac{2\lambda}{\pi}$$

we get $\lambda=\frac{\pi}{2}$ (Wallis' formula).

□

Problem 23. *Consider the infinite product $\prod_{n=-\infty}^{\infty}\left(1+\frac{z}{n-\lambda}\right)e^{-\frac{z}{n-\lambda}}$ with $\lambda\in\mathbb{C}\setminus\mathbb{Z}$.*

(a) *Prove that $|(1-z)e^z-1|\leq |z|^2$ for $|z|<1$.*
(b) *Show that infinite product converges normally in $\mathcal{O}(\mathbb{C})$ to a function $f_\lambda(z)$ having a zero at $n+\lambda$ for any $n\in\mathbb{Z}$.*
(c) *Find $\alpha>0$ and $\beta\in\mathbb{C}$ such that $\sin(\alpha(z+\beta))$ and $f_\lambda(z)$ define the same divisor.*
(d) *For α,β as in (c), find explicitly a function $v_\lambda(z)$ such that $\sin(\alpha(z+\beta))=e^{v_\lambda(z)}f_\lambda(z)$.*

Solution. (a) For any $|z|<1$, we have

$$|(1-z)e^z-1|=\left|\sum_{n\geq 0}\frac{z^n}{n!}-\sum_{n\geq 0}\frac{z^{n+1}}{n!}-1\right|\leq\sum_{n\geq 2}\left(\frac{1}{(n-1)!}-\frac{1}{n!}\right)|z|^n=|z|^2,$$

as the series $\sum_{n\geq 2}\left(\frac{1}{(n-1)!}-\frac{1}{n!}\right)$ telescopes to 1. (See **Problem** 27 (a).)

(b) For any compact $K\subset\mathbb{C}$, take $n_K\in\mathbb{N}$ such that $n_K-|\lambda|>\|z\|_K$. Then for any $z\in K$ and $|n|\geq n_K$, we have $\left|\frac{z}{n-\lambda}\right|\leq\frac{|z|}{|n|-|\lambda|}\leq\frac{\|z\|_K}{n_K-|\lambda|}<1$, hence by (a)

$$\left|\left(1+\frac{z}{n-\lambda}\right)e^{-\frac{z}{n-\lambda}}-1\right|\leq\left|\frac{z}{n-\lambda}\right|^2\leq\frac{\|z\|_K^2}{(|n|-|\lambda|)^2}.$$

Since $\sum_{|n|\geq 1}\frac{1}{(|n|-|\lambda|)^2} < +\infty$, the series $\sum_{|n|\geq 1}\left\|\left(1+\frac{z}{n-\lambda}\right)e^{-\frac{z}{n-\lambda}}-1\right\|_K$ is convergent, hence $\prod_{n=-\infty}^{\infty}\left(1+\frac{z}{n-\lambda}\right)e^{-\frac{z}{n-\lambda}}$ defines an entire holomorphic map $f_\lambda(z)$, which has a zero at $n+\lambda$ for any $n\in\mathbb{Z}$.

(c) The zeros of $f_\lambda(z)$ are all simple, as well as those of $\sin(\alpha(z+\beta))$ for $\alpha>0$. It follows that they define the same divisor if and only if $\sin(\alpha(n+\lambda+\beta))=0$ for any $n\in\mathbb{Z}$, for some $\alpha>0$ and $\beta\in\mathbb{C}$, that is, iff $\alpha=\pi$ and $\beta=-\lambda$.

(d) Clearly, since $\sin(\pi(z-\lambda))$ and $f_\lambda(z)$ have the same divisor and $\mathbb{C}$ is simply connected, there exist a function $v_\lambda(z)$ such that $\sin(\pi(z-\lambda))=e^{v_\lambda(z)}f_\lambda(z)$. To find it explicitly, by taking $\partial\log$ of the previous equality, we get

$$\begin{aligned}\pi\cot(\pi(z-\lambda)) &= v'_\lambda(z)+\sum_{n=-\infty}^{\infty}\left(\frac{\frac{1}{n-\lambda}}{1+\frac{z}{n-\lambda}}-\frac{1}{n-\lambda}\right)\\ &= v'_\lambda(z)+\sum_{n=-\infty}^{\infty}\left(\frac{1}{z+n-\lambda}-\frac{1}{n-\lambda}\right)\\ &= v'_\lambda(z)+p.v.\sum_{n=-\infty}^{\infty}\frac{1}{z-\lambda+n}-p.v.\sum_{n=-\infty}^{\infty}\frac{1}{-\lambda+n}\\ &= v'_\lambda(z)+\pi\cot(\pi(z-\lambda))-\pi\cot(-\pi\lambda)\end{aligned}$$

that is, $v'_\lambda(z)=\pi\cot(-\pi\lambda)$. It follows that $v_\lambda(z)=\pi\cot(-\pi\lambda)z+c$ for some $c\in\mathbb{C}$.

Since $f_\lambda(z)=1$, we get $\sin(-\pi\lambda)=e^{v_\lambda(0)}=e^c$, hence $c=\log(\sin(-\pi\lambda))$. □

Problem 24. *Let $\{a_n\}_{n\in\mathbb{N}}\subset\mathbb{C}$ be a sequence of bounded variation,* i.e. *satisfying $\sum_{n\geq 0}|a_{n+1}-a_n|<+\infty$.*

(a) *Show that the series $\sum_{n\geq 0}a_nz^n$ converges normally in $\mathcal{O}(\mathbb{E})$ to a function f.*
(b) *Suppose $a_0=a_1=1$ and set $a=\sum_{n\geq 0}|a_{n+1}-a_n|$ and $E(z)=(1-z)f(z)$. Show that $|E(z)-1|\leq a|z|^2$ for any $z\in\mathbb{E}$.*
(c) *In the situation of (b), suppose moreover that f extends analytically in $\mathbb{C}$. Show that $\prod_{n\geq 1}E(\frac{z}{n})$ converges normally in $\mathcal{O}(\mathbb{C})$ to an holomorphic function g having a zero at any $n\geq 1$.*
(d) *Set $a_n=\frac{1}{n!}$. Show that the sequence $\{a_n\}_{n\in\mathbb{N}}$ is of bounded variation. Then prove that $zg(z)g(-z)=\frac{\sin(\pi z)}{\pi}$ and compute $g(-1)$.*

Solution. (a) For any $k\geq 1$ we have

$$|a_k|=|a_0+\sum_{n=0}^{k-1}(a_{n+1}-a_n)|\leq|a_0|+\sum_{n=0}^{k-1}|a_{n+1}-a_n|\leq|a_0|+\sum_{n\geq 0}|a_{n+1}-a_n|,$$

hence $\{a_n\}_{n\in\mathbb{N}}$ is a bounded sequence. (In fact, in the same way one may prove that $\{a_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, hence convergent.)

For any compact $K\subset\mathbb{E}$ and any $n\geq 0$ we get $\|a_nz^n\|_K=|a_n|\,\|z\|_K^n\leq$ cost. $\|z\|_K^n$. Since $\sum_{n\geq 0}\|z\|_K^n<+\infty$, the series $\sum_{n\geq 0}a_nz^n$ converges normally in $\mathcal{O}(\mathbb{E})$.

(b) For $z \in \mathbb{E}$, we have

$$E(z) = (1-z)f(z) = \sum_{n\geq 0} a_n z^n - \sum_{n\geq 1} a_{n-1} z^n = a_0 + \sum_{n\geq 1} (a_n - a_{n-1}) z^n.$$

Since $a_0 = a_1 = 1$ and $|z| < 1$, we get

$$|E(z) - 1| = \left| \sum_{n\geq 2} (a_n - a_{n-1}) z^n \right| \leq \sum_{n\geq 2} |a_n - a_{n-1}||z|^n \leq a|z|^2.$$

(c) Since f extends analytically on $\mathbb{C}$, $E(\frac{z}{n}) \in \mathcal{O}(\mathbb{C})$ for any $n \in \mathbb{N}$. For a compact $K \subset \mathbb{C}$, take $n_K \in \mathbb{N}$ such that $n_K > \|z\|_K$. Then for any $z \in K$ and $n \geq n_K$, we have $\left|\frac{z}{n}\right| < 1$, hence by (b)

$$\left| E\left(\frac{z}{n}\right) - 1 \right| \leq a \left|\frac{z}{n}\right|^2 \leq a \frac{\|z\|_K^2}{n^2}.$$

Since $\sum_{n\geq n_K} \frac{1}{n^2} < +\infty$, the series $\sum_{n\geq n_K} \left\| E(\frac{z}{n}) - 1 \right\|_K$ converges. It follows that the infinite product $\prod_{n\geq 1} E(\frac{z}{n})$ converges normally in $\mathcal{O}(\mathbb{C})$ to a function g.

Since $E(\frac{z}{n}) = 0$ if $z = n$, it follows that any $n \geq 1$ is a zero of g.

(d) As the sequence $\{\frac{1}{n!}\}_{n\in\mathbb{N}}$ is decreasing, we get a telescoping series

$$\sum_{n\geq 0} \left| \frac{1}{(n+1)!} - \frac{1}{n!} \right| = \sum_{n\geq 0} \left(\frac{1}{n!} - \frac{1}{(n+1)!} \right) = 1.$$

For $a_n = \frac{1}{n!}$, we have $f(z) = e^z$, hence

$$g(z) = \prod_{n\geq 1} \left(1 - \frac{z}{n}\right) e^{\frac{z}{n}}.$$

By using the sine product, we get

$$zg(z)g(-z) = z \prod_{n\geq 1} \left(1 - \frac{z}{n}\right) e^{\frac{z}{n}} \prod_{n\geq 1} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}} = z \prod_{n\geq 1} \left(1 - \frac{z^2}{n^2}\right) = \frac{\sin(\pi z)}{\pi}.$$

Note that

$$g(-z) = \frac{\Delta(z)}{ze^{\gamma z}}$$

where $\Delta(z)$ is the Weierstrass Delta function and γ the Euler-Mascheroni constant, hence

$$g(-1) = \prod_{n\geq 1} \left(1 + \frac{1}{n}\right) e^{-\frac{1}{n}} = \frac{\Delta(1)}{e^\gamma} = e^{-\gamma}.$$

In order to prove the above equality directly, one uses the following computation

$$\begin{aligned}
\log g(-1) &= \sum_{n\geq 1} \left(\log\left(1 + \frac{1}{n}\right) - \frac{1}{n} \right) \\
&= -\sum_{n=1}^{\infty} \left(\frac{1}{n} + \log n - \log(n+1) \right) \\
&= -\lim_{n\longrightarrow +\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log(n+1) \right) = -\gamma
\end{aligned}$$

$\square$

Problem 25. (a) *Let $f \in \mathcal{O}(\mathbb{C})$ satisfy $f(0) = 1$ and $f'(0) = 0$. Show that $\prod_{n\geq 1} f(\frac{z}{n})$ converges normally in $\mathcal{O}(\mathbb{C})$.*

(b) *Let $f(z)$ be the analytic continuation of $\frac{\sin \pi z}{\pi z}$ on $\mathbb{C}$. Show that $\prod_{n\geq 1} f(\frac{z}{2n-1})$ converges normally in $\mathcal{O}(\mathbb{C})$ to a function g and compute the divisor (g).*

(c) *Prove that $g(z) = \prod_{n\geq 1} \cos(\frac{\pi z}{2n})$. (*Hint*: use the sine product)*

Solution. (a) The function $f(z) - 1$ has a zero of order ≥ 2 in 0, hence we may write $f(z) - 1 = a(z)z^2$ for $a \in \mathcal{O}(\mathbb{C})$. For a compact $K \subset \mathbb{C}$, take $n_K \in \mathbb{N}$ such that $n_K \geq \|z\|_K$. Then for any $z \in K$ and $n \geq n_K$, we have $|\frac{z}{n}| \leq 1$ and

$$\left|f\left(\frac{z}{n}\right) - 1\right| = \left|a\left(\frac{z}{n}\right)\right| \left|\frac{z}{n}\right|^2 \leq \|a\|_{\mathbb{E}} \|z\|_K^2 \frac{1}{n^2}.$$

Since $\sum_{n\geq n_K} \frac{1}{n^2} < +\infty$, the series $\sum_{n\geq n_K} \left\|f(\frac{z}{n}) - 1\right\|_K$ converges. It follows that the infinite product $\prod_{n\geq 1} f(\frac{z}{n})$ converges normally in $\mathcal{O}(\mathbb{C})$.

