Basic complex analysis
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1. Complex differentiation

1.1. Complex derivative. The field of complex numbers has a topology compatible with the field structure, i.e. the operations which make it into a field are continuous; this allows us to define a derivative for complex valued functions of one complex variable. Assume that $D \subseteq \mathbb{C}$ and $f : D \to \mathbb{C}$ is a function. Given $z \in D$, which we also assume to be interior to D, for every $w \in D$ different from z we can define the *difference quotient*:

$$\frac{f(w) - f(z)}{w - z} \qquad w \in D \setminus \{z\}$$

(here the field structure of $\mathbb{C}$ is used); we can next try to take the limit as w tends to z; if this limit exists in $\mathbb{C}$, we call it *derivative* (in the complex sense) of f at z, and we denote it $f'(z)$.

Definition. Let $D \subseteq \mathbb{C}$, $f : D \to \mathbb{C}$ a mapping, z in the interior of D. We say that f has a derivative at z, in the complex sense, if the limit :

$$\lim_{w \to z} \frac{f(w) - f(z)}{w - z} \quad \text{exists in } \mathbb{C}.$$

In that case the limit is denoted $f'(z)$, and is called *derivative of f at z*, in the complex sense.

It is easy to see that many elementary facts on real derivatives carry over to complex derivatives: e.g. constants have zero derivative , the identity map $z \mapsto z$ has constant derivative 1, $z \mapsto z^2$ has $2z$ as derivative, for any positive integer m the power $z \mapsto z^m$ has derivative $m\, z^{m-1}$; derivation is linear, Leibniz rule on the derivative of products and quotients holds, as well as the chain rule for the derivative of a composite map: all the proof are strictly analogous to the real case. The complex polynomial $\sum_{n=0}^{m} a_n\, z^n$ has then as complex derivative the polynomial $\sum_{n=1}^{m} n a_n\, z^{n-1}$; the reciprocal map $z \mapsto z^{-1} = 1/z$ has $z \mapsto -1/z^2$ as derivative on $\mathbb{C} \setminus \{0\}$. It is also easy to see, as in the real case, that if f has a derivative at z in the complex sense then it is continuous at z (write $f(w) - f(z) = (w - z)\,(f(w - f(z)/(w - z))$ and recall that the limit of the product is the product of the limits). In the real case the function $x \mapsto |x|$ is the simplest example of a function continuous but without derivative at $x = 0$ (and only for $x = 0$). Clearly then the mapping $z \mapsto |z|$ lacks a derivative at $z = 0$ in the complex sense, too: existence of the limit $\lim_{w \to 0,\, w \in \mathbb{C}} |w|/w$ implies existence of the limit $\lim_{w \to 0,\, w \in \mathbb{R}} |w|/w$, which we know nonexistent, being 1 on the right and -1 on the left. But on the complex field the absolute value has *nowhere* a derivative (see later); here is a first difference of behavior. It is most interesting to observe that an extremely simple function as conjugation, $z \mapsto \bar{z} = \operatorname{Re} z - i \operatorname{Im} z$, has *nowhere* a derivative in the complex sense, at no $z \in \mathbb{C}$. In fact

$$\frac{\bar{w} - \bar{z}}{w - z} = \frac{\overline{w - z}}{w - z}.$$

If $h = w - z = r\, e^{i\vartheta}$, the preceding limit as $w \to z$ exists iff $\bar{h}/h = e^{-2i\vartheta}$ has limit in $\mathbb{C}$ as $r \to 0^+$; but as ϑ varies in $\mathbb{R}$ (or in $]-\pi, \pi]$) the quantity $e^{-2i\vartheta}$, not dependent on r, takes on all complex values of the unit circle $\mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}$.

Example 1.1.0.1. It easily follows that $z \mapsto |z|^2$ admits a derivative in the complex sense only at $z = 0$: since $|z|^2 = z\,\bar{z}$, if its derivative existed at $z \neq 0$, then also conjugation, which is this map divided by z, would have a derivative at z. Then also $z \mapsto |z|$ has not a derivative at $z \neq 0$: otherwise its square admits a derivative.

1.2. Complex and real differentials. Recall that a $\mathbb{C}-$linear space is trivially also an $\mathbb{R}-$linear space, simply by restricting the scalar field to $\mathbb{R}$. And if X, Y are $\mathbb{C}-$vector spaces, and $T : X \to Y$ is a $\mathbb{C}-$ linear map, plainly T is also an $\mathbb{R}-$linear map between the $\mathbb{R}-$spaces X, Y; but not always an $\mathbb{R}-$linear map is also $\mathbb{C}-$linear: e.g. the maps $z \mapsto \bar{z}$ (conjugation), $z \mapsto \operatorname{Re} z$ (real part) $z \mapsto \operatorname{Im} z$ (imaginary part) are $\mathbb{R}-$linear, but not $\mathbb{C}-$linear , as self–maps of the $\mathbb{C}-$linear space $\mathbb{C}$. As a $\mathbb{C}-$linear space on itself $\mathbb{C}$ is of course one–dimensional, and every non–zero complex number is a base for it; the *standard base* is 1, and every linear self–map of $\mathbb{C}$ is of course the multiplication by a given complex number $\alpha \in \mathbb{C}$: in fact if $T : \mathbb{C} \to \mathbb{C}$ is $\mathbb{C}-$linear then

$$T(z) = T(z\,1) = z\,T(1) = z\,\alpha, \quad \text{with } \alpha = T(1).$$

To sum up: $\operatorname{End}_{\mathbb{C}}(\mathbb{C}) := \operatorname{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C})$ is isomorphic to $\mathbb{C}$ (this holds in general for every field, considered as a vector space over itself) and the nonzero $\mathbb{C}-$linear self–maps of $\mathbb{C}$ are exactly the *rotohomoteties* of the plane (with positive determinant, see below).

As an $\mathbb{R}-$vector space $\mathbb{C}$ is $2-$dimensional, and among the possible bases $\{1, i\}$ is the standard one; to every $T \in \operatorname{End}_{\mathbb{R}}(\mathbb{C})$ we may then associate a 2×2 real matrix:

$$\begin{bmatrix} T(1) & T(i) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{where} \quad T(1) = a_{11} + i\,a_{21},\, T(i) = a_{12} + i\,a_{22}$$

(the columns are the components of the vectors $T(1)$, $T(i) \in \mathbb{C}$ in the choosen basis $\{1, i\}$). The $\mathbb{C}-$linear operators are $T(z) = z(a + ib)$, where $\alpha = a + ib$, and $a, b \in \mathbb{R}$; since $T(1) = a + ib$ and $T(i) = i(a + ib) = -b + ia$, the matrices which correspond to $\mathbb{C}-$linear maps are those of the form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \qquad \text{equivalently,} \quad a_{11} = a_{22},\, a_{12} = -a_{21}.$$

Notice that if $T(z) = \alpha\,z$ is $\mathbb{C}-$linear, as an $\mathbb{R}$ linear map we have $\det T = a^2 + b^2 = |\alpha|^2$

If X and Y are normed $\mathbb{C}-$linear spaces, $D \subseteq X$, $f : D \to Y$ is a map, and z is in the interior of D, we say that f is *differentiable at z, in the complex sense*, if there exists a $\mathbb{C}-$linear continuous map $T : X \to Y$such that:

$$\lim_{w \to z,\, w \in D} \frac{f(w) - f(z) - T(w - z)}{\|w - z\|_X} = 0;$$

and f is said to be differentiable at z in *the real sense* if the same holds, but with T supposed to be $\mathbb{R}-$linear only, and not necessarily $\mathbb{C}-$linear. Specializing to the case $X = \mathbb{C} = Y$ we have (exactly as in the case of real–valued functions of one real variable):

Proposition. *Let $D \subseteq \mathbb{C}$, $f : D \to \mathbb{C}$ be a function, and let z be in the interior of D. Then f has a complex derivative at z if and only if it is differentiable at z in the complex sense.*

Proof. Given $\alpha \in \mathbb{C}$, if $T(h) = \alpha\,h$ for every $h \in \mathbb{C}$, T is $\mathbb{C}-$linear self–map of $\mathbb{C}$; and

$$\lim_{w \to z,\, w \in D} \frac{f(w) - f(z) - T(w - z)}{|w - z|} = 0 \quad \text{if and only if} \quad \lim_{w \to z} \frac{f(w) - f(z)}{w - z} = \alpha,$$

trivially: simply observe that

$$\lim\nolimits_{w \to z} \left| \frac{f(w) - f(z) - \alpha\,(w - z)}{w - z} \right| = 0 \iff \lim_{w \to z} \left| \frac{f(w) - f(z)}{w - z} - \alpha \right| = 0 \iff$$
$$\iff \lim_{w \to z} \frac{f(w) - f(z)}{w - z} = \alpha.$$

□

1.2.1. *The Cauchy–Riemann identities.* The linear map T in the preceding definition is the differential of f at z; we know that the differential, if it exists, is uniquely determined (its values on vectors of X are the derivatives of f at z, in the real sense, along these vectors). Thus

Proposition. *A mapping between normed linear $\mathbb{C}-$spaces is differentiable in the complex sense at an interior point of its domain if and only if it is differentiable in the real sense at the same point, and its differential is $\mathbb{C}-$linear.*

Specializing to the case $X = Y = \mathbb{C}$ we get (cfr. 10.1.3):

Proposition. *Let $D \subseteq \mathbb{C}$, let $f : D \to \mathbb{C}$ be a function, with $f = u + i\,v$ and u, v real valued, and let $x + iy$ belong to the interior of D. Then f is $\mathbb{C}-$differentiable at z if and only if u, v are differentiable (in the real sense) at (x, y) and in this point the Cauchy–Riemann identities*

$$\partial_x u(x,y) = \partial_y v(x,y), \quad \partial_y u(x,y) = -\partial_x v(x,y)$$

hold.

Proof. Plainly f is $\mathbb{R}-$differentiable at $x+iy$ if and only if so are u, v at (x, y) (when $\mathbb{C}$ is interpreted as an $\mathbb{R}-$linear space f is a vector valued function with components u, v). The matrix of its $\mathbb{R}-$differential, in the standard basis $\{1, i\}$, is the *jacobian matrix*:

$$\begin{bmatrix} \partial_x u(x,y) & \partial_y u(x,y) \\ \partial_x v(x,y) & \partial_y v(x,y) \end{bmatrix},$$

which, as above seen, represents a $\mathbb{C}-$linear endomorphism of $\mathbb{C}$ if and only if $\partial_x u(x,y) = \partial_y v(x,y)$, $\partial_y u(x,y) = -\partial_x v(x,y)$, exactly the Cauchy–Riemann identities. $\square$

Remark. Notice that the complex derivative is exactly the partial derivative with respect to $x = \operatorname{Re} z$, equivalently to the vector $1 \in \mathbb{C}$: if $f'(z)$ exists we have

$$f'(x+iy) = \partial_x f(x+iy) = \partial_x u(x,y) + i\,\partial_x v(x,y),$$

whereas ($y = \operatorname{Im} z$ second variable):

$$f'(x+iy) = \frac{1}{i}\,\partial_y f(x+iy) = \partial_y v(x,y) - i\,\partial_y u(x,y);$$

of course, by definition, $\partial_y = \partial_i$; in general, if the complex derivative $f'(z)$ exists then at z all derivatives (in the real sense) along every $w \in \mathbb{C}$ exist, and $\partial_w f(z) = f'(z)\,w$:

$$\partial_w f(z) := \lim_{t\to 0,\, t\in\mathbb{R}} \frac{f(z+tw) - f(z)}{t} = \lim_{t\to 0,\, t\in\mathbb{R}} w\left(\frac{f(z+tw) - f(z)}{w\,t}\right) = w\,f'(z).$$

Notice also that, always assuming that the complex derivative $f'(z)$ exists, the real jacobian matrix of f at z has as determinant:

$$\partial_x u(x,y)\partial_y v(x,y) - \partial_x v(x,y)\partial_x u(x,y) = (\partial_x u(x,y))^2 + (\partial_x v(x,y))^2 = |f'(z)|^2,$$

square of the absolute value of the complex derivative at z.

1.2.2. *A very important function: the principal logarithm.* The complex exponential $z \mapsto \exp z = e^z$ induces a continuous bijection of the half–open strip $\{z \in \mathbb{C} : -\pi < \operatorname{Im} z \le \pi\}$ onto the punctured plane $\mathbb{C}_* = \mathbb{C} \setminus \{0\}$. The inverse bijection is the *principal logarithm*:

$$\log z = \log|z| + i\,\arg z; \quad \arg z \text{ is the principal argument, i.e. } \log(x+iy) = \log\sqrt{x^2+y^2} + i\,\arg(x,y)$$

it being understood that $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$; recall that $\arg z = \arg(x+iy) = \arg(x,y)$ is the unique real number in $]-\pi,\,\pi]$ such that $z = |z|\,e^{i\arg z}$. But the exponential does *not* give a homeomorphism of the strip onto the punctured plane (which *in no way* are homeomorphic!): the principal logarithm is in fact *not* continuous at all points of the negative real semiaxis $\mathbb{R}_-$ (to be more precise, its imaginary part arg is not continuous at these points). Let us prove:

. *At all points of the cut plane $\mathbb{C}_- = \mathbb{C} \setminus \mathbb{R}_-$ the principal logarithm admits a complex derivative, and*

$$\frac{d}{dz}\log z = \frac{1}{z} \quad \textit{for every} \quad z \in \mathbb{C} \setminus \mathbb{R}_-.$$

Proof. We have

$$\partial_x u(x,y) = \partial_x \log\sqrt{x^2+y^2} = \frac{x}{x^2+y^2}; \quad \partial_y u(x,y) = \partial_y \log\sqrt{x^2+y^2} = \frac{y}{x^2+y^2},$$

and we recall that outside the negative real semiaxis:

$$\partial_x v(x,y) = \partial_x \arg(x,y) = \frac{-y}{x^2+y^2}; \quad \partial_y v(x,y) = \partial_y \arg(x,y) = \frac{x}{x^2+y^2}.$$

The Cauchy–Riemann identities are then verified, and

$$\frac{d}{dz}\log z = \partial_x u(x,y) + i\partial_x v(x,y) = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2} = \frac{x-iy}{x^2+y^2} = \frac{\bar z}{z\,\bar z} = \frac{1}{z}.$$

$\square$

1.3. Holomorphic functions.

DEFINITION. Let D be open in $\mathbb{C}$. A function $f\colon D \to \mathbb{C}$ is said to be *holomorphic* in D if it admits a complex derivative at every point of D, and the complex derivative is continuous in D.

In other words, we call holomorphic a function of class C^1 in the complex sense . We know that a C^1 function (in the real sense) is $\mathbb{R}-$differentiable; using this and 1.2.1 it is easy to see that:

PROPOSITION. *Let D be open in $\mathbb{C}$, and let $f\colon D \to \mathbb{C}$ be a function; then f is holomorphic if and only if $\partial_x f$, $\partial_y f$ exist, are continuous, and satisfy Cauchy-Riemann identities in all of D, i.e. $\partial_x f(z) = \partial_y f(z)/i$ for every $z \in D$.*

In other words, f is holomorphic if and only if $u = \operatorname{Re} f$ e $v = \operatorname{Im} f$ are both real C^1 functions and verify the identities

$$\partial_x u(x,y) = \partial_y v(x,y); \qquad \partial_y u(x,y) = -\partial_x v(x,y) \qquad \textit{for all } x+iy \in D.$$

Proof. Hinted above. □

REMARK. We hasten to add that our definition of holomorphic function is redundant: continuity of the derivative is superfluous; a theorem of Goursat (see 2.6.2) proves that if $f'(z)$ exists for every $z \in D$, D open in $\mathbb{C}$, then f is holomorphic on D.

But in almost all verifications of holomorphy for a function continuity of the derivative comes easily, so that this theorem is rarely used in practice. From the theoretical point of view it espresses however a very remarkable fact.

Functions that are sums of power series are holomorphic in the interior of their disc of convergence: we know in fact that in the open convergence disc the sum of a power series is termwise differentiable, that is, its derivative is the sum of the series of the derivatives, a power series with the same radius of convergence, hence again with a complex differentiable, and in particular continuous, sum.

Differentiating termwise the exponential series $\exp z = \sum_{n=0}^{\infty} z^n/n!$ we get the fundamental fact that the complex exponential exp is a holomorphic function and that $\exp' z = \exp z$, for every $z \in \mathbb{C}$.

Polinomials, exponential, cosine and sine, hyperbolic or not, the gaussian function $z \mapsto e^{-z^2}$, are sum of power series with infinite radius of convergence, and hence are holomorphic on all of $\mathbb{C}$; such functions are called *entire*; the preceding is a list of typical entire functions much in use. We have just proved that the principal logarithm is holomorphic on the cut plane $\mathbb{C} \setminus \mathbb{R}_-$. The function $z \mapsto 1/(z^2+1)$ is holomorphic on $\mathbb{C} \setminus \{-i, i\}$. The set of all functions $f\colon D \to \mathbb{C}$ holomorphic on D will often be denoted by $\mathcal{O}(D)$ (or $H(D)$). With the usual pointwise addition and multiplication of functions $\mathcal{O}(D)$ is a ring, and also a vector space on the field $\mathbb{C}$, hence is a $\mathbb{C}$-algebra (the reader not acquainted with the notion of $\mathbb{C}$-algebra can just ignore this remark).

1.3.1. *Another proof of* $\log' z = 1/z$. We derive this formula directly from the definition of the principal logarithm as a local inverse to exp, and $\exp' z = \exp z$: given z in the cut plane $\mathbb{C}_-$, since this set is open, for w close enough to z we also have $w \in \mathbb{C}_-$; since log is a homeomorphism of $\mathbb{C}_-$ onto the open strip $\{z \in \mathbb{C} : -\pi < \operatorname{Im} z < \pi\}$, with exp as inverse, we can use the change of variable $\log w = \zeta$ obtaining

$$\lim_{w \to z} \frac{\log w - \log z}{w - z} = \lim_{\zeta \to \log z} \frac{\zeta - \log z}{\exp \zeta - \exp(\log z)} = \lim_{\zeta \to \log z} \frac{1}{(\exp \zeta - \exp(\log z))/(\zeta - \log z)} = \frac{1}{\exp'(\log z)} = \frac{1}{\exp(\log z)} = \frac{1}{z}.$$

1.3.2. *Locally constant functions and functions with zero derivative.* We say that a function $h : X \to Y$, with X a topological space and Y a set, is *locally constant* at a point $c \in X$ if there exists a neighborhood U of c in X such that h is constant on U, i.e. $h(x) = h(c)$ for every $x \in U$; we say that h is locally constant on X if it is locally constant at every $c \in X$. Every locally constant $h : X \to Y$ is constant on the connected components of the domain space X (prove it! if you need help, look at the end of the section). We recall that if X is an open subset of $\mathbb{R}^n$ then a function $h : X \to \mathbb{R}$ is locally constant if and only if all its partial derivatives vanish identically on X. Hence:

PROPOSITION. *If D is an open subset of $\mathbb{C}$ a function $f : D \to \mathbb{C}$ is locally constant if and only if its complex derivative is identically zero.*

Proof. If f is locally constant, trivially it has a complex derivative everywhere zero on D. And if f has zero complex derivative, then it is $\mathbb{R}-$differentiable with zero differential, and as recalled above this implies that f is locally constant on D (partial derivatives of $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ are identically zero). $\square$

EXERCISE 1.3.2.1. Let $D \subseteq \mathbb{C}$ be open , $f : D \to \mathbb{C}$ a function, $z \in D$ and let $f'(z)$ exist. Prove that the conjugate function $\bar{f}$ of f, defined by $\bar{f}(w) = \overline{f(w)}$ for every $w \in D$, has a complex derivative at z if and only if $f'(z) = 0$. Deduce that f and $\bar{f}$ are both holomorphic on D if and only if f is locally constant on D.

Solution. First method By the hypotesis, $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ are differentiable at z and verify the Cauchy–Riemann identities there, so that $\partial_x u(z) = \partial_y v(z)$, $\partial_y u(z) = -\partial_x v(z)$. The function $\bar{f}$ has a complex derivative at z if and only if $\operatorname{Re} \bar{f} = u$ and $\operatorname{Im} \bar{f} = -v$, known to be differentiable at z, verify also Cauchy–Riemann identities, i.e. $\partial_x u(z) = -\partial_y v(z)$ e $\partial_y u(z) = \partial_x v(z)$; these identities are compatible with the previous ones if and only if all these derivatives are zero.

Second method Alternatively, we may try directly with the definition:

$$\frac{\overline{f(w)} - \overline{f(z)}}{w - z} = \frac{\overline{f(w)} - \overline{f(z)}}{\bar{w} - \bar{z}}\,\frac{\bar{w} - \bar{z}}{w - z} = \overline{R(w)}\,\frac{\bar{w} - \bar{z}}{w - z};$$

where $R(w) = (f(w) - f(z))/(w - z)$ is the difference quotient of f at z. As w tends to z we have that $\overline{R(w)}$ tends to $\overline{f'(z)}$, while the second factor assumes all values in the unit circle. Then the limit exists if and only if $\overline{f'(z)} = 0$, in which case it is 0.

The last assertion then immediately follows from the preceding proposition. $\square$

Solutions to the following exercises are at the end of the section.

EXERCISE 1.3.2.2. Let $D \subseteq \mathbb{C}$ be open, $f : D \to \mathbb{C}$ holomorphic. Prove that if f assumes only real, or only purely imaginary values, then f is locally costant on D.

EXERCISE 1.3.2.3. Let $D \subseteq \mathbb{C}$ be open, and let D^* be its symmetric with respect to the real axis, i.e. $D^* = \{\bar{z} : z \in D\}$. For every $f : D \to \mathbb{C}$ define $f^* : D^* \to \mathbb{C}$ by $f^*(z) = \overline{f(\bar{z})}$; assume that f has a complex derivative at $z \in D$. Prove that then so does f^* at $\bar{z}$, and express $(f^*)'(\bar{z})$ by $f'(z)$ (this may be done directly with the definition, or by the Cauchy–Riemann identities). Prove that instead $g : D^* \to \mathbb{C}$ defined by $g(z) = f(\bar{z})$ has a complex derivative at z if and only if $f'(z) = 0$ (see also exercise 1.3.2.1); deduce that if f is holomorphic then g is holomorphic if and only if f is locally constant.

Let us call *region* of $\mathbb{C}$ any connected open subset of $\mathbb{C}$.

EXERCISE 1.3.2.4. Prove that if $f : D \to \mathbb{C}$ is holomorphic and D is open in $\mathbb{C}$ then $|f|$ is locally constant if and only if f is locally constant.

REMARK. If X is a connected topological space, and $f : X \to \mathbb{R}$ is continuous, constancy of $|f|$ implies trivially constancy of f (if $|f(x)| = 0$ per ogni $x \in X$, then f itself is identically zero, and if $|f(x)| = k > 0$ we have $f(x) = \pm k$ for every $x \in X$, but connectedness of X and continuity of f imply that f cannot assume positive and negative values without being 0, so that $|f|$ also assumes the value 0, and cannot be a constant). By contrast, a continuous complex–valued function may well be of constant absolute value without being constant, e.g. the winding map $t \mapsto e^{it}$ from $\mathbb{R}$ to $\mathbb{C}$ is continuous with modulus constantly 1, but it is not a constant; in general, whatever the continuous map $g : X \to \mathbb{R}$, the map $f(x) = e^{i\,g(x)}$ is also continuous and of constant modulus 1. The preceding exercise expresses the remarkable fact that for a holomorphic function on a region constancy of the absolute value implies constancy of the function.

1.3.3. *Local constancy implies global constancy on connected sets.* Prove that if $h : X \to Y$ is locally costant on the topological space X, then h is constant on every connected subset of X.

Solution. Let $Z = h(X)$ be the image of X by h; consider the partition of X given by the fibers of h, i.e. $\{h^{\leftarrow}(y) : y \in Z\}$. Local constancy of h is equivalent to the assertion that every fiber $h^{\leftarrow}(y) = \{x \in X : h(x) = y\}$ is open in X. But if a topological space X is a disjoint union of open sets, each of them is clearly also closed (trivially: the complement of one of them is the union of the others, union of open sets, hence open), so they are all clopen (=open and closed) sets. If $C \subseteq X$ is connected, a clopen subset of X either has empty intersection with C, or contains all of C. $\square$

1.3.4. *Solution of the exercises.*

Solution. (Exercise 1.3.2.2) To assert that the function f assumes real values only is equivalent to assert that $v(x,y) = \operatorname{Im} f(x+iy)$ is identically zero; then $\partial_x v(x,y) = \partial_y v(x,y) = 0$ identically in D, and Cauchy-Riemann identities imply then $\partial_x u(x,y) = \partial_y u(x,y) = 0$ identically in D, if $u(x,y) = \operatorname{Re} f(x+iy)$. One could also use exercise 1.3.2.1: if f is real, then it coincides with its conjugate, which is holomorphic iff $f'(z) = 0$ for all $z \in D$. Analogous proof for $v = \operatorname{Im} f$. □

Solution. (Exercise 1.3.2.3) *First proof, directly*:

$$\frac{f^*(w) - f^*(z)}{w - z} = \frac{\overline{f(\bar{w})} - \overline{f(\bar{z})}}{w - z} = \overline{\left(\frac{f(\bar{w}) - f(\bar{z})}{\bar{w} - \bar{z}}\right)};$$

as w tends to z, $\bar{w}$ tends to $\bar{z}$ so that the term in parentheses tends to $f'(\bar{z})$, and the preceding limit is $\overline{f'(\bar{z})}$. We have proved that f^* is holomorphic, and that

$$(f^*)'(z) = \overline{f'(\bar{z})}.$$

Second proof, with CR identities If $u(x,y) = \operatorname{Re} f(x+iy)$ and $v(x,y) = \operatorname{Im} f(x+iy)$, we have

$$f^*(x+iy) = u(x,-y) - iv(x,-y) = U(x,y) + iV(x,y);$$

then

$$\begin{aligned} \partial_x U(x,y) &= \partial_x u(x,-y); \qquad & \partial_y U(x,y) &= -\partial_y u(x,-y); \\ \partial_x V(x,y) &= -\partial_x v(x,-y); \qquad & \partial_y V(x,y) &= \partial_y v(x,-y); \end{aligned}$$

and since f is holomorphic it verifies Cauchy-Riemann identities; thus

$$\partial_x u(x,-y) = \partial_y v(x,-y); \qquad \partial_y u(x,-y) = -\partial_y v(x,-y), \quad \text{for every } x+iy \in D,$$

which implies that f^* verifies in turn the Cauchy-Riemann identities:

$$\partial_x U(x,y) = \partial_y V(x,y); \quad \partial_y U(x,y) = -\partial_x V(x,y), \quad \text{per ogni } x+iy \in D,$$

so that f^* is holomorphic; moreover

$$(f^*)'(x+iy) = \partial_x U(x,y) + i\partial_x V(x,y) = \partial_x u(x,-y) + i(-\partial_x v(x,-y)) = \overline{f'(x-iy)}.$$

To prove that g holomorphic implies f locally constant we may then observe that $g = \overline{(f^*)}$, and that clearly f^* is locally constant iff so is f.

□

Solution. (Exercise 1.3.2.4) We may as well assume that D is a region of $\mathbb{C}$ and prove that on D constancy of $|f|$ implies constancy of f; moreover, if $|f| = 0$ then $f = 0$. So we may exclude this case, and assume that f is never zero.

First proof Trivially local constancy of f implies that of $|f|$. And if $|f|$ is constant so is $|f|^2 = f\,\bar{f}$; note that $|f|^2$ is holomorphic, being constant, so that $|f|^2/f = \bar{f}$ is also holomorphic. But from 1.3.2.1 this implies that $f'(z) = 0$ for every $z \in D$, that is, f is constant.

Second Proof Since $|f|^2$ is a constant function partial derivatives of $|f(x+iy)|^2 = (u(x,y))^2 + (v(x,y))^2$ are identically zero:

$$\begin{cases} u(x,y)\partial_x u(x,y) + v(x,y)\partial_x v(x,y) & = 0 \\ u(x,y)\partial_y u(x,y) + v(x,y)\partial_y v(x,y) & = 0 \end{cases} \quad \text{(canceling factor 2).}$$

This may be considered as a 2×2 homogeneous linear system in the unknowns u, v, whose matrix of coefficients is the transpose of the jacobian matrix of f (considered as a function from $D \subseteq \mathbb{R}^2$ to $\mathbb{R}^2$). Since f is never 0, the system has non trivial solutions for every $x+iy \in D$, and by Cramer's theorem, this implies that the determinant of the system matrix, which as observed in 1.2.1 is $|f'(x+iy)|^2$, must be zero, so that $f'(z) = 0$ for every $z \in D$, equivalent to constancy of f on the region D. □

1.4. Holomorphic and harmonic functions. Recall that a function $w : E \to \mathbb{C}$, with E open in $\mathbb{R}^n$, is said to be *harmonic* if it is of class C^2 and its *laplacian* $\Delta w(x) := \sum_{k=1}^n \partial_k^2 w(x)$, is identically zero on E (the laplacian in the case of a real–valued w coincides with the *divergence of the gradient* of w, so that physicists often denote it $\nabla^2 w(x) = \nabla \cdot \nabla w(x)$). Real and imaginary part of holomorphic functions are harmonic, if of class C^2 (as is always the case, 2.5.2). By Cauchy–Riemann identities this is immediate:

$$\partial_x u(x,y) = \partial_y v(x,y) \quad \partial_y u(x,y) = -\partial_x v(x,y)$$

assuming that we can differentiate both sides we get

$$\partial_x^2 u(x,y) = \partial_x\partial_y v(x,y) \quad \partial_y^2 u(x,y) = -\partial_y\partial_x v(x,y),$$

and summing up

$$\Delta u(x,y) = \partial_x\partial_y v(x,y) - \partial_y\partial_x v(x,y) = 0$$

by Schwartz's theorem ; the same for v. Thus every holomorphic function is harmonic. The converse is clearly false: $f(x+iy) = u(x,y) + iv(x,y)$ is harmonic iff so are u and v, but this does not imply that $u + iv$ is holomorphic: for instance the conjugation $z \mapsto \bar{z} = x - i\,y$ is clearly harmonic, but not holomorphic .