(b) Since $f(0) = 1$ and $f'(0) = 0$, by (a) $\prod_{n\geq 1} f(\frac{z}{2n-1})$ converges normally in $\mathcal{O}(\mathbb{C})$ to a function g. Since $f(z)$ has a first order zero at any non-zero integer, $g(z) = 0$ iff $z \in (2n-1)\mathbb{Z} \setminus \{0\}$ for some integer $n \geq 1$. It follows that

$$(g)(z) = \begin{cases} \#\{\text{positive odd integers which divide } z\} & \text{if } z \in \mathbb{Z} \setminus \{0\}, \\ 0 \quad \text{otherwise.} \end{cases}$$

(c) For any $n \geq 1$ and $z \neq 0$, the sine product formula gives

$$\frac{\sin\left(\pi\frac{z}{2n-1}\right)}{\pi\frac{z}{2n-1}} = \prod_{m\geq 1}\left(1 - \frac{\left(\frac{z}{2n-1}\right)^2}{m^2}\right) = \prod_{m\geq 1}\left(1 - \frac{\left(\frac{z}{m}\right)^2}{(2n-1)^2}\right)$$

hence for any $z \neq 0$ we get

$$\prod_{n\geq 1} f\left(\frac{z}{2n-1}\right) = \prod_{n\geq 1} \frac{\sin\left(\pi\frac{z}{2n-1}\right)}{\pi\frac{z}{2n-1}} = \prod_{n,m\geq 1}\left(1 - \frac{\left(\frac{z}{m}\right)^2}{(2n-1)^2}\right).$$

The result follows by recalling that for any $m \geq 1$

$$\prod_{n\geq 1}\left(1 - \frac{4\left(\frac{z}{2m}\right)^2}{(2n-1)^2}\right) = \cos\left(\pi\frac{z}{2m}\right)$$

(see [7, Lecture 14]), and by the uniqueness theorem.

Note that one can also use directly the formula

$$\cos\left(\pi\frac{z}{2m}\right) = \frac{\sin\left(\pi\frac{z}{2n-1}\right)}{2\sin\left(\pi\frac{z}{2m}\right)}.$$

$\square$

Problem 26. *For $n \in \mathbb{N}$, set $E_n(z) = (1-z)e^{p_n(z)}$ with $p_n(z) = z + \frac{z^2}{2} + \cdots + \frac{z^n}{n}$ for $n \geq 1$ and $p_0(z) = 0$.*

(a) *Show that $-\frac{E_n'(z)}{z^n}$ extends to an entire function $F_n(z)$ satisfying $|F_n(z)| \leq F_n(|z|)$.*

(b) *Prove that*

$$|E_n(z)-1| \le |z|^{n+1}\int_0^1 t^n F_n(t|z|)dt$$

for any $z\in\mathbb{C}$ *and deduce that* $|E_n(z)-1|\le |z|^{n+1}$ *if* $|z|\le 1$.

(c) *Let* $\{a_n\}_{n\ge 0}\subset\mathbb{C}^\times$ *be a sequence satisfying* $\lim_{n\to\infty}|a_n| = +\infty$. *Show that for any choice of a sequence* $\{k_n\}_{n\ge0}\subset\mathbb{N}$ *satisfying* $\sum_{n\ge0}\left(\frac{r}{|a_n|}\right)^{k_n+1} < +\infty$ *for any* $r>0$, *the infinite product* $\prod_{n\ge0}E_{k_n}\left(\frac{z}{a_n}\right)$ *converges normally in* $\mathcal{O}(\mathbb{C})$ *to a function having a zero at any* a_n.

(d) *Show that* $\prod_{n\ge0}\left(1-\frac{z}{2^n}\right)$ *converges normally in* $\mathcal{O}(\mathbb{C})$ *to a function* f *satisfying* $f(z)=(1-z)f(\frac{z}{2})$.

(e) *Let* g *be an entire function satisfying* $g(z)=(1-z)g(\frac{z}{2})$ *and* $g(0)\neq 0$. *Show that* $g=g(0)f$.

Solution. (a) We have $E_n'(z) = -z^n e^{p_n(z)}$, hence $F_n(z)=e^{p_n(z)}$ and

$$|F_n(z)| = e^{\mathrm{Re}(p_n(z))}\le e^{|p_n(z)|}\le F_n(|z|)$$

(b) Since $E_n(0)=1$, we get

$$|E_n(z)-1| = \left|\int_{[0,z]}E_n'(w)dw\right| \le |z|^{n+1}\int_0^1 t^n|F_n(tz)|dt \le |z|^{n+1}\int_0^1 t^nF_n(t|z|)dt$$

For $|z|\le 1$, we have $F_n(t|z|)\le F_n(t)$ for any $t\ge 0$, hence

$$|E_n(z)-1|\le |z|^{n+1}\int_0^1 t^nF_n(t)dt = |z|^{n+1}\int_0^1 -E_n'(t)dt = |z|^{n+1}$$

as $E_n(1)=0$.

(c) For a compact $K\subset\mathbb{C}$, take $n_K\in\mathbb{N}$ such that $|a_{n_K}|\ge \|z\|_K$. Then for any $z\in K$ and $n\ge n_K$, we have $|\frac{z}{a_n}|\le 1$. By (b) we get for any $k_n\in\mathbb{N}$

$$\left|E_{k_n}\left(\frac{z}{a_n}\right)-1\right| \le \left|\frac{z}{a_n}\right|^{k_n+1}\le\left(\frac{\|z\|_K}{|a_n|}\right)^{k_n+1}$$

Since $\sum_{n\ge n_K}\left(\frac{\|z\|_K}{|a_n|}\right)^{k_n+1}<+\infty$, the series $\sum_{n\ge n_K}\left\|E_{k_n}(\frac{z}{a_n})-1\right\|_K$ converges. It follows that the infinite product $\prod_{n\ge0}E_{k_n}\left(\frac{z}{a_n}\right)$ converges normally in $\mathcal{O}(\mathbb{C})$ to a function having a zero at any a_n .

(d) Since $\lim_{n\to\infty}2^n=+\infty$ and $\sum_{n\ge0}\frac{1}{2^n}<+\infty$, we may chose $k_n=0$ for any $n\in\mathbb{N}$. As $E_0(z)=1-z$, it follows by (c) that $\prod_{n\ge0}\left(1-\frac{z}{2^n}\right)$ converges normally in $\mathcal{O}(\mathbb{C})$ to a function f satisfying

$$f(z)=(1-z)\prod_{n\ge1}\left(1-\frac{z}{2^n}\right) = (1-z)f\left(\frac{z}{2}\right)$$

(e) By induction, g must satisfy for any $n\in\mathbb{N}$

$$g(z)=\prod_{k=0}^{n}\left(1-\frac{z}{2^k}\right)g\left(\frac{z}{2^{n+1}}\right)$$

By taking $n \to +\infty$, we get

$$g(z) = \prod_{n\geq 0} \left(1 - \frac{z}{2^n}\right) g(0)$$

$\square$

Problem 27. *Set $E(z) = (1-z)e^z$ for any $z \in \mathbb{C}$.*

(a) *Show that $|E(z) - 1| \leq |z|^2$ for any $|z| \leq 1$.*

(b) *Show that the infinite products*

$$\prod_{n\geq 1} E\left(-\frac{z}{2n}\right) \quad \textit{and} \quad \prod_{n\geq 0} E\left(\frac{z}{2n+1}\right)$$

converge normally in $\mathcal{O}(\mathbb{C})$ to functions $a(z)$ and $b(z)$, respectively.

(c) *Show that for any $z \in \mathbb{C}$*

$$za(z)b(z) = 2^{z+1}\sqrt{\pi}\Delta\left(\frac{z}{2}\right)\Delta\left(\frac{1-z}{2}\right)$$

*where Δ is the Weierstress' Delta function. (*Hint*: use the duplication formula)*

(d) *Show that*

$$a(z)b(z) = 2^z \prod_{n\geq 1}\left(1 + (-1)^n \frac{z}{n}\right)$$

where the infinite product is convergent (not normally convergent!) in $\mathcal{O}(\mathbb{C})$. Express the value of $\prod_{n\geq 1}\left(1 + (-1)^n\frac{z}{n}\right)$ in term of the Gamma function.

Solution. (a) This is a particular case of **Problem** 4 (b) and of **Problem** [?] (b). Choosing for example the first method, we have for any $|z| \leq 1$

$$|E(z) - 1| = |e^z - ze^z - 1| = \left|\sum_{n\geq 2}\left(\frac{1}{n!} - \frac{1}{(n-1)!}\right) z^n\right| \leq \sum_{n\geq 2}\left(\frac{1}{(n-1)!} - \frac{1}{n!}\right)|z|^n = |z|^2,$$

as the series $\sum_{n\geq 2}\left(\frac{1}{(n-1)!} - \frac{1}{n!}\right)$ telescopes to 1.

(b) For a compact $K \subset \mathbb{C}$, take $n_K \in \mathbb{N}$ such that $2n_K > \|z\|_K$. Then for any $z \in K$ and $n \geq n_K$, we have $|\frac{z}{2n+1}| < |\frac{z}{2n}| < 1$, hence by (a)

$$\left|E\left(-\frac{z}{2n}\right) - 1\right| \leq \left|\frac{z}{2n}\right|^2 \leq \frac{\|z\|_K^2}{(2n)^2} \quad \text{and} \quad \left|E\left(\frac{z}{2n+1}\right) - 1\right| \leq \left|\frac{z}{2n+1}\right|^2 \leq \frac{\|z\|_K^2}{(2n+1)^2}$$

Since $\sum_{n\geq n_K}\frac{1}{n^2} < +\infty$, the series $\sum_{n\geq n_K}\left\|E(-\frac{z}{2n}) - 1\right\|_K$ and $\sum_{n\geq n_K}\left\|E(\frac{z}{2n+1}) - 1\right\|_K$ converge. It follows that the infinite products $\prod_{n\geq 1} E\left(-\frac{z}{2n}\right)$ and $\prod_{n\geq 0} E\left(\frac{z}{2n+1}\right)$ converge normally in $\mathcal{O}(\mathbb{C})$ to functions $a(z)$ and $b(z)$, respectively.

(c) Recall that $\Delta(z) = ze^{\gamma z}G(z)$ with γ the Euler-Mascheroni's constant and

$$G(z) = \prod_{n\geq 1}\left(1 + \frac{z}{n}\right)e^{-\frac{z}{n}}$$

The duplication formula for the Gamma function in terms of Δ reads as

$$\Delta(2z) = \frac{\sqrt{\pi}}{2^{2z-1}}\Delta\left(z + \frac{1}{2}\right)\Delta(z)$$

From this, one gets

$$\sqrt{\pi}2^{z+1}\Delta\left(-\frac{z}{2}+\frac{1}{2}\right)\Delta\left(\frac{z}{2}\right)=\frac{\Delta(-z)}{\Delta\left(-\frac{z}{2}\right)}\Delta\left(\frac{z}{2}\right)=\frac{zG(-z)}{G\left(-\frac{z}{2}\right)}G\left(\frac{z}{2}\right)=za(z)b(z)$$

(d) One has

$$a(z)b(z)=\lim_{k\longrightarrow+\infty}(1-z)\prod_{n=1}^{k}\left(1+\frac{z}{2n}\right)\left(1-\frac{z}{2n+1}\right)e^{z\left(\frac{1}{2n+1}-\frac{1}{2n}\right)}$$

$$=\lim_{k\longrightarrow+\infty}e^{z\sum_{n=1}^{2k+1}(-1)^n\frac{1}{n}}\prod_{n=1}^{2k+1}\left(1+(-1)^n\frac{z}{n}\right)=e^{z\log 2}\prod_{n\geq1}\left(1+(-1)^n\frac{z}{n}\right)$$

By (c), one has

$$z2^z\prod_{n\geq1}\left(1+(-1)^n\frac{z}{n}\right)=2^{z+1}\sqrt{\pi}\Delta\left(\frac{z}{2}\right)\Delta\left(\frac{1-z}{2}\right)$$

Recalling that $\Gamma(z)=\frac{1}{\Delta(z)}$, one finally gets

$$\prod_{n\geq1}\left(1+(-1)^n\frac{z}{n}\right)=\frac{2\sqrt{\pi}}{z\Gamma\left(\frac{z}{2}\right)\Gamma\left(\frac{1-z}{2}\right)}=\frac{\sqrt{\pi}}{\Gamma\left(1+\frac{z}{2}\right)\Gamma\left(\frac{1-z}{2}\right)}$$

$\square$

Problem 28. *Consider the infinite product*

$$\prod_{n\geq1}\left(\frac{n}{n+1}\right)^z\left(1+\frac{z}{n}\right)$$

(a) *Show that it converges normally in $\mathcal{O}(\mathbb{C})$ to a function $f(z)$.*
(b) *Show that $f(z)=\Delta(z+1)$, where Δ denotes the Weierstress' Delta function.*

Solution. (a) Set

$$f_n(z)=\left(\frac{n}{n+1}\right)^z\left(1+\frac{z}{n}\right)=e^{-z\log\left(1+\frac{1}{n}\right)}\left(1+\frac{z}{n}\right)$$

Then by the mean value theorem, for any $r>0$ and $z\in\overline{B_r(0)}$ we have

$$|f_n(z)-1|=|f_n(z)-f_n(0)|\leq\|f_n'\|_{[0,z]}\,|z|$$
$$\leq e^{r\log\left(1+\frac{1}{n}\right)}r\left\|\frac{1}{n}-\log\left(1+\frac{1}{n}\right)\left(1+\frac{z}{n}\right)\right\|_{\overline{B_r(0)}}$$

For $n\to+\infty$, we have $\log\left(1+\frac{1}{n}\right)=\frac{1}{n}-\frac{1}{2}\frac{1}{n^2}+o(\frac{1}{n^2})$, hence the right hand term is $\sim C_r\frac{1}{n^2}$ for a constant C_r depending on r. Since $\sum_{n\geq1}\frac{1}{n^2}<+\infty$, the series $\sum_{n\geq1}\|f_n(z)-1\|_{\overline{B_r(0)}}$ converges. It follows that the infinite product $\prod_{n\geq1}\left(\frac{n}{n+1}\right)^z\left(1+\frac{z}{n}\right)$ converges normally in $\mathcal{O}(\mathbb{C})$ to a function $f(z)$.

(b) Recall that $\Delta(z)=ze^{\gamma z}G(z)$ with $\gamma=\lim_{k\longrightarrow\infty}\left(\sum_{n=1}^{k}\frac{1}{n}-\log k\right)$ the Euler-Mascheroni's constant and

$$G(z)=\prod_{n\geq1}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}$$

Thanks to functional equation $\Delta(z) = z\Delta(z+1)$, it is enough to prove that $f(z) = e^{\gamma z}G(z)$. As

$$e^{-\frac{z}{n}} = e^{-\frac{z}{n}+z\log\left(1+\frac{1}{n}\right)}\left(1+\frac{1}{n}\right)^{-z} = e^{z\left[-\frac{1}{n}-\log n+\log(n+1)\right]}\left(\frac{n}{n+1}\right)^{z}$$

we get

$$\begin{aligned}
e^{\gamma z}G(z) &= \lim_{k\to+\infty} e^{\gamma z}\prod_{n\geq 1}^{k}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}} \\
&= \lim_{k\to+\infty} e^{z\left[\gamma-\sum_{n=1}^{k}\frac{1}{n}+\log(k+1)\right]}\prod_{n\geq 1}^{k}\left(1+\frac{z}{n}\right)\left(\frac{n}{n+1}\right)^{z} \\
&= \lim_{k\to+\infty} e^{z\left[\gamma-\left(\sum_{n=1}^{k}\frac{1}{n}-\log k\right)+\log\left(1+\frac{1}{k}\right)\right]}\prod_{n\geq 1}^{k}\left(1+\frac{z}{n}\right)\left(\frac{n}{n+1}\right)^{z} = f(z)
\end{aligned}$$

Note that one can also use Gauss' product formula

$$\Delta(z) = \lim_{n\to+\infty}\frac{z(z+1)\dots(z+n)}{n!n^z}$$

□

Problem 29. *Let $\Gamma(z)$ denote the Gamma function.*

(a) *Show that $\Gamma(z+n) \sim n^z\Gamma(n)$ for $n \to +\infty$ and any $z \in \mathbb{C}\setminus\mathbb{Z}_{\leq 0}$.*

(b) *Let $f \in \mathcal{O}(\{Rez > 0\})$ satisfy $f(z+1) = zf(z)$ and $f(z+n) \sim n^z f(n)$ for $n \to +\infty$. Show that $f(z) = f(1)\Gamma(z)$.*

Solution. 22 (a) For any $z \in \mathbb{C}\setminus\mathbb{Z}_{\leq 0}$, we have

$$\lim_{n\to+\infty}\frac{\Gamma(z+n)}{n^z\Gamma(n)} = \lim_{n\to+\infty}\frac{\Gamma(z+n+1)}{(n+1)^z\Gamma(n+1)} = \lim_{n\to+\infty}\frac{(z+n)\dots(z+1)z\Gamma(z)}{(1+\frac{1}{n})^z n^z n!} = 1$$

by the Gauss product representation

$$\frac{1}{\Gamma(z)} = \lim_{n\to\infty}\frac{z(z+1)\dots(z+n)}{n!n^z}.$$

(b) The functional equation $f(z+1) = zf(z)$ allows to analytically extend the function $g(z) = \frac{f(z)}{\Gamma(z)}$ to an entire function, also denoted by g, satisfying $g(z+1) = g(z)$. We have

$$\frac{g(z)}{g(1)} = \frac{g(z+n)}{g(n)} = \lim_{n\to\infty}\frac{g(z+n)}{g(n)} = \lim_{n\to\infty}\frac{f(z+n)}{n^z f(n)}\frac{n^z\Gamma(n)}{\Gamma(z+n)} = 1$$

by (a) and $\lim_{n\to+\infty}\frac{f(z+n)}{n^z f(n)} = 1$, hence g is constant, *i.e.* $\frac{f(z)}{\Gamma(z)} = f(1)$. □

Problem 30. (a) *Find all $\alpha \in \mathbb{C}$ for which $f_\alpha(z) = \int_0^{+\infty}(1-e^{-zt})t^{-\alpha-1}dt$ defines an holomorphic function on $\{Rez > 0\}$.*

(b) *For all $\alpha \in \mathbb{C}$ as in (a), compute explicitly $f'_\alpha(z)$. Derive the explicit value of $f_\alpha(z)$.*

(c) *For any $\alpha \in \mathbb{C}$ as in (a), prove that $f_\alpha(z) \sim -\frac{\Gamma(-\alpha)\Gamma(\alpha+z)}{\Gamma(z)}$ for $z \to \infty$. (*Hint*: use the Stirling formula.)*

Solution. (a) Take a compact $K \subset \{\mathrm{Re} z > 0\}$. For $t \to 0^+$ we have

$$\left\|(1-e^{-zt})t^{-\alpha-1}\right\|_K = \left\|1-e^{-zt}\right\|_K t^{-\mathrm{Re}\,\alpha-1} \sim \|z\|_K t^{-\mathrm{Re}\,\alpha}$$

hence $\int_0^1 \|(1-e^{-zt})t^{-\alpha-1}\|_K \, dt < +\infty$ if and only if Re $\alpha < 1$.

For $t \to +\infty$ we have

$$\left\|1-e^{-zt}\right\|_K t^{-\mathrm{Re}\,\alpha-1} \sim t^{-\mathrm{Re}\,\alpha-1}$$

hence $\int_1^{+\infty} \|(1-e^{-zt})t^{-\alpha-1}\|_K \, dt < +\infty$ if and only if Re $\alpha > 0$. From a known theorem (see [7, Lecture 10]), it follows that $f_\alpha(z)$ defines an holomorphic function on $\{\mathrm{Re} z > 0\}$ if and only if Re $\alpha \in]0,1[$.

(b) For Re $\alpha \in]0,1[$ and $z \in \{\mathrm{Re} z > 0\}$ we have

$$f'_\alpha(z) = \int_0^{+\infty} e^{-zt}t^{-\alpha}dt = \mathcal{L}(t^{-\alpha})(z) = z^{\alpha-1}\Gamma(1-\alpha)$$

where $\mathcal{L}$ stands for the Laplace transform (for the computation, see [7, Lecture 23]). It follows that

$$f_\alpha(z) = \int_{[1,z]} f'_\alpha(w)dw + f_\alpha(1) = -\Gamma(-\alpha)z^\alpha + \Gamma(-\alpha) + f_\alpha(1)$$

By a direct computation, we get $f_\alpha(1) = \int_0^{+\infty}(1-e^{-t})t^{-\alpha-1}dt = -\Gamma(-\alpha)$, hence $f_\alpha(z) = -\Gamma(-\alpha)z^\alpha$.