1.4.1. *Conjugate harmonics.* As observed at the end of the previous section, we cannot take at random two functions $u, v : D \to \mathbb{R}$, even harmonic ones, and hope that $u + iv$ is holomorphic: the Cauchy Riemann conditions must also hold.

We tackle the following problem: given an open $D \subseteq \mathbb{C}$, and a real $u : D \to \mathbb{R}$, which we assume of class C^1, can we say that u is the real part of a holomorphic function? in other words, does there exist $v \in C^1(D, \mathbb{R})$ such that $f = u + iv$ is holomorphic? Cauchy–Riemann identities say that v exists iff the *differential form*

$$(-\partial_y u(x,y))\,dx + (\partial_x u(x,y))\,dy$$

is exact in D; every primitive of this form is in this case one of the required functions v (clearly v is determined up to locally constant additive terms). Assuming the form of class C^1, equivalently u of class C^2, a necessary condition for exactness is closedness, in this case $-\partial_y^2 u = \partial_x^2 u$, equivalent to u being harmonic: every primitive v of the form, when it exists, is said to be a *conjugate harmonic* of u. Locally conjugate harmonics always exist if u is harmonic; but not always in all of D, unless D is simply connected. For instance the function $u(x,y) = \log\sqrt{x^2+y^2} = \log|z|$, real part of the principal logarithm, does not admit a conjugate harmonic on the punctured plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ (conjugate harmonic which is the function $\arg(x,y)$ on the cut plane $\mathbb{C}_- = \mathbb{C} \setminus \mathbb{R}_-$, a star–shaped set, hence simply connected).

EXERCISE 1.4.1.1. Given the function $u(x,y) = x^2 - y^2$, verify that it is harmonic on $\mathbb{C}$, find a conjugate harmonic v on $\mathbb{C}$, and express $u + iv$ in terms of the complex variable $z = x + iy$.

Solution. Harmonicity of u is trivial; and

$$(-\partial_y u(x,y))\,dx + (\partial_x u(x,y))\,dy = 2y\,dx + 2x\,dy,$$

has as primitives $2xy + k$, $k \in \mathbb{R}$ constant. Then

$$x^2 - y^2 + i\,2xy = (x+iy)^2 = z^2$$

is holomorphic. □

EXERCISE 1.4.1.2. Let $u(x,y) = \cos x \cosh y$. Verify that it is harmonic on $\mathbb{C}$, find a conjugate harmonic v on $\mathbb{C}$, and express $u + iv$ in terms of the complex variable $z = x + iy$.

Solution. Harmonicity of u is trivial; and

$$(-\partial_y u(x,y))\,dx + (\partial_x u(x,y))\,dy = -\cos x \sinh y\,dx - \sin x \cosh y\,dy,$$

and it is immediate that $v(x,y) = -\sin x \sinh y$ is a primitive. By Euler's formulas:

$$\cos x \cosh y - i\,\sin x \sinh y = \frac{e^{ix}+e^{-ix}}{2}\,\frac{e^{y}+e^{-y}}{2} - i\,\frac{e^{ix}-e^{-ix}}{2i}\,\frac{e^{y}-e^{-y}}{2} =$$
$$\frac{1}{4}(e^{ix+y} + e^{ix-y} + e^{-ix+y} + e^{-ix-y} - e^{ix+y} + e^{ix-y} + e^{-ix+y} - e^{-ix-y}) =$$
$$\frac{1}{4}(e^{ix-y} + e^{-ix+y} + e^{ix-y} + e^{-ix+y}) = \frac{e^{ix-y}+e^{-ix+y}}{2} = \frac{e^{iz}+e^{-iz}}{2} = \cos z.$$

□

EXERCISE 1.4.1.3. In the right half–plane $E = \{z \in \mathbb{C} : \mathrm{Re}\, z > 0\}$ define $u(x, y) = \arctan(y/x)$; verify that u is harmonic on E, find a conjugate harmonic v on E, and express $u + iv$ in terms of the complex variable $z = x + iy$.

Solution. A tedious verification proves harmonicity of u; we retain from it the partial derivatives of u:

$$\partial_x u(x, y) = \frac{-y/x^2}{1 + (y/x)^2} = \frac{-y}{x^2 + y^2}; \quad \partial_y u(x, y) = \frac{1/x}{1 + (y/x)^2} = \frac{x}{x^2 + y^2};$$

so that

$$(-\partial_y u(x, y))\, dx + (\partial_x u(x, y))\, dy = \frac{-x}{x^2 + y^2}\, dx + \frac{-y}{x^2 + y^2}\, dy,$$

with primitive $-(1/2)\log(x^2 + y^2)$, so that:

$$f(z) = f(x + iy) = \arctan(y/x) - i \log\sqrt{x^2 + y^2} = -i\, \log z.$$

where log is the principal logarithm (in fact on E we have $\arg(x, y) = \arctan(y/x)$). □

2. The Cauchy integral formula and its consequences

2.1. Path integrals. Given an open subset D of $\mathbb{C}$, we call *path* in D every function $\alpha : [a, b] \to D$ which is continuous and piecewise C^1, where $[a, b]$ is a compact interval of $\mathbb{R}$. Thus the continuous function $\alpha : [a, b] \to D$ is a path iff there is a subdivision $a = a_0 < a_1 < \cdots < a_m = b$, and functions $\alpha_k \in C^1([a_{k-1}, a_k], D)$ such that $\alpha_k|[a_{k-1}, a_k] = \alpha|[a_{k-1}, a_k]$, for $k = 1, \ldots, m$. The set $[\alpha] = \alpha([a, b])$ is called *trace* of the path α; it is of course to be kept distinguished from the path α itself.

If the function f is defined and continuous on the trace $[\alpha]$ of a path α, then the *integral of f over α* can be defined:

$$\int_\alpha f(z)\, dz := \int_a^b f(\alpha(t))\, \alpha'(t)\, dt.$$

(notice that the right–hand side is the integral of a function with at most a finite set of jump discontinuities on the compact interval $[a, b]$; hence the integral exists, in the Riemann sense).

EXAMPLE 2.1.0.4. Compute

$$\int_\alpha |z| z\, dz \quad \text{where} \quad \alpha(t) = e^{2\pi i t}, \quad t \in [0, 1/4].$$

Solution. Applying the definition we get for α

$$\int_\alpha |z|z\, dz = \int_0^{1/4} |e^{2\pi i t}|\, e^{2\pi i t}\, e^{2\pi i t}(2\pi i)\, dt = 2\pi i \int_0^{1/4} e^{4\pi i t}\, dt = \left[\frac{e^{4\pi i t}}{2}\right]_{t=0}^{t=1/4} = \frac{-1-1}{2} = -1.$$

□

EXAMPLE 2.1.0.5. Compute

$\displaystyle\int_\beta |z|z\, dz,$ where β is the polygonal path $[1, 0, i]$ given by the join of the line segments $[1, 0]$ and $[0, i]$.

Solution. We have (the parametrization of $[1, 0]$ is $t \mapsto 1 - t$ and that of $[0, i]$ is $t \mapsto it$, both with $t \in [0, 1]$)

$$\int_\beta |z|z\, dz = \int_{[1,0]} |z|z\, dz + \int_{[0,i]} |z|z\, dz = \int_0^1 |1 - t|(1 - t)(-dt) + \int_0^1 |it|(it)(i\, dt) =$$

$$-\int_0^1 (1 - t)^2\, dt - \int_0^1 t^2\, dt = \left[\frac{(1-t)^3}{3}\right]_0^1 - \left[\frac{t^3}{3}\right]_0^1 = -\frac{1}{3} - \frac{1}{3} = -\frac{2}{3}$$

□

. FUNDAMENTAL INEQUALITY *Let $f : D \to \mathbb{C}$ be continuous, and let $\alpha : [a, b] \to D$ be a path in D. Then we have:*

$$\left|\int_\alpha f(z)\, dz\right| \le \int_\alpha |f(z)|\, |dz| \le \|f\|_\alpha\, V(\alpha),$$

where $|dz| = |\alpha'(t)|\, dt$ is the "element of length" of α, $V(\alpha) = \int_\alpha |dz| = \int_a^b |\alpha'(t)|\, dt$ is the length of the path α, and, finally $\|f\|_\alpha = \max\{|f(z)| : z \in [\alpha]\} = \max\{|f(\alpha(t))| : t \in [a, b]\}$ is the sup–norm of f on the trace $[\alpha]$ of α.

Proof. By definition of line integral and the fundamental inequality for complex valued functions of one real variable we have:

$$\left|\int_\alpha f(z)\,dz\right| = \left|\int_a^b f(\alpha(t))\,\alpha'(t)\,dt\right| \le \int_a^b |f(\alpha(t))|\,|\alpha'(t)|\,dt;$$

this last integral may be written also $\int_\alpha |f(z)|\,|dz|$, and clearly we have

$$\int_\alpha |f(z)|\,|dz| \le \int_\alpha \|f\|_\alpha\,|dz| = \|f\|_\alpha \int_\alpha |dz| = \|f\|_\alpha\,V(\alpha).$$

□

EXERCISE 2.1.0.6. Let D be open in $\mathbb{C}$ and let $f, f_n : D \to \mathbb{C}$ be continuous functions; let $\alpha : [a,b] \to D$ be a path in D, and assume that the sequence of functions f_n converges to f uniformly on the trace $[\alpha]$ of α: this is equivalent to say that $\lim_n \|f - f_n\|_\alpha = 0$. Prove that then

$$\lim_n \int_\alpha f_n(z)\,dz = \int_\alpha f(z)\,dz.$$

Solution. In fact

$$\left|\int_\alpha f(z)\,dz - \int_\alpha f_n(z)\,dz\right| = \left|\int_\alpha (f(z) - f_n(z))\,dz\right| \le \int_\alpha |f(z) - f_n(z)|\,|dz| \le$$
$$\int_\alpha \|f - f_n\|_\alpha\,|dz| = \|f - f_n\|_\alpha\,\mathrm{length}(\alpha);$$

letting n tend to infinity the last term tends to zero and we conclude.

□

2.2. Small arc of circle and big arc of circle lemmata. These lemmata will be needed in the sequel for applications to the residue theorem. But they are a nice exercise on limit of integrals, and as such are appropriately placed here. Given $c \in \mathbb{C}$, and two rays starting at c, that is $c + r\,e^{i\alpha}$, $c + r\,e^{i\beta}$, $r > 0$, with $\alpha, \beta \in \mathbb{R}$, and $\alpha < \beta$, the *angle of vertex c* which has these two rays as sides is

$$A(c, [\alpha, \beta] = \{c + t\,e^{i\vartheta} : t \ge 0,\ \vartheta \in [\alpha, \beta]\}.$$

EXAMPLE 2.2.0.7. $A(0, [-\pi/2, \pi/2]$ is the right half–plane $\{\mathrm{Re}\, z \ge 0\}$, $A(0, [0, \pi]$ is the upper half–plane $\{\mathrm{Im}\, z \ge 0\}$, etc. For every $c \in \mathbb{C}$, if $\beta - \alpha \ge 2\pi$ then $A(c, [\alpha, \beta] = \mathbb{C}$.

. *Let $A = A(c, [\alpha, \beta]$ be an angle as above, and let $f : A \setminus \{c\} \to \mathbb{C}$ be continuous. For every $r > 0$ we set $\gamma_r(\vartheta) = c + r\,e^{i\vartheta}$, $\vartheta \in [\alpha, \beta]$. Then:*

- SMALL ARC OF CIRCLE LEMMA *If*

$$\lim_{z \to c,\, z \in A} (z - c)\,f(z) = \lambda \in \mathbb{C} \quad \text{then} \quad \lim_{r \to 0^+} \int_{\gamma_r} f(z)\,dz = i\lambda\,(\beta - \alpha).$$

- LARGE ARC OF CIRCLE LEMMA *If*

$$\lim_{z \to \infty,\, z \in A} z\,f(z) = \lambda \in \mathbb{C} \quad \text{then} \quad \lim_{r \to +\infty} \int_{\gamma_r} f(z)\,dz = i\lambda\,(\beta - \alpha).$$

Proof. Observe that

$$\lim_{z \to \infty,\, z \in A} z\,f(z) = \lambda \iff \lim_{z \to \infty,\, z \in A} (z - c)\,f(z) = \lambda$$

(in fact, if the limit of $z\,f(z)$ is finite when z tends to infinity then $f(z)$ tends to 0). Considering $g(z) = f(z) - \lambda/(z - c)$, we then have that in both cases $(z - c)\,g(z)$ tends to 0. Then

$$\int_{\gamma_r} f(z)\,dz = \int_{\gamma_r} \left(\frac{\lambda}{z - c} + g(z)\right)\,dz = \int_{\gamma_r} \frac{\lambda}{z - c}\,dz + \int_{\gamma_r} g(z)\,dz;$$

the first of these integrals is easily computed:

$$\int_{\gamma_r} \frac{\lambda}{z - c}\,dz = \int_\alpha^\beta \frac{\lambda}{r\,e^{i\vartheta}} r\,i\,e^{i\vartheta}\,dt = i\lambda(\beta - \alpha);$$

so that the conclusion is reached if the second integral tends to 0; the standard inequality gives

$$\left|\int_{\gamma_r} g(z)\,dz\right| \le \int_{\gamma_r} |g(z)|\,|dz| \le \int_{\gamma_r} \|g\|_r\,|dz| = (\beta - \alpha) r\,\|g\|_r,$$

where $\|g\|_r = \|g\|_{\gamma_r} = \max\{|g(r + e^{i\vartheta})| : \vartheta \in [\alpha, \beta]\}$. Since $(z - c)g(z)$ tends to 0 (in the case of the small arc, for $z \to c$, equivalently for $r \to 0^+$; in the case of the large arc, for $z \to \infty$, equivalently for $r \to +\infty$) we of course have $\lim_{r\to 0^+} r\,\|g\|_r = 0$ in the case of the small arc, $\lim_{r\to+\infty} r\,\|g\|_r = 0$ in the case of the large arc, and in both cases we conclude. □