(c) It is enough to prove that

$$\lim_{z\to\infty} \frac{z^\alpha \Gamma(z)}{\Gamma(z+\alpha)} = 1$$

By the Stirling formula (see [5, XIV.2]), we have $\Gamma(z) = \sqrt{2\pi}z^{z-\frac{1}{2}}e^{-z}e^{\mu(z)}$ with $\lim_{z\to\infty}\mu(z) = 0$. It follows that for $z \to \infty$

$$\frac{z^\alpha\sqrt{2\pi}z^{z-\frac{1}{2}}e^{-z}e^{\mu(z)}}{\sqrt{2\pi}(z+\alpha)^{z+\alpha-\frac{1}{2}}e^{-(z+\alpha)}e^{\mu(z+\alpha)}} \sim \frac{z^\alpha z^{z-\frac{1}{2}}}{(z+\alpha)^{z+\alpha-\frac{1}{2}}e^{-\alpha}} \sim \left(1+\frac{\alpha}{z}\right)^{-z}\left(1+\frac{\alpha}{z}\right)^{\frac{1}{2}-\alpha} e^\alpha \sim 1$$

□

Problem 31. *Let $f \in \mathcal{C}^\infty(\mathbb{R})$ be a Schwartz function, that is,* $\sup_{t\in\mathbb{R}} |t^m f^{(l)}(t)| < +\infty$ *for any $m, l \in \mathbb{N}$.*

(a) *Show that $\int_0^{+\infty} f(t)t^{z-1}dt$ defines an holomorphic function $\Gamma_f(z)$ on $\{Rez > 0\}$.*
(b) *Prove that $\Gamma_f(z)$ extends analytically to a meromorphic function on $\mathbb{C}$ with at most simple poles.(*Hint: *integrate by parts.)*
(c) *Assume that f extends to an holomorphic function on a disc $B_r(0)$ for some $r > 1$. Find an explicit partial fraction decomposition for $\Gamma_f(z)$ on $\mathbb{C}$.*
(d) *Let $\Delta(z)$ denote the Weierstrass' Delta function. Show that $\Delta(z)\Gamma_f(z)$ extends to an entire function and compute its value at any $k \in \mathbb{Z}_{\leq 0}$.*
(e) *Set $f(t) = e^{-t^2}$. Prove that $\Gamma_f(2z) = \frac{2^{2z}}{\sqrt{\pi}}\Gamma_f(z)\Gamma_f(z+1)$.*

Solution. (a) For a compact $K \subset \{\mathrm{Re}z > 0\}$, set $\rho_K = \max_K \mathrm{Re}z > 0$ and $\mu_K = \min_K \mathrm{Re}z > 0$. Then for any $t > 0$, we have

$$\left\| f(t)t^{z-1} \right\|_K = |f(t)| \max_{z\in K} t^{\mathrm{Re}z-1} = \begin{cases} |f(t)|t^{\rho_K-1} & \text{if } t > 1 \\ |f(t)|t^{\mu_K-1} & \text{if } 0 < t \leq 1 \end{cases}$$

$$\leq m_K(t) = \begin{cases} \sup_{t\in\mathbb{R}} \left(|f(t)|t^{[\rho_K]+2}\right) t^{-2} = Ct^{-2} & \text{if } t > 1 \\ \sup_{t\in\mathbb{R}} \left(|f(t)|\right) t^{\mu_K-1} = Dt^{\mu_K-1} & \text{if } 0 < t \leq 1. \end{cases}$$

(here $[\cdot]$ denotes the integral part, and C, D positive constant). Since $f(t)t^{z-1}$ is continuous in $\mathbb{R}_{>0} \times \{\mathrm{Re}z > 0\}$ and holomorphic in z, and $\int_0^{+\infty} m_K(t)\, dt < +\infty$, it follows that the given integral defines an holomorphic function $\Gamma_f(z)$ on $\{\mathrm{Re}z > 0\}$.

(b) Note first that for any $m, l \in \mathbb{N}$ and $t > 0$, $\mathrm{Re}z > -n$ we have

$$|f^{(l)}(t)t^{z+n}| = |f^{(l)}(t)|t^{\mathrm{Re}z+n} \leq \begin{cases} \sup_{t\in\mathbb{R}} \left(|f^{(l)}(t)|t^{[\mathrm{Re}z]+n+2}\right) t^{-1} \to 0 & \text{as } t \to +\infty \\ \sup_{t\in\mathbb{R}} \left(|f^{(l)}(t)|\right) t^{\mathrm{Re}z+n} \to 0 & \text{as } t \to 0+ \end{cases}$$

By integrating by parts, we get

$$\Gamma_f(z) = \int_0^{+\infty} f(t)t^{z-1}dt = \frac{1}{z}f(t)t^z|_0^{+\infty} - \frac{1}{z}\int_0^{+\infty} f'(t)t^z dt = -\frac{1}{z}\int_0^{+\infty} f'(t)t^z dt$$

and the right-hand side defines an holomorphic function on $\{\mathrm{Re}z > -1\} \setminus \{0\}$. By iterating, we get

$$\Gamma_f(z) = \frac{(-1)^n}{z(z+1)\dots(z+n)} \int_0^{+\infty} f^{(n+1)}(t)t^{z+n}dt$$

which defines an holomorphic function on $\{\mathrm{Re}z > -n-1\} \setminus \{0, -1, \dots, -n\}$. This allows to analytically extend Γ_f to a meromorphic function on $\mathbb{C}$ with at most simple poles at any $k \in \mathbb{Z}_{\leq 0}$.

(c) From the computation in (a), it follows that $G_f(z) = \int_1^{+\infty} f(t)t^{z-1}dt$ is entire, as $m_K(t)$ does not depend on K for $t > 1$. As f extends to an holomorphic function on a disc $B_r(0)$ with $r > 1$, we may consider its Taylor series $\sum_{n\geq 0} \frac{f^{(n)}(0)}{n!}t^n$at $t = 0$. Then we get the following partial fraction decomposition

$$\begin{aligned} \Gamma_f(z) &= \int_0^1 f(t)t^{z-1}dt + G_f(z) \\ &= \int_0^1 \sum_{n\geq 0} \frac{f^{(n)}(0)}{n!} t^{z+n-1}dt + G_f(z) \\ &= \sum_{n\geq 0} \frac{f^{(n)}(0)}{n!} \int_0^1 t^{z+n-1}dt + G_f(z) = \sum_{n\geq 0} \frac{f^{(n)}(0)}{n!}\frac{1}{z+n} + G_f(z). \end{aligned}$$

where we have used that the series is normally convergent in $[0, 1]$.

(d) Take $k = -n \in \mathbb{Z}_{\leq 0}$. Then $\Gamma_f(z)$ has at most a simple pole at $-n$ by (b), whereas $\Delta(-n) = 0$. It follows that $-n$ is a removable singularity for $\Delta(z)\Gamma_f(z)$. Moreover

$$\lim_{z\longrightarrow -n} \Delta(z)\Gamma_f(z) = \lim_{z\longrightarrow -n} \frac{\Gamma_f(z)}{\Gamma(z)} = \frac{res_{-n}\Gamma_f(z)}{res_{-n}\Gamma(z)} = \frac{\frac{f^{(n)}(0)}{n!}}{\frac{(-1)^n}{n!}} = (-1)^n f^{(n)}(0)$$

where the residue of $\Gamma_f(z)$ at $-n$ follows from (c), or from (b) by a direct computation.

(e) One easily checks that $f(t) = e^{-t^2}$ is a Schwartz function, and one has

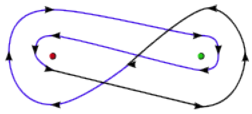
$$\Gamma_f(2z) = \int_0^{+\infty} e^{-t^2} t^{2z-1} dt = \frac{1}{2}\int_0^{+\infty} e^{-s} s^{z-1} ds = \frac{1}{2}\Gamma(z)$$

by change of variable $t = \sqrt{s}$. Then the formula to prove follows directly form Legendre's duplication formula

$$\Gamma\left(\frac{z}{2}\right)\Gamma\left(\frac{z+1}{2}\right) = \frac{\sqrt{\pi}}{2^{z-1}}\Gamma(z).$$

□

Problem 32. *Let γ the path in $\mathbb{C}\setminus\{0,1\}$ drawn below*



(the left dot corresponds to 0 and the right one to 1). Set $f(\zeta, z, w) = \zeta^{z-1}(1-\zeta)^{w-1}$.

(a) *Prove that $f(\cdot, z, w)$ defines a continuos function on $|\gamma|$ for any $z, w \in \mathbb{C}$. Deduce that*

$$F(z, w) = \int_\gamma f(\zeta, z, w)\, d\zeta$$

defines an entire function in each argument.

(b) *Prove that*

$$\Gamma(z+w)F(z,w) = (1 - e^{2\pi i z})\Gamma(z)(1 - e^{2\pi i w})\Gamma(w)$$

*as meromorphic functions on $\mathbb{C}$. (*Hint*: prove the equality for $Rez, Rew > 0$.)*

Solution. (a) Choose arguments for ζ and $1-\zeta$, and take $P \in |\gamma|$, say $P = \gamma(a) = \gamma(b)$ for a parametrization $\gamma\colon [a,b] \to \mathbb{C}$. As $f(\zeta, z, w)$ changes by a factor $e^{2\pi i z}$ (resp. $e^{2\pi i w}$) when $\arg\zeta$ (resp. $\arg(1-\zeta)$) is increased by 2π, and the path circled 0 and 1 twice in opposite directions, the value of $f(\gamma(b), z, w)$ is the same as that of $f(\gamma(a), z, w)$. Hence $f(\cdot, z, w)$ defines a continuous function on $|\gamma|$ for any $z, w \in \mathbb{C}$.

Since the integral exists for any $z, w \in \mathbb{C}$ and $f(\zeta, z, w)$ is holomorphic both in z and w, by a theorem on functions defined by integral along a path (see [7, Lecture 10]), $F(z, w)$ is entire on each argument.

(b) For $\mathrm{Re}z, \mathrm{Re}w > 0$, we have

$$\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} = B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}dt;$$

by the identity principle is thus enough to prove that

$$F(z,w) = (1 - e^{2\pi i z})(1 - e^{2\pi i w})\int_0^1 t^{z-1}(1-t)^{w-1}dt$$

for $\mathrm{Re}z, \mathrm{Re}w > 0$. As f defines an holomorphic function on $\mathbb{C}$ deprived by two half-lines (depending on the choice of the arguments) of initial points 0 and 1, we may deform γ to a path made by four straight line segments γ_i, $i = 1,2,3,4$ along the real axis, two circles δ_i, $i = 1,2$ around 0 and two circles δ_i, $i = 3,4$ around 1, without changing the integral.

Since $\mathrm{Re}z, \mathrm{Re}w > 0$, for any $i = 1,2,3,4$ we have

$$\left|\int_{\delta_i} \zeta^{z-1}(1-\zeta)^{w-1}\, d\zeta\right| \le \int_{\delta_i} |\zeta|^{\mathrm{Re}z-1}|1-\zeta|^{\mathrm{Re}w-1}\, d|\zeta| \to 0$$

as the radius of δ_i tends to 0. Whence, by letting the length of the γ_i's tend to 1 and by taking into account the appearing factors $e^{\pm 2\pi iz}$ and $e^{\pm 2\pi iw}$, we get

$$\begin{aligned}F(z,w) &= \int_{\gamma_1} f(\zeta,z,w)\,d\zeta + e^{2\pi iw}\int_{\gamma_2} f(\zeta,z,w)\,d\zeta \\ &\quad + e^{2\pi iz}e^{2\pi iw}\int_{\gamma_3} f(\zeta,z,w)\,d\zeta + e^{2\pi iz}\int_{\gamma_4} f(\zeta,z,w)\,d\zeta \\ &= \int_0^1 t^{z-1}(1-t)^{w-1}dt - e^{2\pi iw}\int_0^1 t^{z-1}(1-t)^{w-1}dt \\ &\quad + e^{2\pi iz}e^{2\pi iw}\int_0^1 t^{z-1}(1-t)^{w-1}dt - e^{2\pi iz}\int_0^1 t^{z-1}(1-t)^{w-1}dt \\ &=(1-e^{2\pi iz})(1-e^{2\pi iw})\int_0^1 t^{z-1}(1-t)^{w-1}dt.\end{aligned}$$

□

Problem 33. *Consider the double sided series $\sum_{n\in\mathbb{Z}}\frac{1}{(z+n)^k}$ for $k\in\mathbb{N}$.*

(a) *Prove that for any $k\geq 2$, the series converges normally to a meromorphic function $\epsilon_k(z)$ on $\mathbb{C}$, periodic of period 1 and having a pole of order k at any $n\in\mathbb{Z}$.*

(b) *Show that $\epsilon_k(z)=\frac{(-1)^k}{(k-1)!}\partial_z^{(k-2)}\frac{\pi^2}{\sin^2(\pi z)}$ for any $k\geq 2$.*

(c) *Show that $\epsilon_k(z)=\frac{(-1)^k}{(k-1)!}(2\pi i)^k\sum_{n\geq 1}n^{k-1}e^{2\pi inz}$ in $\{Im\ z>0\}$, with normal convergence. (*Hint: *one has $\pi\cot(\pi z)=\pi i\frac{e^{2\pi iz}+1}{e^{2\pi iz}-1}=\dots$)*

(d) *Show that $\epsilon_k(z)=\frac{1}{z^k}+(-1)^k 2\sum_{2n\geq k}\binom{2n-1}{k-1}\zeta(2n)z^{2n-k}$ for $|z|<1$.*

(e) *Show that $(n+\frac{1}{2})\zeta(2n)=\sum_{k+l=n}\zeta(2k)\zeta(2l)$ for any $n\geq 2$. (*Hint: *use $\frac{\pi^2}{\sin^2(\pi z)}=\pi^2(\cot^2(\pi z)+1)$.)*

Solution. (a) For any compact $K\subset\mathbb{C}$, set $m_K=\|z\|_K$. Then for any $k\geq 1$ and any $n>m_K$, one has

$$\left\|\frac{1}{(z\pm n)^k}\right\|_K=\sup_{z\in K}\frac{1}{|z\pm n|^k}\leq\sup_{z\in K}\frac{1}{||z|-n|^k}\leq\frac{1}{(n-m_K)^k}.$$

with $\frac{1}{(z\pm n)^k}$ having no pole in K. Since $\sum_{n>m_K}\frac{1}{(n-m_K)^k}<+\infty$ for any $k\geq 2$, it follows that the series

$$\sum_{n\in\mathbb{Z}}\frac{1}{(z+n)^k}=\sum_{n\geq 0}\frac{1}{(z+n)^k}+\sum_{n\geq 1}\frac{1}{(z-n)^k}$$

converges normally to a meromorphic function $\epsilon_k(z)$ on $\mathbb{C}$, which has a pole of order k at any $n\in\mathbb{Z}$. One has $\epsilon_k(z+1)=\sum_{n\in\mathbb{Z}}\frac{1}{(z+1+n)^k}=\epsilon_k(z)$ by rearranging the terms.

(b) Since the series $\sum_{n\in\mathbb{Z}}\frac{1}{(z+n)^k}$ is normally convergent we may derive term by term, hence $\partial_z\epsilon_k(z)=-k\epsilon_{k+1}(z)$ for any $k\geq 2$, so that $\epsilon_k(z)=\frac{(-1)^k}{(k-1)!}\partial_z^{(k-2)}\epsilon_2(z)$. It remains to prove that $\epsilon_2(z)=\frac{\pi^2}{\sin^2(\pi z)}$. Fort this, it is enough to apply differentiation to both sides of the Mittag-Leffler's partial fractions decomposition

$$\pi\cot(\pi z)=\frac{1}{z}+\sum_{n\in\mathbb{Z}\setminus\{0\}}\left[\frac{1}{z+n}+\frac{1}{n}\right]$$

(the series being normally convergent).

(c) For $z \in \{\mathrm{Im}\ z > 0\}$, we have $|e^{2\pi iz}| = e^{-2\pi \mathrm{Im}\ z} < 1$, hence

$$\pi\cot(\pi z) = \pi i \frac{e^{2\pi iz}+1}{e^{2\pi iz}-1} = \pi i\left(1 - \frac{2}{1-e^{2\pi iz}}\right) = \pi i - 2\pi i \sum_{n\geq 0} e^{2\pi inz}$$

(with normal convergence). By applying differentiation to both sides, we get $\frac{\pi^2}{\sin^2(\pi z)} = (2\pi i)^2 \sum_{n\geq 1} ne^{2\pi inz}$ and we conclude by using (b).

(d) Since $z = 0$ is the unique pole of $\epsilon_k(z)$ in $\mathbb{E}$, and it is of order k with principal part $\frac{1}{z^k}$, the holomorphic function $\epsilon_k(z) - \frac{1}{z^k}$ admits a Taylor series in $\mathbb{E}$ centered at $z = 0$, *i.e.* $\epsilon_k(z) = \frac{1}{z^k} + \sum_{m\geq 0} a_m z^m$ for $|z| < 1$ (with normal convergence). We have

$$a_m = \frac{1}{m!}\partial_z^m\left[\epsilon_k(z) - \frac{1}{z^k}\right]_{z=0} = \frac{1}{m!}\sum_{l\in\mathbb{Z},l\neq 0}\partial_z^m\left[\frac{1}{(z+l)^k}\right]_{z=0} = (-1)^m \sum_{l\in\mathbb{Z},l\neq 0}\binom{k-1+m}{k-1}\frac{1}{l^{k+m}}$$

(the series being normally convergent). If $k+m$ is odd, the terms $\pm\frac{1}{l^{k+m}}$ of the double series cancel out, so that we may suppose $k+m = 2n$ for $n \geq 1$. It follows that

$$a_{2n-k} = (-1)^k 2\sum_{l\geq 1}\binom{k-1+m}{k-1}\frac{1}{l^{2n}} = (-1)^k 2\binom{2n-1}{k-1}\zeta(2n)$$

for any $2n \geq k$.

Note that one may also use (b) and apply differentiation to both sides of $\pi\cot(\pi z) = \frac{1}{z} - 2\sum_{n\geq 1}\zeta(2n)z^{2n-1}$ for $|z| < 1$.

(e) From (b), we get $\epsilon_2(z) = \frac{\pi^2}{\sin^2(\pi z)} = \pi^2(\cot^2(\pi z)+1)$. Let $z \in \mathbb{E}$. By (d) we have

$$\frac{\pi^2}{\sin^2(\pi z)} = \frac{1}{z^2} + 2\sum_{n\geq 1}(2n-1)\zeta(2n)z^{2n-2}$$

and from the Taylor expansion $\pi\cot(\pi z) = \frac{1}{z} - 2\sum_{n\geq 1}\zeta(2n)z^{2n-1}$ and by using Cauchy's product formula for series, we get

$$\pi^2 + \pi^2\cot^2(\pi z) = \pi^2 + \frac{1}{z^2} - 4\sum_{n\geq 1}\zeta(2n)z^{2n-2} + 4\sum_{n\geq 2}\sum_{k+l=n}\zeta(2k)\zeta(2l)z^{2n-2}.$$

By comparing the coefficients of the same power of z, for any $n \geq 2$ we get

$$2(2n-1)\zeta(2n) = -4\zeta(2n) + 4\sum_{k+l=n}\zeta(2k)\zeta(2l)$$

hence the result. Note that, by comparing the coefficients of the zero-th power of z, we recover also $\zeta(2) = \frac{\pi^2}{6}$ □

Problem 34. *Let $\mathbb{E} = \{|z| < 1\}$ denote the unit disk, $\zeta(z)$ the Riemann Zeta function and γ the Euler-Mascheroni constant.*

(a) *Show that the series $\sum_{n=1}^{\infty}\left(\frac{1}{z+n} - \frac{1}{n}\right)$ converges normally in $\mathcal{M}(\mathbb{C})$ to a function $f(z)$ having a simple pole at any $k \in \mathbb{Z}_{\leq -1}$ with $res_k f = 1$.*

(b) *Prove that $f(z) = \sum_{m=1}^{\infty}(-1)^m\zeta(m+1)z^m$ for any $z \in \mathbb{E}$.*

(c) *Set $g(z) = f(z) + \frac{1}{z}$. Prove that $g(z+1) = f(z)$ and $g(z) - g(1-z) = \pi\cot(\pi z)$.*

(d) *For any $z \in \mathbb{C}\setminus\mathbb{R}_{\leq 0}$, set $G(z) = \int_{[1,z]} g(\xi)d\xi$, where $[1,z]$ denotes the oriented line segment joining 1 with z. Show that $G(z) = \log z - \sum_{n=1}^{\infty}\left(\frac{z}{n} - \log(1+\frac{z}{n})\right) + \gamma$.*

(e) *Show that* $G(z+1) = -\sum_{m=2}^{\infty} \frac{(-1)^m}{m}\zeta(m)z^m$ *for any* $z \in \mathbb{E}$. *(*Hint: *use (b) and (c).)*

(f) *Prove that* $l(z) = \gamma - \gamma z - G(z)$ *is a logarithm of* $\Gamma(z)$ *on* $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

(g) *Prove that the series* $\sum_{m=2}^{\infty} \frac{(-1)^m}{m}\zeta(m)$ *is convergent. Then compute its sum. (*Hint: *compute* $\lim_{x \to 1^-} G(x+1)$.*)*

Solution. (a) For any compact $K \subset \mathbb{C}$, set $m_K = \|z\|_K$. Then $\frac{1}{z+n} - \frac{1}{n}$ has no pole in K for any $n > m_K$, and the series $\sum_{n > m_K} \left\|\frac{1}{z+n} - \frac{1}{n}\right\|_K$ is convergent, since for $z \in K$ we have

$$\left|\frac{1}{z+n} - \frac{1}{n}\right| = \left|\frac{z}{n(z+n)}\right| \leq \frac{|z|}{n\,||z|-n|} \leq \frac{m_K}{n(n-m_K)} \quad \text{for any } n > m_K$$

and $\sum_{n > m_K} \frac{m_K}{n(n-m_K)} < +\infty$.

It follows that the series $\sum_{n=1}^{\infty}\left(\frac{1}{z+n} - \frac{1}{n}\right)$ converges normally in $\mathcal{M}(\mathbb{C})$ to a meromorphic function $f(z)$ having a simple pole at any $k \in \mathbb{Z}_{\leq -1}$ with

$$\operatorname{res}_k f = \lim_{z \to k}(z-k)f(z) = \lim_{z \to k}\left(1 - \frac{(z-k)}{-k} + (z-k)\sum_{n \geq 1, n \neq -k}\left(\frac{1}{z+n} - \frac{1}{n}\right)\right) = 1.$$

(b) For any $|z| < n$, we have

$$\frac{1}{z+n} - \frac{1}{n} = \frac{1}{n}\left(\frac{1}{1+\frac{z}{n}} - 1\right) = \sum_{m \geq 1}\frac{(-1)^m}{n}\left(\frac{z}{n}\right)^m,$$

hence, by normal convergence, for any $|z| < 1$ we get

$$\sum_{n \geq 1}\left(\frac{1}{z+n} - \frac{1}{n}\right) = \sum_{n \geq 1}\sum_{m \geq 1}\frac{(-1)^m}{n}\left(\frac{z}{n}\right)^m = \sum_{m \geq 1}(-1)^m z^m \sum_{n \geq 1}\frac{1}{n^{m+1}} = \sum_{m \geq 1}(-1)^m \zeta(m+1) z^m$$

(c) By rearranging the terms, we get

$$g(z+1) = \frac{1}{z+1} + \sum_{n \geq 1}\left(\frac{1}{z+1+n} - \frac{1}{n}\right) = \frac{1}{z+1} - 1 + \sum_{n \geq 2}\left(\frac{1}{z+n} - \frac{1}{n}\right) = f(z)$$

and

$$\begin{aligned} g(z) - g(1-z) &= \frac{1}{z} + f(z) - f(-z) \\ &= \frac{1}{z} + \sum_{n \geq 1}\left(\frac{1}{z+n} - \frac{1}{n}\right) - \sum_{n \geq 1}\left(\frac{1}{-z+n} - \frac{1}{n}\right) \\ &= \frac{1}{z} + \sum_{n \geq 1}\left(\frac{1}{z+n} + \frac{1}{z-n}\right) = \pi\cot(\pi z) \end{aligned}$$