2.3. Complex forms and real forms. If $z = x+iy$, with $x, y \in \mathbb{R}$, and $f(x+iy) = u(x, y)+i\,v(x, y)$, with u, v real valued functions, and finally $\alpha(t) = p(t) + i\,q(t)$, with $p(t), q(t) \in \mathbb{R}$, then:

$$\int_\alpha f(z)\,dz = \int_a^b (u(p(t), q(t)) + i\,v(p(t), q(t)))\,(p'(t) + i\,q'(t))\,dt =$$
$$\int_a^b (u(p(t), q(t))\,p'(t) - v(p(t), q(t))\,q'(t))dt + i\int_a^b (v(p(t), q(t))\,p'(t) + u(p(t), q(t))\,q'(t))\,dt;$$

Formally we can write:

$$f(z)\,dz = (u(x, y) + i\,v(x, y))(dx + idy) = (u(x, y)\,dx - v(x, y)\,dy) + i(v(x, y)\,dx + u(x, y)\,dy),$$

that is, the complex differential form $f(z)\,dz$ can formally be written as above, with two real differential forms as real and imaginary parts. The preceding computation shows that the integral over a path satisfies:

$$\int_\alpha f(z)\,dz = \int_\alpha (u(x, y)\,dx - v(x, y)\,dy) + i\int_\alpha (v(x, y)\,dx + u(x, y)\,dy),$$

and hence that the preceding decomposition of $f(z)\,dz$, however formal, can be used for the computations of line integrals. Next, assuming $u, v \in C^1(D, \mathbb{R})$, and hence that $f \in C^1(D, \mathbb{C})$ in the real sense:

. **The forms $u\,dx - v\,dy$ and $v\,dx + u\,dy$ are closed differential forms (**i.e., we have $\partial_y u = -\partial_x v$ and $\partial_y v = \partial_x u$**) if and only if $f = u + i\,v$ verifies the Cauchy–Riemann identities**.

It follows that everything we know on integrals of closed real differential forms can be used for complex forms $f(z)\,dz$, if f is holomorphic. In particular, we have the following fundamental result, which we shall repeatedly use:

Theorem. *Let D be an open subset of $\mathbb{C}$, and let $f : D \to \mathbb{C}$ be holomorphic. If α, β are loops in D, which are loop–homotopic in D, then*

$$\int_\alpha f(z)\,dz = \int_\beta f(z)\,dz.$$

2.3.1. *Primitives.* Given an open subset D of $\mathbb{C}$, and a continuous function $f : D \to \mathbb{C}$, we say that a function $F : D \to \mathbb{C}$ is a *primitive in the complex sense*, or a complex primitive, or simply a primitive of f on D if F is complex–differentiable for every $z \in D$, and for every $z \in D$ we have $F'(z) = f(z)$. Any complex primitive F of the continuous function f is then holomorphic. Moreover, if $F(x+iy) = U(x, y)+iV(x, y)$ and $f(x+iy) = u(x, y)+i\,v(x, y)$, with f continuous, then F is a complex primitive of f if and only if U, V are C^1 (in the real sense) and

$$\partial_x U(x, y) = \partial_y V(x, y) = u(x, y), \text{ and } -\partial_y U(x, y) = \partial_x V(x, y) = v(x, y), \quad \text{for every } x + iy \in D.$$

The easy verification is left to the reader. In other words:

Theorem. *The continuous function $f : D \to \mathbb{C}$, $f(x + iy) = u(x, y) + i\,v(x, y)$, admits a complex primitive on D if and only if the differential forms*

$$u(x, y)\,dx - v(x, y)\,dy \qquad v(x, y)\,dx + u(x, y)\,dy$$

are exact on D; and every complex primitive F of f is of the form $F(x + iy) = U(x, y) + iV(x, y)$, with U a primitive of $u\,dx - v\,dy$ and V a primitive of $v\,dx + u\,dy$.

The function f admits a primitive on D if and only if the integral of f is zero on every loop of D:

$$\int_\gamma f(z)\,dz = 0 \qquad \textit{for every loop } \gamma \textit{ in } D.$$

Two complex primitives F and G of the same function $f : D \to \mathbb{C}$ differ by a function whose derivative is 0, and hence by a locally constant function.

Again, the proof is left to the reader, who has only to recall what he knows about exact forms.

The knowledge of a primitive of f gives of course immediately all curvilinear integrals of f:

. The Fundamental Theorem of Calculus for complex functions *If the continuous function $f : D \to \mathbb{C}$ has a primitive F on D, then for every path $\alpha : [a, b] \to D$ in D we have*

$$\int_\alpha f(z)\, dz = F(\alpha(b)) - F(\alpha(a)).$$

The proof again can be done using real differential forms.

2.3.2. *Corollary: another proof of the characterization of locally constant functions.* In 1.3.2 we have observed that a function $f : D \to \mathbb{C}$, with D an open subset of $\mathbb{C}$, is locally constant if and only if it is complex differentiable, with derivative identically zero. The only part of this statement which is not completely trivial is the fact that $f'(z) = 0$ implies the local constancy of f; given $c \in D$, pick $\delta > 0$ such that the ball $B(c, \delta[$ is contained in D; since the ball $B(c, \delta[$ is convex the segment $[c, z]$ is contained in it, for every $z \in B(c, \delta[$; by the above theorem we have

$$f(z) - f(c) = \int_{[c,z]} f'(\zeta)\, d\zeta = \int_{[c,z]} 0\, d\zeta = 0,$$

so that $f(z) = f(c)$, for every $z \in B(c, \delta[$, proving the constancy of f on $B(c, \delta[$.

Exercise 2.3.2.1. *Mean value theorem for holomorphic functions*) Let D be open in $\mathbb{C}$ and let $f : D \to \mathbb{C}$ be holomorphic; assume that the segment $[a, b] = \{\zeta \in \mathbb{C} : \zeta = a + t(b - a),\, t \in [0, 1]\}$ is contained in D. Then

$$|f(b) - f(a)| \le \|f'\|_{[a,b]}\, |b - a|,$$

where, as usual, $\|f'\|_{[a,b]} = \max\{|f'(\zeta)| : \zeta \in [a, b]\}$.

Exercise 2.3.2.2. Let D be open in $\mathbb{C}$. Prove that if $f : D \to \mathbb{C}$ is holomorphic and $c \in D$ then

$$f'(c) = \lim_{z,\, w \to c, z \ne w} \frac{f(w) - f(z)}{w - z};$$

and deduce from it that if $f'(c) \ne 0$ then f is locally injective at c, that is, there exists a neighborhood U of c such that $f_{|U}$ is one–to–one.

Solution. We can write, for $w, z \in B(c, \delta[\subseteq D$

$$\frac{f(w) - f(z)}{w - z} = \frac{1}{w - z} \int_{[z,w]} f'(\zeta)\, d\zeta = \frac{1}{w - z} \int_0^1 f'(z + t(w - z))\,(w - z)\, dt = \int_0^1 f'(z + t(w - z))\, dt,$$

so that

$$\left|\frac{f(w) - f(z)}{w - z} - f'(c)\right| = \left|\int_0^1 f'(z + t(w - z))\, dt - \int_0^1 f'(c)\, dt\right| = \left|\int_0^1 (f'(z + t(w - z)) - f'(c))\, dt\right| \le$$
$$\le \int_0^1 |f'(z + t(w - z)) - f'(c)|\, dt;$$

By continuity of f' at c, given $\varepsilon > 0$ for some $\delta = \delta_\varepsilon > 0$ we have $|f'(\zeta) - f'(c)| \le \varepsilon$ if $|\zeta - c| \le \delta$; by convexity of the disc $B(c, \delta[$ we have $z + t(w - z) \in B(c, \delta[$ if $z, w \in B(c, \delta[$ and $t \in [0, 1]$ so that the last integral is smaller than ε if $z, w \in B(c, \delta[$, an the conclusion is reached. Local injectivity is now a trivial argument by contradiction. □

Example 2.3.2.3. *(important)* In $D = \mathbb{C} \setminus \{c\}$, where c is a given complex number, the functions $h_p(z) = (z - c)^p$, where p is an integer, admit a primitive for every $p \ne -1$, and that primitive is $z \mapsto (z - c)^{p+1}/(p + 1)$; this is trivial to verify. That $z \mapsto 1/(z - c)$ has no primitive is due to the fact that if we consider the loop $\gamma_m(t) = c + R\, e^{2\pi i m t}$, $t \in [0, 1]$, where $m \in \mathbb{Z}$, $R > 0$ we have:

$$\int_{\gamma_m} \frac{dz}{z - c} = \int_0^1 \frac{2\pi i m\, R\, e^{2\pi i m t}}{R\, e^{2\pi i m t}}\, dt = \int_0^1 2\pi i m\, dt = 2\pi i m \quad (\ne 0 \text{ if } m \ne 0).$$

The loop γ_m is the circle of center c and radius R, described m times; for $m = 1$ it is called simply *the circle of center c and radius R.*

Taking $c = 0$ we see that $1/z$ has no primitive in the *punctured plane* $\mathbb{C}_* = \mathbb{C} \setminus \{0\}$; but if we restrict the domain of $1/z$ to the *slit plane* $\mathbb{C}_- = \mathbb{C} \setminus \mathbb{R}_-$, where $\mathbb{R}_- = \{x \in \mathbb{R} : x \le 0\}$, which is simply connected, then $1/z$ has as a primitive the *principal logarithm* $z \mapsto \log z$, uniquely determined in $\mathbb{C}_-$ by the requirements $\exp(\log z) = z$ and $\operatorname{Im}(\log z) \in]-\pi, \pi[$. We know that we have $\log z = \log |z| + i \arg z$, in other words $\log(x + iy) = \log \sqrt{x^2 + y^2} + i\, \arg(x + iy)$.

2.3.3. *Change of variable formula for line integrals.* The following formula is often useful:

. *Let $f : E \to \mathbb{C}$ be continuous, and $\varphi : D \to E$ be holomorphic. For every path α in D we have*

$$\int_{\varphi\circ\alpha} f(z)\, dz = \int_\alpha f(\varphi(\zeta))\, \varphi'(\zeta)\, d\zeta.$$

Proof. It is a trivial computation; assume that α is parametrized on the interval $[a, b]$; by definition the left hand side is then

$$\int_{\varphi\circ\alpha} f(z)\, dz = \int_a^b f(\varphi(\alpha(t)))\,(\varphi\circ\alpha)'(t)\, dt = \int_a^b f(\varphi(\alpha(t)))\,\varphi'(\alpha(t))\,\alpha'(t)\, dt,$$

and, again by definition, the right–hand side is

$$\int_\alpha f(\varphi(\zeta))\,\varphi'(\zeta)\, d\zeta = \int_a^b f(\varphi(\alpha(t))\,\varphi'(\alpha(t))\,\alpha'(t)\, dt.$$

□

2.3.4. *Local primitives.* We know that closed forms are exact on simply connected sets, in particular on star–shaped sets, or even better on convex sets; hence every function holomorphic on an open simply connected set has a complex primitive on that set. Since open discs are convex, every holomorphic funtion on a disc has a primitive on that disc. Every holomorphic function has then *local primitives*; but $1/z$ on $\mathbb{C}_* = \mathbb{C} \setminus \{0\}$ shows that not always these local primitives can be patched together to give a global primitive. Clearly, if $f : D \to \mathbb{C}$ is of class C^1 in the real sense, then f has local primitives if and only if the forms $u\,dx - v\,dy$, $v\,dx + u\,dy$ are locally exact, equivalently, closed differential forms, equivalently, iff f is holomorphic. We shall later see that a *continuous* function $f : D \to \mathbb{C}$ has local primitives in D iff it is holomorphic.

2.3.5. *Analiticity.* As recalled, in the open disc of convergence the sum of a power series is a holomorphic function, whose derivative is the sum of the derived series. Functions which locally are sum of power series are then holomorphic. Such functions are called *analytic*: if D is an open subset of $\mathbb{C}$, we say that a function $f : D \to \mathbb{C}$ is (complex) analytic in D if for every $c \in D$ there exists a complex power series $\sum_{n=0}^\infty c_n(z-c)^n$ with positive radius of convergence, such that $f(z) = \sum_{n=0}^\infty c_n(z-c)^n$ holds for all z in some disc $B(c, \delta[\subseteq D$. If this holds, then f is complex differentiable in $B(c, \delta[$ infinitely many times, and the power series is exactly the Taylor series of f at c, that is $c_n = f^{(n)}(c)/n!$, as is well–known. In particular, *analytic functions are holomorphic.* Moreover, complex derivatives of analytic functions are also analytic, by the theorem on termwise differentiation of power series.