(d) Since the series is normally convergent, we can integrate term by term, and we get

$$\begin{aligned}
G(z) &= \int_{[1,z]} \left(\frac{1}{\xi} + \sum_{n\geq 1} \left(\frac{1}{\xi+n} - \frac{1}{n} \right) \right) d\xi \\
&= \int_{[1,z]} \frac{1}{\xi}\, d\xi + \sum_{n\geq 1} \int_{[1,z]} \left(\frac{1}{\xi+n} - \frac{1}{n} \right) d\xi \\
&= \log z - \log 1 + \sum_{n\geq 1} \left(\log(z+n) - \frac{z}{n} - \log(1+n) + \frac{1}{n} \right) \\
&= \log z - \sum_{n\geq 1} \left(\frac{z}{n} - \log\left(1 + \frac{z}{n}\right) \right) + \gamma.
\end{aligned}$$

Here we used that $\log(z+n) = \log\left(1+\frac{z}{n}\right) + \log n$, as both z and $\frac{z}{n}$ lie in $\mathbb{C}\setminus\mathbb{R}_{\leq 0}$ for any $n \in \mathbb{N}_{\geq 1}$ and any $z \in \mathbb{C}\setminus\mathbb{R}_{\leq 0}$, and that

$$\sum_{n=1}^{\infty} \left(\log n - \log(1+n) + \frac{1}{n} \right) = \lim_{n\to+\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log(1+n) \right) = \gamma.$$

(e) Since $G(1) = 0$ and for any $z \in \mathbb{E}$

$$(G(z+1))' = g(z+1) = f(z) = \sum_{m\geq 1} (-1)^m \zeta(m+1) z^m,$$

we get for any $z \in \mathbb{E}$

$$G(z+1) = -\sum_{m\geq 2} \frac{(-1)^m}{m} \zeta(m) z^m.$$

(f) By (d), we have $l(z) = -\gamma z - \log z + \sum_{n=1}^{\infty} \left(\frac{z}{n} - \log(1+\frac{z}{n})\right)$, hence for any $z \in \mathbb{C}\setminus\mathbb{R}_{\leq 0}$

$$e^{l(z)} = e^{-\gamma z} z^{-1} \prod_{n\geq 1} e^{\frac{z}{n}} \left(1 + \frac{z}{n}\right)^{-1} = \frac{1}{\Delta(z)} = \Gamma(z).$$

(g) The series $\sum_{m=2}^{\infty} \frac{(-1)^m}{m} \zeta(m)$ is convergent by the Leibniz criterion, since the sequence $\{\frac{\zeta(m)}{m}\}_{m\geq 1} \subset \mathbb{R}_{>0}$ is decreasing, as $\zeta'(x) < 0$ for $x > 1$, and infinitesimal, as $\lim_{m\to+\infty} \zeta(m) = 1$.

By (e) and (f), we get

$$\sum_{m\geq 2} \frac{(-1)^m}{m} \zeta(m) = -\lim_{x\to 1-} G(x+1) = -G(2) = l(2) - \gamma + 2\gamma = \gamma$$

since $l(2) = \log \Gamma(2) = \log 1 = 0$. $\square$

Problem 35. *Let $\{t\}$ denote the fractional part of $t \in \mathbb{R}$.*

(a) *Prove that $f(z) = \int_0^1 \{s^{-1}\} s^z\, ds$ defines an holomorphic function on $\{Re z > -1\}$.*

(b) *Show that there exist a rational function $R(z)$ and $\lambda \in \mathbb{C}^\times$ such that $(z+1)f(z) = R(z) + \lambda\zeta(z+1)$ in $\{Re z > -1\}$ (here $\zeta(z)$ denotes the Riemann Zeta function).*

(c) *Compute $f(n)$ for $n \in \mathbb{N}$ odd.*

Solution. (a) Since $|\{s^{-1}\}| \leq 1$ for any $s \neq 0$, for any compact $K \subset \{\mathrm{Re} z > -1\}$ and any $z \in K$, we have

$$\left|\{s^{-1}\}s^z\right| \leq s^{\mathrm{Re} z} \leq \frac{1}{s^{-\rho}}$$

with $\rho = \max_K \mathrm{Re} z > -1$ and $\int_0^1 t^{-\rho}\, dt < +\infty$. Then $\int_0^1 \{s^{-1}\}s^z\, ds$ defines an holomorphic function in $\mathcal{O}(\{\mathrm{Re} z > -1\})$ (the function $\{s^{-1}\}s^z$ being locally bounded on $]0,1] \times \{\mathrm{Re} z > -1\}$).

(b) Since $\{s^{-1}\} = s^{-1} - [s^{-1}]$ ($[s^{-1}]$ is the integral part of s^{-1}) and by setting $t = s^{-1}$, in $\{\mathrm{Re} z > 0\}$ one gets

$$(z+1)f(z) = (z+1)\int_0^1 s^{z-1}\, ds - (z+1)\int_0^1 [s^{-1}]s^z\, ds = \frac{z+1}{z} - (z+1)\int_1^{+\infty} [t]t^{-z-2}\, dt$$

with

$$(z+1)\int_1^{+\infty} [t]t^{-z-2}\, dt = \mathcal{M}([t])(z+1) = \zeta(z+1)$$

where $\mathcal{M}([t])$ denotes the Mellin transform of $[t]$ and ζ the Riemann Zeta function (see [7, Lecture 22]). It follows that

$$(z+1)f(z) = 1 + \frac{1}{z} - \zeta(z+1) \quad \text{in } \{\mathrm{Re} z > 0\}.$$

Since ζ is a meromorphic function on $\mathbb{C}$ with a unique pole in $z = 1$, which is simple with residue 1, the function $\frac{1}{z} - \zeta(z+1)$ extends analytically on $\mathbb{C}$, and by the identity principle the above equality holds on $\{\mathrm{Re} z > -1\}$.

(c) From the above computation, for any $k \in \mathbb{N}_{\geq 1}$ we have

$$f(k) = \frac{1}{k} - \frac{\zeta(k+1)}{k+1}$$

hence, recalling Euler's identities, for $k = 2n+1$ one gets

$$f(2n+1) = \frac{1}{2n+1} - \frac{\zeta(2(n+1))}{2(n+1)} = \frac{1}{2n+1} + \frac{(-1)^{n+1}}{2(n+1)}\frac{(2\pi)^{2(n+1)}}{2(2(n+1))!}B_{2(n+1)}$$

where $B_{2(n+1)}$ denotes the $2(n+1)$-th Bernoulli's number. $\square$

Problem 36. *Let $\zeta(z)$ denote the Riemann Zeta function.*

(a) *Prove that $(1-2^{1-z})\zeta(z)$ extends to an entire function $\eta(z)$.*

(b) *Show that $\eta(z) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n^z}$ for Re$z > 1$.*

(c) *Let Re$z > 1$. Show that $\eta(z)$ is the Mellin transform of $f(t) = \frac{1+(-1)^{[t]-1}}{2}$. (Here $[t]$ denotes the integral part of t and the Mellin transform of f is defined as $\mathcal{M}f(z) = z\int_1^{+\infty} f(t)t^{-z-1}\, dt$.)*

(d) *Show that the equality in (b) holds for Re$z > 0$. (*Hint*: prove that $\sum_{n\geq 1} \frac{(-1)^{n-1}}{n^z}$ converges in $\mathcal{O}(\{\mathrm{Re} z > 0\})$.)*

Solution. (a) Recall that $\zeta(z)$ is meromorphic with a unique pole in $z = 1$, which is simple. As the function $1 - 2^{1-z} = 1 - e^{(1-z)\log 2}$ is holomorphic on $\mathbb{C}$ with a zero at $z = 1$, it follows that $(1-2^{1-z})\zeta(z)$ extends to an entire function.

(b) First, note that the series is normally convergent in $\mathcal{O}(\{\mathrm{Re} z > 1\})$ and recall that $\zeta(z) = \sum_{n\geq 1} \frac{1}{n^z}$ on $\mathrm{Re} z > 1$. Hence

$$(1-2^{1-z})\zeta(z) = \sum_{n\geq 1} \frac{1}{n^z} - \sum_{n\geq 1} \frac{1}{(2n)^z} - \sum_{n\geq 1} \frac{1}{(2n)^z} = \sum_{n\geq 1} \frac{1}{(2n-1)^z} - \sum_{n\geq 1} \frac{1}{(2n)^z} = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n^z}$$

(c) From (b) we have $\eta(z) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n^z}$ for $\mathrm{Re} z > 1$. As $(-1)^{n-1} = f(n) - f(n-1)$ and $f(0) = 0$, by Abel's summation formula we get

$$\begin{aligned}
\sum_{n=1}^{k} \frac{(-1)^{n-1}}{n^z} &= \frac{f(k)}{k^z} + \sum_{n=1}^{k-1} f(n) \left(\frac{1}{n^z} - \frac{1}{(n+1)^z} \right) \\
&= \frac{f(k)}{k^z} + \sum_{n=1}^{k-1} z \int_n^{n+1} f(t) t^{-z-1} \, dt \\
&= \frac{f(k)}{k^z} + z \int_1^k f(t) t^{-z-1} \, dt.
\end{aligned}$$

Since $f(t)$ is bounded, it follows that $\frac{f(k)}{k^z} \to 0$ in $\mathcal{O}(\{\mathrm{Re} z > 1\})$ as $k \to +\infty$, and $\eta(z) = \mathcal{M}f(z)$.

(d) For any compact $K \subset \{\mathrm{Re} z > 0\}$ and any $z \in K$ we have

$$\left| f(t) t^{-z-1} \right| \leq t^{\mathrm{Re} z - 1} \leq t^{-\rho-1}$$

with $\rho = \min_K \mathrm{Re} z > 0$ and $\int_1^{+\infty} t^{-\rho-1} \, dt < +\infty$. Then $z \int_1^{+\infty} f(t) t^{-z-1} \, dt$ converges normally in $\mathcal{O}(\{\mathrm{Re} z > 0\})$.

Since $\frac{f(k)}{k^z} \to 0$ in $\mathcal{O}(\{\mathrm{Re} z > 0\})$, we get that $\sum_{n\geq 1} \frac{(-1)^{n-1}}{n^z}$ is convergent (but not normally convergent!) in $\mathcal{O}(\{\mathrm{Re} z > 0\})$.

The equality in (b) then follows by the identity principle.

□

Problem 37. *Let μ denote the Möbius μ-function: $\mu(1) = 1$ and, for $n > 1$, $\mu(n) = (-1)^r$ if n is a product of r distinct primes, $\mu(n) = 0$ otherwise.*

(a) *Prove that $\sum_{n\geq 1} \frac{\mu(n)}{n^z}$ defines an holomorphic function $\eta(z)$ on $\{\mathrm{Re} z > 1\}$.*

(b) *Prove that $\eta(z)\zeta(z) = 1$ on $\{\mathrm{Re} z > 1\}$, where $\zeta(z)$ denotes the Riemann Zeta function.* (Hint*: use Euler's product formula.)*

Solution. (a) For any compact $K \subset \{\mathrm{Re} z > 1\}$ and any $z \in K$, we have

$$\left| \frac{\mu(n)}{n^z} \right| \leq \frac{1}{|n^z|} = \frac{1}{n^{\mathrm{Re} z}} \leq \frac{1}{n^\rho},$$

with $\rho = \min_K \mathrm{Re} z > 1$. Hence the series converges normally in $\mathcal{O}(\{\mathrm{Re} z > 1\})$.

(b) By Euler's product formula, we have to show that

$$\prod_{p\in\mathbb{P}} (1 - p^{-z}) = \sum_{n\geq 1} \frac{\mu(n)}{n^z}$$

on $\{\mathrm{Re} z > 1\}$, where $\mathbb{P}$ denotes the set of prime numbers.

Note that, for any prime p, we have

$$1 - p^{-z} = \frac{\mu(1)}{1^z} + \frac{\mu(p)}{p^z} = \sum_{n=p^k, k\geq 0} \frac{\mu(n)}{n^z}$$

We will thus proceed by induction. Let $F \subset \mathbb{P}$ be finite and suppose that

$$\prod_{p\in F}(1-p^{-z}) = \sum_{n\in\mathbb{N}_F} \frac{\mu(n)}{n^z},$$

where $\mathbb{N}_F = \{p_1^{k_1} \dots p_l^{k_l};\, p_i \in F, k_i \geq 0 \text{ for any } i = 1, \dots, l\} \subset \mathbb{N}$.

Let $q \in \mathbb{P} \setminus F$. Then

$$\prod_{p\in F\cup\{q\}} (1-p^{-z}) = (1-q^{-z}) \prod_{p\in F}(1-p^{-z}) = (1-q^{-z}) \sum_{n\in\mathbb{N}_F} \frac{\mu(n)}{n^z} = \sum_{n\in\mathbb{N}_F} \frac{\mu(n)}{n^z} - \sum_{n\in\mathbb{N}_F} \frac{\mu(n)}{(qn)^z}.$$

As $\mu(nq^k) = 0$ for $k > 1$ and $\mu(nq) = \mu(n)\mu(q) = -\mu(n)$ for any $n \in \mathbb{N}$, we get

$$\sum_{n\in\mathbb{N}_{F\cup\{q\}}} \frac{\mu(n)}{n^z} = \sum_{n\in\mathbb{N}_F} \frac{\mu(n)}{n^z} + \sum_{n\in\mathbb{N}_F} \frac{\mu(nq)}{(nq)^z} = \sum_{n\in\mathbb{N}_F} \frac{\mu(n)}{n^z} - \sum_{n\in\mathbb{N}_F} \frac{\mu(n)}{(nq)^z}.$$

We thus proved that

$$\prod_{p\in F\cup\{q\}} (1-p^{-z}) = \sum_{n\in\mathbb{N}_{F\cup\{q\}}} \frac{\mu(n)}{n^z}.$$

By induction, we get that

$$\prod_{p\in F}(1-p^{-z}) = \sum_{n\in\mathbb{N}_F} \frac{\mu(n)}{n^z},$$

for any $F \subset \mathbb{P}$ finite. By passing to the limit we get the result, as $\mathbb{N}_{\mathbb{P}} = \mathbb{N}$ by the fundamental theorem of arithmetic. □

Problem 38. *Let $\mathbb{P}$ denote the set of prime numbers and* log *the principal branch of the logarithm.*

(a) *Show that $|\log(1-z)+z| \leq |z|^2$ for any $|z| \leq \frac{1}{2}$.*
(b) *Let $\{b_p\}_{p\in\mathbb{P}} \subset \mathbb{C}$ be a sequence such that the series $\sum_{p\in\mathbb{P}} b_p$ and $\sum_{p\in\mathbb{P}} |b_p|^2$ are convergent. Prove that the infinite product $\Pi_{p\in\mathbb{P}}(1-b_p)$ converges. (*Hint: *use (a).)*
(c) *Prove that $\sum_{p\in\mathbb{P}} \frac{1}{p} = +\infty$. (*Hint: *use (b)).*
(d) *Show that $\sum_{p\in\mathbb{P}} \frac{1}{p^z}$ defines an holomorphic function $\zeta_{\mathbb{P}}(z)$ on $\{Rez > 1\}$.*
(e) *Prove that $\zeta_{\mathbb{P}}(z)$ is the Mellin transform of $\pi(t) = \#\{p \in \mathbb{P};\, p \leq t\}$. (Recall that the Mellin transform of $f(t)$ is defined as $\mathcal{M}f(z) = z\int_1^{+\infty} f(t)t^{-z-1}\,dt$.)*
(f) *Prove that $\zeta_{\mathbb{P}}(z) - \log\zeta(z)$ extends analytically to a neighbourhood of $\{Rez \geq 1\}$. (*Hint: *use Euler's product formula and (a).)*
(g) *Show that $\zeta_{\mathbb{P}}(z) + \log(z-1)$ extends analytically to a neighbourhood of $\{Rez \geq 1\}$.*

Solution. (a) For any $|z| < 1$, we have

$$|\log(1-z)+z| = \left|-\sum_{n\geq 1} \frac{z^n}{n} + z\right| \leq \sum_{n\geq 2} \frac{|z|^n}{n} \leq \frac{|z|^2}{2} \sum_{n\geq 0} |z|^n = \frac{|z|^2}{2} \frac{1}{1-|z|}.$$

If $|z| \leq \frac{1}{2}$, then $\frac{1}{1-|z|} \leq 2$ and we get the result.

(b) As $\mathbb{P}$ is countable, all results for series and infinite products still hold when the indexes belong to $\mathbb{P}$. In particular, $\Pi_{p\in\mathbb{P}}(1-b_p)$ is convergent iff so is $\sum_{p\in\mathbb{P}}\log(1-b_p)$ (providing $b_p\neq 1$ for $p>>0$). Since $\sum_{p\in\mathbb{P}} b_p$ is convergent, $b_p\to 0$ as $p\to+\infty$, hence there exists $\bar{p}\in\mathbb{P}$ such that $|b_p|\leq\frac{1}{2}$ for $p\geq\bar{p}$ and

$$|\log(1-b_p)+b_p|\leq|b_p|^2$$

by (a). Since $\sum_{p\in\mathbb{P}}|b_p|^2$ and $\sum_{p\in\mathbb{P}} b_p$ are convergent, so are $\sum_{p\in\mathbb{P}}(\log(1-b_p)+b_p)$ and $\sum_{p\in\mathbb{P}}\log(1-b_p)=\sum_{p\in\mathbb{P}}(\log(1-b_p)+b_p)-\sum_{p\in\mathbb{P}} b_p$.

(c) Assume that $\sum_{p\in\mathbb{P}}\frac{1}{p}<+\infty$. Then $\Pi_{p\in\mathbb{P}}(1-\frac{1}{p})$ would be convergent by (b), as $\sum_{p\in\mathbb{P}}\frac{1}{p^2}\leq\sum_{n\geq 1}\frac{1}{n^2}<+\infty$. By Euler's product formula, we have

$$\Pi_{p\in\mathbb{P}}\left(1-\frac{1}{p}\right)=\lim_{z\longrightarrow 1}\Pi_{p\in\mathbb{P}}\left(1-\frac{1}{p^z}\right)=\lim_{z\longrightarrow 1}\frac{1}{\zeta(z)}=0$$

since $z=1$ is a pole of ζ. Contradiction.

(d) For any compact $K\subset\{\mathrm{Re}z>1\}$ and any $z\in K$, we have

$$\sum_{p\in\mathbb{P}}\left|\frac{1}{p^z}\right|=\sum_{p\in\mathbb{P}}\frac{1}{p^{\mathrm{Re}z}}\leq\sum_{n\geq 1}\frac{1}{n^{\mathrm{Re}z}}<+\infty,$$

hence $\sum_{p\in\mathbb{P}}\frac{1}{p^z}$ converges normally in $\mathcal{O}(\{\mathrm{Re}z>1\})$ to an holomorphic function $\zeta_{\mathbb{P}}(z)$.

(e) As $\mathbb{P}$ is countable, we may denote by p_n the n-th prime ($n\geq 1$). By Abel's summation formula we get

$$\begin{aligned}\sum_{n=1}^{k}\frac{1}{p_n^z}=\sum_{n=1}^{k}\frac{n-(n-1)}{p_n^z}&=\frac{k}{p_{k+1}^z}+\sum_{n=1}^{k}\left(\frac{n}{p_n^z}-\frac{n}{p_{n+1}^z}\right)\\&=\frac{k}{p_{k+1}^z}+\sum_{n=1}^{k}z\int_{p_n}^{p_{n+1}}nt^{-z-1}\,dt\\&=\frac{k}{p_{k+1}^z}+z\int_{1}^{p_{k+1}}\pi(t)t^{-z-1}\,dt,\end{aligned}$$

as $\pi(t)=\begin{cases}0 & \text{if } t\in[1,2[\\ n & \text{if } t\in[p_n,p_{n+1}[\end{cases}$. Since $p_k>k$ for any $k\geq 1$, it follows that $\left|\frac{k}{p_{k+1}^z}\right|<\frac{k}{(k+1)^{\mathrm{Re}\,z}}\to 0$ as $k\to+\infty$, and $\zeta_{\mathbb{P}}(z)=\mathcal{M}\pi(t)$.

(f) By Euler's product formula, in $\{\mathrm{Re}z>1\}$ we have

$$\zeta_{\mathbb{P}}(z)-\log\zeta(z)=\sum_{p\in\mathbb{P}}p^{-z}+\sum_{p\in\mathbb{P}}\log\left(1-p^{-z}\right)=\sum_{p\in\mathbb{P}}\left(p^{-z}+\log\left(1-p^{-z}\right)\right)$$

It is enough to show that the right-hand side is normally convergent in a neighbourhood of $\{\mathrm{Re}z\geq 1\}$. Since $|p^{-z}|=\frac{1}{p^{\mathrm{Re}z}}\leq\frac{1}{2}$ in $\{\mathrm{Re}z\geq 1\}$ for all $p\in\mathbb{P}$ for any compact $K\subset\{\mathrm{Re}z\geq 1\}$ and any $z\in K$ we have by (a) and (d)

$$\left|\sum_{p\in\mathbb{P}}p^{-z}+\log\left(1-p^{-z}\right)\right|\leq\sum_{p\in\mathbb{P}}\left|p^{-z}+\log\left(1-p^{-z}\right)\right|\leq\sum_{p\in\mathbb{P}}\left|p^{-z}\right|^2=\sum_{p\in\mathbb{P}}\frac{1}{p^{2\mathrm{Re}z}}<+\infty$$

(g) Recall that $\zeta(z)$ has a unique pole in $z=1$, which is simple, and never vanishes in $\{\mathrm{Re}z\geq 1\}$. It follows that $(z-1)\zeta(z)$ is holomorphic never-vanishing in a neighbourhood of $\{\mathrm{Re}z \geq 1\}$, hence $\log((z-1)\zeta(z)) = \log(z-1) + \log\zeta(z)$ is holomorphic there, and so is $\zeta_{\mathbb{P}}(z) + \log(z-1) = \zeta_{\mathbb{P}}(z) - \log\zeta(z) + \log(z-1) + \log\zeta(z)$ by (f). $\square$

Problem 39. *Let $\zeta(z)$ denote the Riemann Zeta function, $\Delta(z)$ the Weierstrass' Delta function, and B_m the m-th Bernoulli's number.*

(a) *Prove that $\int_0^1 \frac{t^{z-1}}{e^t-1}dt = \sum_{n\geq 0} \frac{B_n}{n!(z+n-1)}$ for $\mathrm{Re}z > 1$.*
(b) *Show that $\int_1^{+\infty} \frac{t^{z-1}}{e^t-1}dt$ defines an entire holomorphic function.*
(c) *Show that $\int_0^{+\infty} \frac{t^{z-1}}{e^t-1}dt$ defines an holomorphic function on $\{\mathrm{Re}z > 1\}$, which extends analitically to a meromorphic function $F(z)$ on $\mathbb{C}$.*
(d) *Show that $\Delta(z)F(z)$ has a unique pole at $z=1$, which is simple.*
(e) *Prove that $\Delta(z)F(z) = \zeta(z)$. (*Hint: *use the change $t = \frac{s}{n}$.)*

Solution. Recall that $\frac{w}{e^w-1}$ has a removable singularity at $w=0$ and has Taylor series at 0

$$\frac{w}{e^w-1} = \sum_{n\geq 0} \frac{B_n}{n!} w^n \qquad \text{for } |w| < 2\pi.$$

(a) First note that for $\mathrm{Re}z > 1$, the integral is absolutely convergent, as

$$\left|\frac{t^{z-1}}{e^t-1}\right| = \frac{t^{\mathrm{Re}z-1}}{e^t-1} \sim t^{\mathrm{Re}z-2} \qquad \text{as } t \to 0+.$$

We have

$$\frac{t^{z-1}}{e^t-1} = \sum_{n\geq 0} \frac{B_n}{n!} t^{z+n-2} \qquad \text{with normal convergence in } [0,1],$$

hence

$$\int_0^1 \frac{t^{z-1}}{e^t-1}dt = \lim_{\epsilon\longrightarrow 0} \int_\epsilon^1 \sum_{n\geq 0} \frac{B_n}{n!} t^{z+n-2}dt = \sum_{n\geq 0} \frac{B_n}{n!} \lim_{\epsilon\longrightarrow 0} \int_\epsilon^1 t^{z+n-2}dt = \sum_{n\geq 0} \frac{B_n}{n!(z+n-1)}$$

for $\mathrm{Re}z > 1$, as $\lim_{\epsilon\longrightarrow 0} \epsilon^{z+n-1}$ for any $n \geq 0$.