A way to construct analytic function which is very important is the following. We are given a path $\alpha : [a, b] \to \mathbb{C}$, and a continuous function $u : [\alpha] \to \mathbb{C}$; if $z \in \mathbb{C} \setminus [\alpha]$ we define

(THE CAUCHY TRANSFORM OF u) $$f_u(z) = \frac{1}{2\pi i}\int_\alpha \frac{u(\zeta)}{\zeta - z}\, d\zeta$$

(the mysterious factor $1/(2\pi i)$ is to be explained later). The Cauchy transform f_u is analytic on its domain $\mathbb{C}\setminus[\alpha]$, as we now prove. If $c \in \mathbb{C}\setminus[\alpha]$, then c has a strictly positive distance δ from the compact set $[\alpha]$; we prove:

. *In the above notations and terminology we have*

$$f_u(z) = \sum_{n=0}^\infty c_n(z-c)^n \quad \textit{if } |z-c| < \delta, \textit{ where for every } n \in \mathbb{N}: \quad c_n = \frac{1}{2\pi i}\int_\alpha \frac{u(\zeta)}{(\zeta - c)^{n+1}}\, d\zeta.$$

Proof. If $\zeta \in [\alpha]$ and $z \in B(c, \delta[$ then $|z - c| < \delta \le |\zeta - c|$ and hence $|z - c|/|\zeta - c| \le |z - c|/\delta < 1$; then we can write:

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - c + c - z} = \frac{1}{(\zeta - c)}\,\frac{1}{1 - (z-c)/(\zeta - c)} = \frac{1}{(\zeta - c)}\sum_{n=0}^\infty \frac{(z-c)^n}{(\zeta - c)^n}$$

(notice that the geometric series in the right–hand side converges, because $|z-c|/|\zeta-c| < 1$). Multiplying both sides by $u(\zeta)/(2\pi i)$ we get:

$$\frac{1}{2\pi i}\frac{u(\zeta)}{\zeta - z} = \frac{1}{2\pi i}\frac{u(\zeta)}{(\zeta - c)}\sum_{n=0}^\infty \frac{(z-c)^n}{(\zeta - c)^n} = \sum_{n=0}^\infty \frac{1}{2\pi i}\frac{u(\zeta)}{(\zeta - c)^{n+1}}(z-c)^n;$$

integrating both sides on the path α:

$$f_u(z) = \frac{1}{2\pi i}\int_\alpha \frac{u(\zeta)}{\zeta - z}\,d\zeta = \int_\alpha \left(\sum_{n=0}^{\infty} \frac{1}{2\pi i}\frac{u(\zeta)}{(\zeta - c)^{n+1}}(z-c)^n\right)\,d\zeta.$$

We wish to exchange $\int_\alpha$ and $\sum_{n=0}^{\infty}$. To do that, we simply observe that the series normally converges on the trace $[\alpha]$ of α:

$$\left|\frac{1}{2\pi i}\frac{u(\zeta)}{(\zeta-c)^{n+1}}(z-c)^n\right| = \frac{1}{2\pi}\frac{|u(\zeta)|}{|\zeta-c|^{n+1}}|z-c|^n = \frac{|u(\zeta)|}{2\pi|\zeta-c|}\frac{|z-c|^n}{|\zeta-c|^n} \le \frac{\|u\|_\alpha}{2\pi\delta}\left(\frac{|z-c|}{\delta}\right)^n,$$

where $\|u\|_\alpha = \max\{|u(\zeta)| : \zeta \in [\alpha]\}$, and as above $\delta = \operatorname{dist}(c, [\alpha])$. Since $|z - c| < \delta$, the series $\sum_{n=0}^{\infty} \|u\|_\alpha/(2\pi\delta)(|z-c|/\delta)^n$ converges. We then can put the integral inside the series (this is essentially exercise 2.1.0.6, applied to the sequence of partial sums of the series) obtaining

$$f_u(z) = \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_\alpha \frac{u(\zeta)}{(\zeta-c)^{n+1}}\,d\zeta\right)(z-c)^n.$$

The proof is concluded. □

2.3.6. *Derivatives of the Cauchy transform.* As observed above the power series whose sum about c is f_u must be the Taylor series of f_u at c; thus we have

$$c_n = \frac{f_u^{(n)}(c)}{n!} = \frac{1}{2\pi i}\int_\alpha \frac{u(\zeta)}{(\zeta-c)^{n+1}}\,d\zeta, \quad \text{for every } n \in \mathbb{N};$$

and since this holds for every $c \in \mathbb{C} \setminus [\alpha]$ we have also the formula:

. Derivatives of the Cauchy transform. *If $z \in \mathbb{C} \setminus [\alpha]$ and $n \in \mathbb{N}$ we have*

$$f_u^{(n)}(z) = \frac{n!}{2\pi i}\int_\alpha \frac{u(\zeta)}{(\zeta-z)^{n+1}}\,d\zeta.$$

Notice that the derivatives of f_u are exactly those that one would obtain by differentiating with respect to z under the integral sign (in fact, one can prove directly that this is admissible, but we shan't do it here); this is trivial for the first derivative, and is left as an exercise for the other derivatives. We have observed that every analytic function is holomorphic; the Cauchy integral formula will prove that, conversely, *every holomorphic function is analytic* (2.5.1). Thus: *for complex functions analyticity and complex differentiability coincide.* A far cry from the real case!

Example 2.3.6.1. Consider the function $f : \mathbb{C} \setminus [0, 1] \to \mathbb{C}$ given by

$$f(z) = \int_{[0,1]} \frac{d\zeta}{\zeta - z} = \int_0^1 \frac{dt}{t-z}.$$

A formal integration gives $f(z) = [\log(t-z)]_{t=0}^{t=1} = \log(1-z) - \log(-z) = \log(1-1/z)$. Observe that if $\log : \mathbb{C} \setminus \mathbb{R}_- \to \mathbb{C}$ is the principal logarithm, then $1 - 1/z = -t$, with $t \in \mathbb{R}_+$ obtains if and only if $z = 1/(1+t)$, that is, exactly iff $z \in [0, 1]$; hence $g(z) = \log(1 - 1/z)$ is holomorphic in $\mathbb{C} \setminus [0, 1]$. And it is exactly the function f defined by the integral: in fact $g'(z) = (1/z^2)/(1-1/z) = 1/(z(z-1))$ and by the formula for the derivatives:

$$f'(z) = \int_0^1 \frac{dt}{(t-z)^2} = \left[\frac{-1}{t-z}\right]_{t=0}^{t=1} = \frac{1}{-z} - \frac{1}{1-z} = \frac{1}{z(z-1)};$$

hence f and g differ by a constant; but we have

$$f(2) = \int_0^1 \frac{dt}{t-2} = -\log 2; \quad g(2) = \log(1-1/2) = -\log 2,$$

and then f and g are everywhere equal on the connected set $\mathbb{C} \setminus [0, 1]$.

Alternatively we can observe that since $-\log(1-w) = \sum_{n=1}^{\infty} w^n/n$ for $|w| < 1$, then we have $g(z) = \sum_{n=1}^{\infty}(-1)/(nz^n)$ for $|z| > 1$; and for $|z| > 1$ we can write

$$\frac{1}{t-z} = \frac{1}{z}\frac{-1}{1-(t/z)} = \sum_{n=0}^{\infty} -\frac{t^n}{z^{n+1}},$$

with the series normally convergent for $t \in [0,1]$; termwise integration is then possible and gives

$$f(z) = \int_0^1 \frac{dt}{t-z} = -\sum_{n=0}^{\infty} \int_0^1 \frac{t^n}{z^{n+1}}\, dt = \sum_{n=1}^{\infty} \frac{-1}{(n+1)z^{n+1}} = g(z) \quad (|z|>1).$$

Thus f and g coincide if $|z|>1$, and the identity theorem will imply that they coincide on all of $\mathbb{C} \setminus [0,1]$.

2.4. Winding number. One of the most important notions in complex analysis is that of *winding number*, the number of times that a given circuit winds around a point. It is strictly related to the complex logarithm, and one might say that it really embodies the difference between complex and real analysis methods.

2.4.1. *Logarithm of a path.*

LEMMA. *Let $\alpha : [a,b] \to \mathbb{C}_*(= \mathbb{C} \setminus \{0\})$ be a path in the punctured plane. Then*

$$\exp\left(\int_\alpha \frac{dz}{z}\right) = \frac{\alpha(b)}{\alpha(a)}.$$

Consequently, if α is a circuit ($\alpha(b) = \alpha(a)$), then

$$\frac{1}{2\pi i}\int_\alpha \frac{dz}{z} \quad \textit{is an integer.}$$

Proof. We have by definition

$$\int_\alpha \frac{dz}{z}\, dz := \int_a^b \frac{\alpha'(\theta)}{\alpha(\theta)}\, d\theta;$$

before continuing the proof let us observe that if $\log \alpha(t)$ could be defined, with the usual properties, we would have

$$\int_a^b \frac{\alpha'(\theta)}{\alpha(\theta)}\, d\theta = [\log \alpha(\theta)]_{\theta=a}^{\theta=b} = \log(\alpha(b)) - \log(\alpha(a)) = \log(\alpha(b)/\alpha(a)) \quad \text{thus} \quad \exp\left(\int_\alpha \frac{dz}{z}\right) = \frac{\alpha(b)}{\alpha(a)},$$

as asserted. But the logarithm function is not globally available, so we must choose a different route.

Put $\beta(t) = \int_a^t (\alpha'(\theta)/\alpha(\theta))\, d\theta$; let us prove that $\alpha(t)\, e^{-\beta(t)}$ is a constant; in fact differentiating:

$$\alpha'(t)\, e^{-\beta(t)} + \alpha(t)\, e^{-\beta(t)}(-\beta'(t)) = e^{-\beta(t)}\left(\alpha'(t) - \alpha(t)\frac{\alpha'(t)}{\alpha(t)}\right) = 0.$$

Thus $\alpha(a)\, e^{-\beta(a)} = \alpha(b)\, e^{-\beta(b)}$; but $\beta(a) = 0$, $\beta(b) = \int_\alpha (dz/z)$, and the first assertion is proved. For the second, simply remember that $\exp(\int_\alpha (dz/z)) = 1$ if and only if $\int_\alpha (dz/z) = 2\pi i m$, with $m \in \mathbb{Z}$. □

2.4.2. *Periods of logarithmic derivatives.*

PROPOSITION. *Let $f : D \to \mathbb{C}$ be holomorphic, where D is an open subset of $\mathbb{C}$, and let γ be a circuit in D. Assume that f is never zero on the trace $[\gamma]$ of γ. Then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\, dz$$

is an integer.

Proof. We have only to apply the preceding lemma to the circuit $\omega = f \circ \gamma$ (we apply in fact a version of 2.3.3). □

The proposition states that a *logarithmic derivative* of a holomorphic function f, that is $f'(z)/f(z)$, when integrated on a circuit gives an integral multiple of $2\pi i$.

2.4.3. *Winding number.* Since in $\mathbb{C}$ every compact set K is closed, its complement $\mathbb{C} \setminus K$ is open, and has open connected components. Moreover, $\mathbb{C} \setminus K$ has only one unbounded component: if $r > 0$ is such that $rB \supseteq K$, then $\mathbb{C} \setminus rB$ is disjoint from K, and being connected it is contained in a component U_∞ of $\mathbb{C} \setminus K$; the other components are disjoint from U_∞, hence contained in rB, and hence bounded.

THEOREM. *Let $\gamma : [a,b] \to \mathbb{C}$ be a circuit in $\mathbb{C}$. Define the function $\operatorname{ind}_\gamma : \mathbb{C} \setminus [\gamma] \to \mathbb{C}$ by the formula*

$$\operatorname{ind}_\gamma(z) = \frac{1}{2\pi i}\int_\gamma \frac{d\zeta}{\zeta - z};$$

then $\operatorname{ind}_\gamma$ is integer valued, constant on every connected component of $\mathbb{C} \setminus [\gamma]$, and zero on the unbounded component.

Proof. That ind_γ is integer valued is the preceding proposition, with $f(\zeta) = \zeta - z$, for every fixed $z \in \mathbb{C} \smallsetminus [\gamma]$. By theorem 2.3.5 the function ind_γ is holomorphic in $\mathbb{C} \smallsetminus [\gamma]$, and its derivative is obtained (as if) by differentiating under the integral sign:

$$\mathrm{ind}'_\gamma(z) = \frac{1}{2\pi i}\int_\gamma \frac{d\zeta}{(\zeta - z)^2}; \quad \text{but} \quad \frac{d}{d\zeta}\left(\frac{-1}{\zeta - z}\right) = \frac{1}{(\zeta - z)^2}$$

and the integration can easily be performed, giving:

$$\mathrm{ind}'_\gamma(z) = \frac{1}{2\pi i}\left(\frac{-1}{\gamma(b) - z} - \frac{-1}{\gamma(a) - z}\right) = 0;$$

(recall that $\gamma(a) = \gamma(b)$, since γ is a circuit). This proves that ind_γ is locally constant, hence constant on the connected components of its domain $\mathbb{C} \smallsetminus [\gamma]$ (a fact that of course also follows directly from continuity and integer–valuedness of ind_γ).

Finally, let $\mu = \max\{|\gamma(t)| : t \in [a, b]\}$, and assume $|z| > \mu$; the standard inequality gives

$$|\mathrm{ind}_\gamma(z)| = \frac{1}{2\pi}\left|\int_\gamma \frac{d\zeta}{\zeta - z}\right| \le \frac{1}{2\pi}\int_\gamma \frac{|d\zeta|}{|\zeta - z|} \le \frac{1}{2\pi}\int_\gamma \frac{|d\zeta|}{|z| - \mu} = \frac{V(\gamma)}{2\pi(|z| - \mu)},$$

where $V(\gamma) = \int_\gamma |d\zeta|$ is the length of γ. It is plain that the last expression tends to zero as $|z|$ tends to infinity. □

2.5. The Cauchy integral formula for a circle. If $B(c, r]$ is the closed disc of center c and radius $r > 0$, by $\partial B(c, r]$ we shall denote the loop $t \mapsto c + re^{it}$, $t \in [0, 2\pi]$, the positively oriented circle of center c and radius r. Observe that by the previous section we have

$$\frac{1}{2\pi i}\int_{\partial B(c,r]} \frac{d\zeta}{\zeta - z} = \begin{cases} 1 & \text{if} \quad |z - c| < r \\ 0 & \text{if} \quad |z - c| > r \end{cases}.$$

(in 2.3.2.3 the value $\mathrm{ind}_{\partial B(c,r]}(c) = 1$ has been computed directly).

. Cauchy integral formula for a circle. *Let D be open in $\mathbb{C}$, and let $f : D \to \mathbb{C}$ be holomorphic. Given $c \in D$, let $r > 0$ be such that $B(c, r] \subseteq D$. Then we have*

$$f(z) = \frac{1}{2\pi i}\int_{\partial B(c,r]} \frac{f(\zeta)}{\zeta - z}\, d\zeta \quad \textit{for every } z \in B(c, r[.$$

Before the proof let us have some comments. This extremely beautiful formula says that in the interior of a disc contained in a region of holomorphy of f, the function f itself is the Cauchy transform of its trace on the boundary of the disc. This fact has very far reaching consequences, e.g. combined with 2.3.5 proves that holomorphic functions are analytic, and even more, the radius of convergence of the Taylor series of a holomorphic function at a point $c \in D$ is at least equal to the distance of c from the boundary of D!

Proof. Given $z \in D$ we consider the function $g : D \to \mathbb{C}$ defined by

$$g(\zeta) = \frac{f(\zeta) - f(z)}{\zeta - z} \quad \text{if } \zeta \in D \smallsetminus \{z\}, \quad g(z) = f'(z).$$

Obviously g is continuous on D, and holomorphic on $D \smallsetminus \{z\}$. If c, r, z are as in the statement of the theorem, we consider the circles $\gamma_\lambda(t) = z + \lambda(\gamma(t) - z)$, for $\lambda \in]0, 1]$, where $\gamma(t) = c + r\, e^{it}$, $t \in [0, 2\pi]$ is the boundary circle of $B(c, r]$ (for fixed $\lambda \in]0, 1]$, γ_λ is obtained from γ by the *homotethy* of center z and ratio λ). If we set

$$I(\lambda) = \int_{\gamma_\lambda} g(\zeta)\, d\zeta, \quad \lambda \in]0, 1],$$

then I is a constant function: in fact γ_λ is clearly homotopic, in $D \smallsetminus \{z\}$ where g is holomorphic, to γ; hence $I(\lambda) = I(1)$, for every $\lambda \in]0, 1]$. Now it is easy to see that $\lim_{\lambda \to 0^+} I(\lambda) = 0$: since g is continuous on the compact disc $B(c, r] \subseteq D$, we have that$\|g\|_{B(c,r]} = \max\{|g(\zeta)| : \zeta \in B(c, r]\}$ exists, and

$$|I(\lambda)| = \left|\int_{\gamma_\lambda} g(\zeta)\, d\zeta\right| \le \int_{\gamma_\lambda} |g(\zeta)|\, |d\zeta| \le \|g\|_{B(c,r]}\, V(\gamma_\lambda) = \|g\|_{B(c,r]} 2\pi\lambda\, r.$$

Since I is a constant, if $\lim_{\lambda \to 0^+} I(\lambda) = 0$ then I is identically zero, in particular $I(1) = 0$; thus

$$\int_{\partial B(c,r]} g(\zeta)\, d\zeta = \int_{\partial B(c,r]} \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta = 0,$$

which implies

$$\int_{\partial B(c,r]} \frac{f(z)}{\zeta - z}\, d\zeta = \int_{\partial B(c,r]} \frac{f(\zeta)}{\zeta - z}\, d\zeta;$$

but

$$\int_{\partial B(c,r]} \frac{f(z)}{\zeta - z}\, d\zeta = f(z) \int_{\partial B(c,r]} \frac{d\zeta}{\zeta - z} = f(z)\, 2\pi i,$$

since the last integral is $2\pi i$ times the winding number of $\partial B(c,r]$ around z.

Remark. Of course the formula

$$\frac{1}{2\pi i} \int_{\partial B(c,r]} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

defines a function which is analytic also in $\mathbb{C} \setminus B(c,r]$; we leave it to the reader to show that in this open set the function is identically 0.

□

2.5.1. *Holomorphic functions are analytic.*

Corollary. *Let D be an open subset of $\mathbb{C}$, and let $f : D \to \mathbb{C}$ be holomorphic. Then f is analytic on D. Moreover, given $c \in D$, the Taylor series of f at c converges to f on every disc centered at c contained in D.*

Proof. Given $c \in D$, let $r > 0$ be such that $B(c,r]$ is contained in D. The Cauchy integral formula asserts that in the interior $B(c,r[$ of the disc $B(c,r]$ the function f is the Cauchy transform of $f_{|\partial B(c,r]}$; by the theorem 2.3.5, f is then analytic in $B(c,r[$, and

$$f(z) = \sum_{n=0}^{\infty} c_n (z-c)^n \quad \text{for } z \in B(c,r[, \text{ with} \quad c_n = \frac{1}{2\pi i} \int_{\partial B(c,r]} \frac{f(\zeta)}{(\zeta - c)^{n+1}}\, d\zeta = \frac{f^{(n)}(c)}{n!}.$$

□

Observe that the preceding corollary implies that the radius of convergence of the Taylor series at $c \in D$ of the holomorphic function $f : D \to \mathbb{C}$ is not less than the distance of c from the complement of D.

2.5.2. *Derivatives of holomorphic functions are holomorphic.*

Corollary. *Let D be an open subset of $\mathbb{C}$, and let $f : D \to \mathbb{C}$ be holomorphic. Then f admits complex derivatives of all order in D, and all these derivatives are then holomorphic functions on D. Moreover, if $B(c,r]$ is a disc contained in D, then for every $z \in B(c,r[$ and every $n \in \mathbb{N}$ we have*

Cauchy formula for derivatives $$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial B(c,r]} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta; \qquad |z - c| < r,\ n \in \mathbb{N}.$$

Proof. The formula is simply formula 2.3.6. We want to emphasize the fact that the derivative of a holomorphic function is analytic and hence also holomorphic. □

2.5.3. *Cauchy formula and holomorphy.* The Cauchy formula actually characterizes holomorphic functions among continuous functions:

Proposition. *Let D be an open subset of $\mathbb{C}$, and let $f : D \to \mathbb{C}$ be continuous. Then f is holomorphic on D if and only if for every closed disc $B(c,r]$ contained in D we have*

$$f(z) = \frac{1}{2\pi i} \int_{\partial B(c,r]} \frac{f(\zeta)}{\zeta - z}\, d\zeta \quad \textit{for every } z \in B(c,r[.$$

Proof. Necessity is 2.5; sufficiency is 2.5.2. □

2.5.4. *Cauchy formula and harmonic functions.* The Cauchy formula gives the value of a holomorphic function in the center of a disc as an average of the values on the boundary of the disc:

$$f(c)=\frac{1}{2\pi i}\int_{\partial B(c,r]}\frac{f(\zeta)}{\zeta-c}\,d\zeta=\frac{1}{2\pi i}\int_0^{2\pi}\frac{f(c+r\,e^{i\vartheta})}{r\,e^{i\vartheta}}\,ir\,e^{i\vartheta}\,d\vartheta=\frac{1}{2\pi}\int_0^{2\pi}f(c+r\,e^{i\vartheta})\,d\vartheta;$$

computing for the circle $\partial B(c,r]=\gamma(\vartheta)=c+r\,e^{i\vartheta}$ the "length element" $|d\zeta|=|d\gamma|=|\gamma'(\vartheta)|\,d\vartheta=r\,d\vartheta$ we can write

$$f(c)=\frac{1}{2\pi r}\int_{\partial B(c,r]}f(\zeta)\,|d\zeta|=\frac{1}{\mathrm{length}(\partial B(c,r])}\int_{\partial B(c,r]}f(\zeta)\,|d\zeta|,$$

thus realizing that the Cauchy formula implies that the value $f(c)$ in the center of the disc is the average of the values on the boundary; this separately happens also for the real and imaginary parts of f, which then enjoy the same property. This property, that the value in the center of the disc is the average of the values on the boundary, characterizes *harmonic* functions (but the proof is not easy, see e.g L.C. Evans, Partial Differential Equations); holomorphic functions are harmonic, but not vice–versa.

2.5.5. *Cauchy estimates.* The Cauchy formula for the derivatives gives immediately:

. Cauchy estimates. *Let $f:D\to\mathbb{C}$ be holomorphic, and let $B(c,r]$ be a closed disc contained in D. Then, for every $n\in\mathbb{N}$:*

$$|f^{(n)}(c)|\le\frac{n!}{r^n}\,\|f\|_{\partial B(c,r]}\quad\textit{where}\quad\|f\|_{\partial B(c,r]}=\max\{|f(\zeta)|:\zeta\in\partial B(c,r]\}.$$

Proof. It is simply the fundamental estimate applied to the formula for derivatives:

$$\begin{aligned}|f^{(n)}(c)|&=\frac{n!}{2\pi}\left|\int_{\partial B(c,r]}\frac{f(\zeta)}{(\zeta-c)^{n+1}}\,d\zeta\right|\le\frac{n!}{2\pi}\int_{\partial B(c,r]}\frac{|f(\zeta)|}{r^{n+1}}\,|d\zeta|\\&\le\frac{n!}{2\pi}\int_{\partial B(c,r]}\frac{\|f\|_{\partial B(c,r]}}{r^{n+1}}\,|d\zeta|=\frac{n!}{r^n}\,\|f\|_{\partial B(c,r]}.\end{aligned}$$

□

2.5.6. *A theorem of Liouville.* From the Cauchy estimates one immediately gets:

. Liouville's theorem *A bounded entire holomorphic function is constant.*

Proof. We know (2.5.1) that every entire function is on all of $\mathbb{C}$ the sum of its Mc Laurin series:

$$f(z)=c_0+c_1\,z+\dots+c_n\,z^n+\dots\qquad z\in\mathbb{C}.$$

with $c_n=f^{(n)}(0)/n!$. For every $r>0$ we can estimate c_n by the values of f on the circle $\partial(rB)=\{z\in\mathbb{C}:|z|=r\}$, obtaining

$$|c_n|=\frac{|f^{(n)}(0)|}{n!}\le\frac{\|f\|_{\partial(rB)}}{r^n}\le\frac{\|f\|_{\mathbb{C}}}{r^n}\quad\text{for every } r>0 \text{ and every } n\in\mathbb{N};$$

where we put $\|f\|_{\mathbb{C}}=\sup\{|f(z)|:z\in\mathbb{C}\}$; $\|f\|_{\mathbb{C}}$ is finite by hypothesis. Letting r tend to infinity, if $n\ge1$ also r^n tends to infinity. Thus $c_n=0$ for $n\ge1$, hence $f(z)=c_0$ for all $z\in\mathbb{C}$. □

The significance of Liouville's theorem is illustrated by the following corollary:

Corollary. *(*Fundamental theorem of algebra*) Every complex polynomial of degree at least one has at least one zero in $\mathbb{C}$.*

Proof. Let $p(z)=a_m\,z^m+\dots+a_0$ be a polynomial with $a_m\ne0$ and $m\ge1$. Observe that $\lim_{z\to\infty}p(z)=\infty$ (we can write $|p(z)|=|a_m||z|^m|1+a_{m-1}/(a_m\,z)+\dots+a_0/(a_m\,z^m)|$, from which the previous statement clearly follows). Then we have $|p(z)|\ge1$ for $|z|>R>0$. If $p(z)\ne0$ for every $z\in\mathbb{C}$, then $f(z)=1/p(z)$ is an entire function; if $|z|>R$ we have $|f(z)|=1/|p(z)|\le1$; and on the compact disc $R\,B=\{z\in\mathbb{C}:|z|\le R\}$ the continuous function $|f|$ has certainly a maximum. Thus f is bounded, hence constant by Liouville's theorem; but this implies that also $p(z)=1/f(z)$ is constant, a contradiction. □

Exercise 2.5.6.1. Prove the following generalization of Liouville's theorem: if $f\in H(\mathbb{C})$ is entire, and there exists α,k, with $k>0$ and $m\le\alpha<m+1$, $m\in\mathbb{N}$, such that $|f(z)|\le k\,|z|^\alpha$ for $|z|$ large enough (say $|z|>R$), then f is a polynomial of degree at most m.

2.5.7. *An existence theorem for zeroes.* For future use, and as a nice application of the first Cauchy estimate, we prove the following:

. *If $f : D \to \mathbb{C}$ is holomorphic, $B(c,r] \subseteq D$, and $|f(z)| > |f(c)|$ for every $z \in \partial B(c,r]$, then f has at least one zero in the interior $B(c,r[$ of $B(c,r]$.*

Proof. If not, the open set $D \smallsetminus Z(f)$, where $g(z) = 1/f(z)$ is defined and holomorphic, contains $B(c,r]$, so that the first Cauchy estimate can be applied to g on $B(c,r]$, giving $|g(c)| \le \|g\|_r$. But $|g(c)| = 1/|f(c)|$ and $\|g\|_r = 1/|f(z)|$ for some $z \in \partial B(c,r]$, so that $|f(c)| \ge |f(z)|$ for this $z \in \partial B(c,r]$, contradicting the hypothesis. □

2.6. Local primitives and holomorphy. It is now very easy to prove the following:

PROPOSITION. *Le D be open in $\mathbb{C}$, and let $f : D \to \mathbb{C}$ be continuous. Then f is holomorphic on D if and only if f has local primitives.*

Proof. If f has local primitives, by 2.5.2 it is locally holomorphic, i.e., holomorphic. And if f is holomorphic on D, it has primitives on every convex open subset, in particular on every open disc, of D. (2.3.4) □

COROLLARY. *Let D be open in $\mathbb{C}$ and let $f : D \to \mathbb{C}$ be continuous. Then f is holomorphic if and only if for every loop γ whose trace $[\gamma]$ is contained in a disc contained in D we have*

$$\int_\gamma f(z)\,dz = 0.$$

Proof. Since discs are convex this condition is equivalent to the assertion that the continuous function f has local primitives on all of D. □

EXERCISE 2.6.0.1. Let D be an open subset of $\mathbb{C}$, and let $f_n \in \mathcal{O}(D)$ be a sequence of functions holomorphic on D which converges uniformly on D to a function $f : D \to \mathbb{C}$. Then f is holomorphic on D.

Solution. Clearly f is continuous. And if γ is a loop with trace contained in some convex subset of D then $\int_\gamma f_n(z)\,dz = 0$ for every n; by uniform convergence on $[\gamma]$ we have

$$\int_\gamma f(z)\,dz = \lim_{n\to\infty} \int_\gamma f_n(z)\,dz = \lim_{n\to\infty} 0 = 0,$$

(2.1.0.6) so that, by the above corollary, f is holomorphic on D. □

2.6.1. *Morera's theorem.* Many call "Morera's condition" the above characterization of holomorphy via local primitives. In some textbooks this other condition, concerning integrals on the boundary of triangles, which we are going to describe, is called "Morera's theorem". Given three points $a, b, c \in \mathbb{C}$ the *(oriented) triangle* $T = \Delta(a,b,c)$ with vertices a, b, c is the convex hull of the set $\{a,b,c\}$, that is

$$\Delta(a,b,c) = \{\lambda\, a + \mu\, b + \nu\, c :\ \lambda + \mu + \nu = 1,\ \lambda,\ \mu,\ \nu \ge 0\}.$$

The *boundary* ∂T of the triangle $T = \Delta(a,b,c)$ is the closed polygonal path $[a,b,c,a]$ made of the three line segments $[a,b]$, $[b,c]$, $[c,a]$.

. MORERA'S THEOREM. *Let D be open in $\mathbb{C}$, and let $f : D \to \mathbb{C}$ be continuous. Then f is holomorphic on D if and only if for every triangle $T \subseteq D$ we have $\int_{\partial T} f(z)\,dz = 0$.*

Proof. Necessity If f is holomorphic in D, then f has local primitives on every open convex subset of D. Given a triangle $T \subseteq D$, there exists $\delta > 0$ such that $T_\delta = \{z \in \mathbb{C} :\ \mathrm{dist}(z,T) < \delta\}$ is contained in D (simply take $\delta < \min\{\mathrm{dist}(w, \mathbb{C} \smallsetminus D) :\ w \in T\}$), and this set is convex, as it is easy to verify; since $[\partial T] \subseteq T_\delta$, we have $\int_{\partial T} f(z)\,dz = 0$.