(b) For a compact $K \subset \mathbb{C}$, set $\rho_K = \max_K \mathrm{Re}z \geq 0$. Then for any $z \in K$ and $t \geq 1$, we have

$$\left|\frac{t^{z-1}}{e^t-1}\right| = \frac{t^{\mathrm{Re}z-1}}{e^t-1} \leq \frac{t^{\rho_K-1}}{e^t-1} = o(t^{-2}) \qquad \text{as } t \to +\infty$$

hence $\int_1^{+\infty} \frac{t^{\rho_K-1}}{e^t-1}\, dt < +\infty$. Since $\frac{t^{z-1}}{e^t-1}$ is continuos in $\mathbb{R}_{\geq 1} \times \mathbb{C}$ and holomorphic in z, it follows that the given integral defines an entire holomorphic function.

(c) By (b), it is enough to prove that

$$\int_0^1 \frac{t^{z-1}}{e^t-1}dt$$

defines an holomorphic function on $\{\mathrm{Re}z > 1\}$.

For a compact $K \subset \{\mathrm{Re}z > 1\}$, set $\mu_K = \min_K \mathrm{Re}z > 1$. Then for any $z \in K$ and $t \in]0,1]$, we have

$$\left|\frac{t^{z-1}}{e^t-1}\right| = \frac{t^{\mathrm{Re}z-1}}{e^t-1} \leq \frac{t^{\mu_K-1}}{e^t-1} \sim t^{\mu_K-2} \qquad \text{as } t \to 0+,$$

hence $\int_0^1 \frac{t^{\mu_K-1}}{e^t-1}\, dt < +\infty$. Since $\frac{t^{z-1}}{e^t-1}$ is continuos in $]0,1] \times \{\mathrm{Re}z > 1\}$ and holomorphic in z, the result then follows.

By (a), in $\{\mathrm{Re} z > 1\}$ we have

$$\int_0^{+\infty} \frac{t^{z-1}}{e^t - 1} dt = \sum_{n \geq 0} \frac{B_n}{n!(z+n-1)} + \int_1^{+\infty} \frac{t^{z-1}}{e^t - 1} dt.$$

It remains to prove that the series $\sum_{n\geq 0} \frac{B_n}{n!(z+n-1)}$ is normally convergent in $\mathcal{M}(\mathbb{C})$ to a meromorphic function on $\mathbb{C}$.

For any compact $K \subset \mathbb{C}$, set $m_K = \|z\|_K$. Then $\frac{1}{(z+n-1)}$ has no pole on K for any $n > m_K + 1$ and the series $\sum_{n>m_K+1} \left\| \frac{B_n}{n!(z+n-1)} \right\|_K$ is convergent, since for $z \in K$ we have

$$\left| \frac{B_n}{n!(z+n-1)} \right| \leq \frac{|B_n|}{n!} \frac{1}{||z| - (n-1)|} \leq \frac{|B_n|}{n!} \frac{1}{(n-1-m_K)} \leq \frac{|B_n|}{n!}$$

and $\sum_{n>m_K+1} \frac{|B_n|}{n!} \leq \sum_{n\geq 0} \frac{|B_n|}{n!} t^n|_{t=1} < +\infty$.

(d) By (c), we have

$$F(z) = \sum_{n\geq 0} \frac{B_n}{n!(z+n-1)} + \int_1^{+\infty} \frac{t^{z-1}}{e^t-1} dt \quad \in \mathcal{M}(\mathbb{C}),$$

hence $F(z)$ has a simple pole at any $k \in \mathbb{Z}_{\leq 1}$. Since $\Delta(z)$ is entire and it has a simple zero at any $k \in \mathbb{Z}_{\leq 0}$, the meromorphic function $\Delta(z)F(z)$ has removable singularities at any $k \in \mathbb{Z}_{\leq 0}$ and a unique pole at $z = 1$, which is simple.

(e) Since both sides of the equation are meromorphic on $\mathbb{C}$, by the identity principle it is enough to prove it for $\mathrm{Re} z > 1$. In this case we have

$$\begin{aligned}
\Delta(z)F(z) &= \Delta(z) \int_0^{+\infty} t^{z-1} e^{-t} \frac{1}{1-e^{-t}} dt \\
&= \Delta(z) \int_0^{+\infty} t^{z-1} \sum_{n\geq 1} e^{-tn} dt \\
&= \Delta(z) \sum_{n\geq 1} \int_0^{+\infty} t^{z-1} e^{-tn} dt \\
&= \Delta(z) \sum_{n\geq 1} \frac{1}{n^z} \int_0^{+\infty} t^{z-1} e^{-t} dt = \Delta(z)\Gamma(z)\zeta(z) = \zeta(z)
\end{aligned}$$

where we have used that the series is normally convergent in $\mathbb{R}_{>0}$. □

Formulæ

Pompeiu's formula: $$f(z)=\frac{1}{2\pi i}\left[\int_{\partial K}\frac{f(w)}{w-z}dw+\int_K\frac{\partial_{\bar w}f(w)}{w-z}dw\wedge d\bar w\right]\quad\text{for}\begin{cases}f\in\mathcal{C}^1(\Omega),\\ \Omega\underset{\text{open}}{\subset}\mathbb{C},\\ z\in\overset{\circ}{K}\subset K\subset\Omega,\\ K\underset{\text{compact}}{\subset}\mathbb{C}\end{cases}$$

Euler's Gamma function: $$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt\quad\text{for }\mathrm{Re}z>0$$

Euler's Beta function: $$B(z,w)=\int_0^1 t^{z-1}(1-t)^{w-1}dt=\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}\quad\text{for }\mathrm{Re}z,\mathrm{Re}w>0$$

Euler's supplement: $$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin(\pi z)}$$

Legendre's duplication formula: $$\Gamma\left(\frac{z}{2}\right)\Gamma\left(\frac{z+1}{2}\right)=\frac{\sqrt{\pi}}{2^{z-1}}\Gamma(z)$$

Multiplication formula: $$\Gamma(mz)=\frac{m^{mz}}{(\sqrt{2\pi})^{m-1}\sqrt{m}}\prod_{k=0}^{m-1}\Gamma(z+k/m)\quad\text{for }m\geq 2$$

Weierstrass' Delta function: $$\Delta(z)=ze^{\gamma z}\prod_{n\geq 1}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}\quad\text{with }\gamma=\lim_{n\to\infty}\left(\sum_{k=1}^n\frac{1}{k}-\log n\right)$$

Gauss' product representation: $$\frac{1}{\Gamma(z)}=\Delta(z)=\lim_{n\to\infty}\frac{z(z+1)\dots(z+n)}{n!n^z}$$

Mittag-Leffler's partial fractions decompositions:

$$\pi\cot(\pi z)=p.v.\sum_{n\in\mathbb{Z}}\frac{1}{z-n}=\frac{1}{z}+\sum_{n\in\mathbb{Z}\setminus\{0\}}\left[\frac{1}{z-n}+\frac{1}{n}\right]=\frac{1}{z}+\sum_{n\geq 1}\frac{2z}{z^2-n^2}=\sum_{n\in\mathbb{Z}}\frac{z}{z^2-n^2}$$

$$\frac{\pi}{\sin(\pi z)}=p.v.\sum_{n\in\mathbb{Z}}\frac{(-1)^n}{z-n}=\frac{1}{z}+\sum_{n\in\mathbb{Z}\setminus\{0\}}(-1)^n\left[\frac{1}{z-n}+\frac{1}{n}\right]=\frac{1}{z}+\sum_{n\geq 1}\frac{(-1)^n 2z}{z^2-n^2}=\sum_{n\in\mathbb{Z}}\frac{(-1)^n z}{z^2-n^2}$$

Sine product: $$\frac{\sin(\pi z)}{\pi}=z\prod_{n\geq 1}\left(1-\frac{z^2}{n^2}\right)$$

Bernoulli's numbers: $$\frac{z}{e^z-1}=\sum_{n\geq 0}\frac{B_n}{n!}z^n\quad\text{for }|z|<2\pi$$

$$B_0=1,\ B_1=-\frac{1}{2},\ B_{2n+1}=0\text{ for }n\geq 1,\ \sum_{k=0}^n\binom{n+1}{k}B_k=0$$

$$\text{Euler's identities:} \qquad \zeta(2n) = (-1)^{n+1}\frac{(2\pi)^{2n}}{2(2n)!}B_{2n} \text{ for } m \geq 1$$

$$\text{Euler's product formula:} \qquad \zeta(z) = \prod_{p\,:\,\text{prime}} \frac{1}{1-p^{-z}} \qquad \text{for Re}z > 1$$

$$\text{Riemann's integral representation:} \qquad \zeta(z) = -\frac{\Gamma(1-z)}{2\pi i}\int_\gamma \frac{(-w)^{z-1}}{e^w - 1}dw$$

$$\text{Riemann's relation:} \qquad \zeta(z) = \left[2(2\pi)^{z-1}\sin(\frac{\pi z}{2})\Gamma(1-z)\right]\zeta(1-z)$$

References

[1] R. B. Ash, W. P. Novinger, *Complex Variables. Second Edition*, Dover Books on Mathematics (2007).
[2] J. Bak, D.J.F. Newman, *Complex Analysis. Third Edition*, UTM, Springer-Verlag, Berlin (2010).
[3] G. De Marco, *Basic Complex Analysis*, Università di Padova, self-published (2011).
[4] ———, *Selected Topics of Complex Analysis*, Università di Padova, self-published (2012).
[5] T. W. Gamelin, *Complex Analysis*, UTM, Springer-Verlag, Berlin (2001).
[6] E. Freitag, R. Busam, *Complex Analysis. Second Edition*, Universitext, Springer-Verlag, Berlin (2009).
[7] P. Polesello, *Complex Analysis*, Master degree in Mathematics Lecures Notes (2019).
[8] R. Remmert, *Theory of Complex Functions*, Graduate Texts in Mathematics, Springer-Verlag, Berlin (1991).
[9] ———, *Classical Topics in Complex Function Theory*, Graduate Texts in Mathematics, Springer-Verlag, Berlin (1991).
[10] W. Rudin, *Real and Complex Analysis. Third edition*, McGraw-Hill (1986).
[11] J.-P. Schneiders, *Fonctions de Variables Complexes*, Universitè de Liége, self-published (2010).

Dipartimento di Matematica "Tullio Levi-Civita"; Università di Padova; Via Trieste, 63; 35121 Padova; Italy

E-mail address: pietro@math.unipd.it