Sufficiency We prove that f has local primitives in D. Given $c \in D$, let $r > 0$ be such that $B(c,r[\subseteq D$. Define $F : B(c,r[\to \mathbb{C}$ by

$$F(z) = \int_{[c,z]} f(\zeta)\,d\zeta, \quad \text{where, as usual, } [c,z] \text{ denotes the segment of origin } c \text{ and extreme } z;$$

notice that the definition makes sense, since $B(c,r[$ is convex, and hence $[c,z] \subseteq B(c,r[\subseteq D$ for every $z \in B(c,r[$. Let us prove that $F'(z) = f(z)$ for every $z \in B(c,r[$. We have

$$\frac{F(w)-F(z)}{w-z} - f(z) = \frac{1}{w-z}\left(\int_{[c,w]} f(\zeta)\,d\zeta - \int_{[c,z]} f(\zeta)\,d\zeta\right) - f(z);$$

the triangle $T = \Delta(c,z,w)$ is contained in $B(c,r[$, which is convex; then

$$0 = \int_{\partial\Delta(c,z,w)} f(\zeta)\,d\zeta = \int_{[c,z]} f(\zeta)\,d\zeta + \int_{[z,w]} f(\zeta)\,d\zeta + \int_{[w,c]} f(\zeta)\,d\zeta,$$

which implies

$$\int_{[c,w]} f(\zeta)\,d\zeta = -\int_{[w,c]} f(\zeta)\,d\zeta = \int_{[c,z]} f(\zeta)\,d\zeta + \int_{[z,w]} f(\zeta)\,d\zeta;$$

thus we can write:

$$\frac{F(w)-F(z)}{w-z} - f(z) = \frac{F(w)-F(z)-f(z)(w-z)}{w-z} = \frac{1}{w-z}\left(\int_{[z,w]} f(\zeta)\,d\zeta - f(z)(w-z)\right);$$

it is now easy to conclude: since $f(z)(w-z) = \int_{[z,w]} f(z)\,d\zeta$, we obtain:

$$\left|\frac{F(w)-F(z)-f(z)(w-z)}{w-z}\right| = \frac{1}{|w-z|}\left|\int_{[z,w]} (f(\zeta)-f(z))\,d\zeta\right| \le \frac{1}{|w-z|}\int_{[z,w]} |f(\zeta)-f(z)|\,|d\zeta|;$$

since f is continuous at z, given $\varepsilon > 0$ we find $\delta > 0$ such that $B(z,\delta[\subseteq B(c,r[$ and $|f(\zeta)-f(z)| \le \varepsilon$ if $\zeta \in B(z,\delta[$; then, if $w \in B(z,\delta[$ we have:

$$\int_{[z,w]} |f(\zeta)-f(z)|\,|d\zeta| \le \varepsilon \int_{[z,w]} |d\zeta| = \varepsilon|w-z|,$$

($|w-z|$ is the length of the segment $[z,w]$); thus

$$\left|\frac{F(w)-F(z)-f(z)(w-z)}{w-z}\right| \le \frac{\varepsilon|w-z|}{|w-z|} = \varepsilon \quad \text{if } |w-z| < \delta.$$

□

2.6.2. *Goursat's theorem.*

Theorem. (Goursat) *Let $D \subseteq \mathbb{C}$ be open, and let $f : D \to \mathbb{C}$ be complex–differentiable at every point of D. Then f is holomorphic.*

Proof. We prove that for every triangle $T \subseteq D$ we have

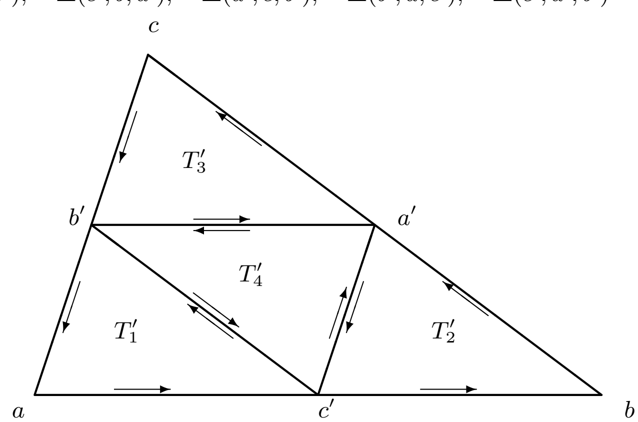
$$\int_{\partial T} f(z)\,dz = 0;$$

the conclusion then follows from Morera's theorem. Argue by contradiction and assume that there is a triangle $T = \Delta(a,b,c)$ such that $\int_{\partial T} f(z)\,dz = I \neq 0$. Taking the midpoints of the edges

$$a' = \frac{b+c}{2}; \qquad b' = \frac{a+c}{2}; \qquad c' = \frac{a+b}{2},$$

we consider the four triangles T_j', $j = 1,2,3,4$

$$\Delta(a,c',b'); \quad \Delta(c',b,a'); \quad \Delta(a',c,b'); \quad \Delta(b',a,c'); \quad \Delta(c',a',b')$$



A simple check proves that

$$I = \int_{\partial T} f(z)\,dz = \sum_{j=1}^{4} \int_{\partial T'_j} f(z)\,dz;$$

(the integrals on the inner edges cancel with each other). Then for at least one $j \in \{1,2,3,4\}$ we get

$$\left| \int_{\partial T'_j} f(z)\,dz \right| \ge \frac{|I|}{4};$$

let us call T_1 the triangle T'_j with least index $j \in \{1,2,3,4\}$ for which this holds; if $I_1 = \int_{\partial T_1} f(z)\,dz$ we then get $|I_1| \ge |I|/4$. Observe also that if $d = \operatorname{diam}(T)$(=longest side of T) we have $\operatorname{diam}(T_1) = d_1 = d/2$.

Repeating the procedure with T_1 in place of T we get another triangle $T_2 \subseteq T_1$, such that letting $I_2 = \int_{\partial T_2} f(z)\,dz$ we have $|I_2| \ge |I_1|/4$, and $\operatorname{diam}(T_2) = \operatorname{diam}(T_1)/2 = d/2^2$. Inductively we costruct a sequence $(T_k)_{k\in\mathbb{N}}$ of triangles such that:

$$T_{k+1} \subseteq T_k; \quad \operatorname{diam}(T_{k+1}) = \operatorname{diam}(T_k)/2; \quad |I_{k+1}| \ge |I_k|/4,$$

where for every $k = 1,2,3,\dots$ we set: $I_k = \int_{\partial T_k} f(z)\,dz$. Thus, for $k = 1,2,3,\dots$:

$$\operatorname{diam}(T_k) = \frac{d}{2^k}; \quad |I_k| \ge \frac{|I|}{4^k}, \quad \text{equivalently} \quad 4^k|I_k| \ge |I|.$$

Let us show that $\lim_{k\to\infty} 4^k|I_k| = 0$; this implies $|I| = 0$, a contradiction. By compactness of the triangles the sequence T_k has a non–empty intersection, i.e. there is $c \in \mathbb{C}$ such that $c \in T_k$, for every $k = 1,2,3,\dots$. Since f is differentiable at c we may write

$$f(z) = f(c) + f'(c)(z-c) + \sigma(z)\,(z-c) \quad \text{with} \quad \lim_{z\to c} \sigma(z) = 0.$$

Observe that for every triangle T_k we have

$$\int_{\partial T_k} f(z)\,dz = \int_{\partial T_k} (f(c) + f'(c)(z-c) + \sigma(z)\,(z-c))\,dz = \int_{\partial T_k} \sigma(z)\,(z-c)\,dz,$$

since $z \mapsto f(c) + f'(c)\,(z-c)$ is holomorphic, and its integral on ∂T_k is zero. Now, given $\varepsilon > 0$ there is $\delta > 0$ such that $|\sigma(z)| \le \varepsilon$ for $|z-c| \le \delta$. Since $\lim_{k\to\infty} d_k = 0$, for k large, say $k \ge k_\varepsilon$, we have $d_k = \operatorname{diam}(T_k) \le \delta$, so that for k large $T_k \subseteq B(c,\delta]$ and $|\sigma(z)| \le \varepsilon$ for $z \in T_k$.

Then

$$|I_k| = \left| \int_{\partial T_k} f(z)\,dz \right| \le \int_{\partial T_k} |\sigma(z)|\,|z-c|\,|dz| \le \int_{\partial T_k} d_k\,|\sigma(z)|\,|dz| \le d_k \int_{\partial T_k} \varepsilon\,|dz| \le \varepsilon\,d_k 3\,d_k \le 3\frac{d^2}{4^k}\,\varepsilon,$$

since $\operatorname{length}(\partial T_k) \le 3d_k$. We get $4^k|I_k| \le 3\varepsilon\,d^2$ for $k \ge k_\varepsilon$, and the proof ends. □

2.7. Identity theorem.

2.7.1. *Order af a holomorphic function at a point.* Recall that in first year calculus a definition of "functions of the same order as x tends to c" was given: two functions defined in a neighborhood of c were said to be of the same order as x tends to c if the limit $\lim_{x\to c} f(x)/g(x)$ existed finite and nonzero. And comparison scales were used, to help in the computation of limits, the most common being the scale of integer powers of the independent variable. The same definition is given for complex holomorphic functions; here the situation is much simpler, given the high regularity of these functions.

Definition. If D is an open subset of $\mathbb{C}$, $f : D \to \mathbb{C}$ is holomorphic, c is a point of D and m is an integer, we say that f has order m at c if

$$\lim_{z\to c} \frac{f(z)}{(z-c)^m} \quad \text{exists and is a nonzero complex number.}$$

If f has order m at c we write $\operatorname{ord}(f, c) = m$. Notice that f has order 0 at c if and only if $f(c) \neq 0$ (this is true for every f continuous at c). If f is identically 0 on neighborhood of c, then clearly the limit above is always zero, for every integer m, and the order cannot exist (sometimes the zero function is said to have $+\infty$ as order). A holomorphic function not locally zero at c always has an order at c, as we now prove. Notice that this contrasts with the real case, where a function does not necessarily have an order with respect to the scale of integer powers of the independent variable (e.g $x \mapsto x^2 \sin(1/x)$ does not have an order at 0 with respect to the powers x^m, for $x \to 0$). Clearly the order, if it exists, is unique (prove it!).

Proposition. *Let D be open in $\mathbb{C}$, let $f : D \to \mathbb{C}$ be holomorphic, and let c be a point in D. Then, unless f is identically zero on a neighborhood of c, there exist a function g holomorphic on D which does not vanish at c, and an integer $m \geq 0$ such that $f(z) = (z-c)^m g(z)$.*

Proof. The key fact here is analiticity of f: if $r = \operatorname{dist}(c, \mathbb{C} \setminus D)$ in the disc $B(c, r[\subseteq D$ we have:

$$f(z) = c_0 + c_1(z-c) + \cdots + c_n(z-c)^n + \ldots \qquad \text{with} \quad c_n = \frac{f^{(n)}(c)}{n!},\ n \in \mathbb{N}.$$

If all the coefficients c_n are 0, then f is identically zero on a neighborhood of c, and as observed above in this case f does not have an order at c. If not, let m be the smallest natural n for which $c_n \neq 0$. We then have in $B(c, r[$:

$$f(z) = c_m(z-c)^m + c_{m+1}(z-c)^{m+1} + \cdots = (z-c)^m \left(c_m + c_{m+1}(z-c) + \ldots\right)$$

define $g(z) = f(z)/(z-c)^m$ if $z \in D \setminus \{c\}$, $g(c) = c_m$; then g is holomorphic in $D \setminus \{c\}$; and on $B(c, r[$ we have

$$g(z) = c_m + c_{m+1}(z-c) + \cdots = \sum_{n=m}^{\infty} c_n (z-c)^{n-m},$$

the sum of a converging power series, and hence holomorphic. □

If f, g and m are as above, then clearly

$$\lim_{z\to c} \frac{f(z)}{(z-c)^m} = \lim_{z\to c} g(z) = g(c) \neq 0,$$

proving that $\operatorname{ord}(f, c) = m$. This proves that a non locally zero holomorphic function has an order. Moreover this order is also the smallest index of the non vanishing coefficients of the Taylor series of f at c, and hence also the smallest index of the derivatives of f not vanishing at c: $\operatorname{ord}(f, c) = 0$ means $f(c) \neq 0$, $\operatorname{ord}(f, c) = m > 0$ means that $f(c) = \cdots = f^{(m-1)}(c) = 0$, but $f^{(m)}(c) \neq 0$.

The reader is advised to work through the following trivial but important:

Exercise 2.7.1.1. Let f and g be holomorphic on a neighborhood of c. Prove that then $\operatorname{ord}(fg, c) = \operatorname{ord}(f, c) + \operatorname{ord}(g, c)$, and that $\operatorname{ord}(f+g, c) \geq \operatorname{ord}(f, c) \wedge \operatorname{ord}(g, c)$, with equality if $\operatorname{ord}(f, c) \neq \operatorname{ord}(g, c)$; it is assumed that $\operatorname{ord}(0, c) = +\infty$ and that $\infty + m = \infty + \infty = \infty$ for every integer m. Prove also that if $\operatorname{ord}(f, c) = m > 0$ then $\operatorname{ord}(f', c) = m - 1$.

Solution. Barring the trivial case in which one or both the functions f, g are identically 0 around c, we have $f(z) = (z-c)^m f_1(z)$ and $g(z) = (z-c)^n g_1(z)$, with f_1, g_1 holomorphic and nonzero at c, if $m = \operatorname{ord}(f, c)$ and $n = \operatorname{ord}(g, c)$. Thus $f(z)\, g(z) = (z-c)^{m+n} f_1(z)\, g_1(z)$. And if $m \neq n$, say $m < n$, then

$(f+g)(z) = (z-c)^m(f_1(z)+(z-c)^{n-m}\,g_1(z))$; computing at c we get $f_1(c)+(c-c)^{n-m}\,g_1(c) = f_1(c) \neq 0$; thus $\mathrm{ord}(f+g,c) = m = \min\{m,n\}$. Finally, assuming $m \geq 1$ we have

$$f'(z) = m(z-c)^{m-1}\,f_1(z) + (z-c)^m\,f_1'(z) = (z-c)^{m-1}(m\,f_1(z) + (z-c)\,f_1'(z)),$$

and since $m\,f_1(c) + (c-c)\,f_1'(z) = m\,f_1(c) \neq 0$ we conclude. Easy examples show that if $\mathrm{ord}(f,c) = \mathrm{ord}(g,c)$ then we may have $\mathrm{ord}(f+g,c) > \mathrm{ord}(f,c) = \mathrm{ord}(g,c)$; take e.g. $f(z) = z$ and $g(z) = -\sin z$; both functions have order 1 at 0, but $z - \sin z$ has order 3 at 0. □

2.7.2. *Zeroes, order, multiplicity.* If $D \subseteq \mathbb{C}$ and $f : D \to \mathbb{C}$ is continuous, then the *zero set* $Z_D(f) = Z(f) = \{z \in D : f(z) = 0\}$ is clearly a closed subset of D, in the relative topology of D. If f is only continuous, then any closed subset G of D is the zero–set of a continuous function (e.g, the function $z \mapsto \mathrm{dist}(z, G)$). The zero set of a holomorphic function has instead a very peculiar structure, which we now investigate. An element of $Z(f)$ is a zero of f; we say that $c \in Z(f)$ is an *isolated zero* of f if it is an isolated point of $Z(f)$, equivalently, $c \in Z(f)$ is not an accumulation point of $Z(f)$, meaning as usual that there exists a nbhd U of c such that $U \cap Z(f) = \{c\}$.

The *multiplicity* of f at c, written $\nu(f,c)$, is defined as $\mathrm{ord}(f - f(c), c)$: it is defined iff f is not locally constant at c; of course if $f(c) = 0$ then $\mathrm{ord}(f,c) = \nu(f,c)$, so that, whenever $\mathrm{ord}(f,c) \geq 1$ then $\mathrm{ord}(f,c) = \nu(f,c)$, and the integer $\mathrm{ord}(f,c) = \nu(f,c) = m \geq 1$ is also called *multiplicity of* c as a zero of f. This multiplicity of a zero is to be compared with the analogous notion given in algebra for a polynomial.

2.7.3. *The zero set lemma.* We first recall an elementary topological fact: in a metrizable space, more generally in a Hausdorff topological space, the set of accumulation points of a subset is a closed set. That is, if X is metrizable, $S \subseteq X$, and S' is the set of all accumulation points of S, then S' is closed in X (see below for the proof).

Lemma. *Let D be open in $\mathbb{C}$, and let $f : D \to \mathbb{C}$ be holomorphic. For a point $c \in D$ the following are equivalent:*

(i) *c is an accumulation point of the zero–set $Z(f)$ of f.*
(ii) *c is contained in the (topological) interior of $Z(f)$.*
(iii) *We have $f^{(n)}(c) = 0$ for all $n = 0, 1, 2, 3, \ldots$.*

Moreover, if D is a region (i.e. a connected open set) then f is identically zero on D if and only if $Z(f)$ has at least one accumulation point belonging to D.

Proof. Clearly $c \in Z(f)$ by continuity. (i) implies (ii): if c is not in the interior of $Z(f)$ then f has at c a finite order m, and by 2.7.1 we have $f(z) = (z-c)^m\,g(z)$, with g holomorphic on D and $g(c) \neq 0$. By continuity of g at c we have $g(z) \neq 0$ for all $z \in B(c, \delta[$, for $\delta > 0$ small enough; then clearly $Z(f) \cap B(c, \delta[= \{c\}$, proving that c is isolated in $Z(f)$.

(ii) implies (iii): if f is identically zero on some open disc centered at c then clearly all derivatives of f are zero on this disc, in particular at the center.

(iii) implies (i) If f and all its derivatives are 0 at c, then the order of f at c is infinite, that is, f is zero on a neighborhood of c, meaning that c is in the interior of $Z(f)$, in particular of accumulation for $Z(f)$.

We have proved that the set of accumulation points of $Z(f)$ in D, clearly closed in D as recalled above, is also the topological interior of $Z(f)$ in D, hence also open in D. Then this set is either empty, or a connected component of D. □

Proof that the set of accumulation points is closed In fact, remember that x is an accumulation points of S if and only if every nbhd of x contains infinitely many points of S (this is due to the metrizability of the space, it is true also in every Hausdorff topological space). Given an accumulation point p of S', every open nbhd U of p contains points of S'; since U is open, it is a nbhd of every one of its points, in particular of the points of S' it contains, and as such it contains infinitely many points of S. We have proved that every open nbhd of p contains infinitely many points of S. Then p is an accumulation point of S, and hence is in S'; and then S' is closed, since it contains all its accumulation points.

2.7.4. *The identity theorem.* There is a class of results which can collectively be described as "identity theorems" or "identity principles". Generally speaking these theorems infer the equality of two functions on a set by their equality on some special subset. For instance: for polynomial functions of one variable we know that two polynomial functions of degree at most m which agree on $m+1$ distinct points agree on the whole coefficient ring, when this is a commutative integral domain with infinitely many elements. A more complicated principle can be stated for polynomial functions of more than one variable. And two continuous functions on a topological space X with values in a Hausdorff space Y agree on X, if they agree on a dense subset of X. The list might continue. For holomorphic functions we have the following:

. THE IDENTITY THEOREM. *Let $D \subseteq \mathbb{C}$ be a region (i.e. D is an open connected subset of $\mathbb{C}$), and let $f, g : D \to \mathbb{C}$ be holomorphic. The following are equivalent*

(i) *f and g agree on D (i.e., $f(z) = g(z)$ for every $z \in D$).*
(ii) *f and g agree on some non empty open subset of D.*
(iii) *f and g agree on a subset of D which has an accumulation point in D.*
(iv) *For some point $c \in D$ we have $f^{(n)}(c) = g^{(n)}(c)$, for every $n \in \mathbb{N}$.*

Proof. Simply apply the zero–set lemma to the difference $h = f - g$. $\square$

REMARK. The most important equivalence is at any rate that of (i) and (iii): to verify equality of f and g on a region, it is enough to verify equality on a subset with an accumulation point belonging to D.

It is hard to overemphasize the importance of the identity theorem. It gives to every holomorphic function a precise individuality. For instance, any two entire functions which coincide on the reals must coincide on all of $\mathbb{C}$: the complex exponential is completely determined by the real exponential, if we want the extension to $\mathbb{C}$ to be complex–differentiable (which seems to be a natural requirement).

EXERCISE 2.7.4.1. Is there an entire function $f : \mathbb{C} \to \mathbb{C}$ such that $f(x) = 1/(1 + x^2)$ for $x \in \mathbb{R}$?

Solution. The formula $f(z) = 1/(1 + z^2)$ clearly defines a holomorphic function $f : \mathbb{C} \setminus \{-i, i\} \to \mathbb{C}$ wich extends f. Since $\mathbb{C} \setminus \{-i, i\}$ is connected, and its subset $\mathbb{R}$ has accumulation points in $\mathbb{C} \setminus \{-i, i\}$, this is the only holomorphic extension of $x \mapsto 1/(1 + x^2)$ over $\mathbb{C} \setminus \{-i, i\}$. There cannot be an entire extension: in fact $\lim_{z \to \pm i} 1/(1+z^2) = \infty$; any entire extension $f : \mathbb{C} \to \mathbb{C}$ has to coincide with $1/(1+z^2)$ for $z \in \mathbb{C} \setminus \{-i, i\}$, and this function does not have a continuous extension to $\mathbb{C}$. $\square$

The Riemann zeta function is defined on the open half–plane $E = \{\operatorname{Re} z > 1\}$ by the formula

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z},$$

where $n^z := \exp(z \log n)$ is the principal value ot the power n^z. It can be easily proved that this function is holomorphic on this half–plane; and a clever argument due to Riemann, involving also the Γ function, proves that ζ has a holomorphic extension to the punctured plane $\mathbb{C} \setminus \{1\}$, no longer expressible as the sum of the above series for $\operatorname{Re} z \leq 1$, of course; moreover $\lim_{z \to 1} \zeta(z) = \infty$. Since $\mathbb{C} \setminus \{1\}$ is connected, this is the *only possible* holomorphic extension of ζ to $\mathbb{C} \setminus \{1\}$; and ζ cannot have an entire extension to $\mathbb{C}$.

2.7.5. *Exercise: rings of holomorphic functions.* Given an open set $D \subseteq \mathbb{C}$ the set $\mathcal{O}(D)$ of all holomorphic functions $f : D \to \mathbb{C}$ is a commutative ring, and also a $\mathbb{C}-$algebra, under pointwise operations: clearly the constant 1 is the multiplicative identity, and the units (the multiplicatively invertible elements) are then the functions with an empty zero–set.

. *Let D be open in $\mathbb{C}$. The ring $\mathcal{O}(D)$ is an integral domain (i.e., it has no zero divisors) if and only if D is connected.*

Proof. Let D be a region; we assume $fg = 0$, with $f, g \in \mathcal{O}(D)$ and $f \neq 0$, and prove that $g = 0$. If $f \neq 0$, then $f(c) \neq 0$ for some $c \in D$; by continuity $f(z) \neq 0$ for all $z \in U$, where U is an open set containing c. Since $f(z)\, g(z) = 0$ for all $z \in D$, this implies $g(z) = 0$ for all $z \in U$; the identity theorem then implies that g is identically 0 on D.

To conclude, observe that if $D = A \cup B$, with A, B disjoint open non–empty, then $fg = 0$ if f is the function defined as $f(z) = 1$ for $z \in A$, $f(z) = 0$ for $z \in B$, and $g(z) = 1 - f(z)$; and f, g are both holomorphic and not the zero function. $\square$

As in any ring, given $f, g \in \mathcal{O}(D)$ we say that g divides f, or that f is divisible by g in the ring $\mathcal{O}(D)$ if there exists $h \in \mathcal{O}(D)$ such that $f = g\, h$. Then:

EXERCISE 2.7.5.1. Given $f \in \mathcal{O}(D)$ and $c \in D$ such that f is not locally zero at c, prove that $\operatorname{ord}(f, c) = m$ if and only if f is divisible by $(z - c)^m$, and non divisible by $(z - c)^{m+1}$.

Prove that if D is connected then $f, g \in \mathcal{O}(D)$ divide each other (that is f divides g and g divides f) iff they are *associate*, that is $f = u\, g$ where $u \in \mathcal{O}(D)$ is a unit. This in fact holds in every integral domain; assuming it holds in $\mathcal{O}(D)$, can you prove that D is connected?

But in the ring $\mathcal{O}(D)$ an element f can have infinitely many divisors which are non trivial, i.e. neither units of the ring, nor associates of f: for instance in $\mathcal{O}(\mathbb{C})$ the function $\sin z$ is divisible by all functions $z - k\pi$ (with $k \in \mathbb{Z}$), and more generally by every polynomial whose zeroes are simple and belong to the set $\mathbb{Z}\,\pi$: $\mathcal{O}(\mathbb{C})$ is not a gaussian (i.e., unique factorization) domain, it is not noetherian and it has no "finitistic" condition whatsoever.

Exercise 2.7.5.2. Find the order at 0 of the functions

$$z - \sin z; \quad \sin z - z\cos z; \quad \cosh z - \cos z; \quad e^{\alpha z} - \cos z; \quad 1 - e^{-z^2}.$$

Exercise 2.7.5.3. Discuss the existence of funtions holomorphic in a neighborhood of $z = 0$ with the required properties; write a formula to prove existence, otherwise prove non–existence:

(i) $f(1/n) = f(-1/n) = 1/n^2$, for infinitely many $n \in \mathbb{N}$.
(ii) $f(1/n) = (-1)^n/n$, for $n \geq 1$.
(iii) $f(1/n) = 1/(n^2 - 1)$, for infinitely many $n \in \mathbb{N}$.
(iv) $|f(1/n)| \leq 1/2^n$, for infinitely many $n \in \mathbb{N}$.

(a hint for (iv): evaluate $\mathrm{ord}(f, 0) \ldots$).

Exercise 2.7.5.4. Let D be a region, and lef $f : D \to \mathbb{C}$ be holomorphic. Assume that there exist $c \in D$ and $m > 0$ such that $f^{(k)}(c) = 0$ for $k > m$. Prove that then f is a polynomial of degree at most m.

Exercise 2.7.5.5. Let D be a region of $\mathbb{C}$. Denote by $U(D)$ the set of all holomorphic functions $f : D \to \mathbb{C}$ which are zero–free, that is, with empty zero–set; $U(D)$ is a group under pointwise multiplication of functions, the group of units (i.e., multiplicatively invertible elements) of the ring $\mathcal{O}(D)$ of all holomorphic functions on D. Let $\omega : U(D) \to \mathcal{O}(D)$ be the map defined by $\omega(f) = f'/f$: that is, ω assigns to every zero–free function its *logarithmic derivative* f'/f.

(i) Prove that ω is homomorphism of the multiplicative group $(U(D), \cdot)$ into the additive group $(\mathcal{O}(D), +)$,
(ii) Determine $\mathrm{Ker}(\omega)$.
(iii) Assume that $0 \in D$, and that $f, g \in U(D)$ are such that $f'(1/n)/f(1/n) = g'(1/n)/g(1/n)$ for infinitely many $n \in \mathbb{N}$; prove that then there exists $k \in \mathbb{C} \setminus \{0\}$ such that $f(z) = k\,g(z)$, for every $z \in D$.

2.8. The open mapping theorem. Given topological spaces X, Y, a (continuous) mapping $f : X \to Y$ is said to be *open* if for every open subset A of X the image set $f(A)$ is open in Y. This notion can be localized: given $c \in X$, we say that a function $f : X \to Y$ is open at c if for every neighborhood U of c in X the image $f(U)$ is a neighborhood of $f(c)$ in Y. It is clear that $f : X \to Y$ is open if and only if it is open at every point $x \in X$.

Remark. Beginners often confuse this concept of mapping open at c with continuity; but continuity at c means, backwards, that the *inverse image of every neighborhood of $f(c)$ is a neighborhood of c*. For instance the map $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sin(1/x)$ for $x \neq 0$, and $f(0) = 0$ is open, but not continuous, at $c = 0$. And the sine, considered as a function from $\mathbb{R}$ to $\mathbb{R}$ is everywhere continuous and even infinitely differentiable, but is not open at points of the set $\pi/2 + \mathbb{Z}\,\pi$ (and the image of the whole domain $\mathbb{R}$ under the real sine function, the interval $[-1, 1] = \sin(\mathbb{R})$, is closed, but not open, in $\mathbb{R}$).

2.8.1. *Open mapping theorem, local version.* Let us prove

. *If D is an open subset of $\mathbb{C}$, $f : D \to \mathbb{C}$ is holomorphic and not locally constant at $c \in D$, then f is open at c.*

Proof. Since f is not locally constant at c, c is an isolated zero for $f - f(c)$, that is, there is a disc $B(c, r] \subseteq D$ such that $Z(f - f(c)) \cap B(c, r] = \{c\}$; and every given neighborhood U of c contains such a disc. Then $\min\{|f(z) - f(c)| : z \in \partial B(c, r]\} = \mu > 0$. We prove that $B(f(c), \mu/2[$ is contained in $f(B(c, r[)$, thus concluding the proof. We use 2.5.7: given $w \in B(f(c), \mu/2[$ we have, for $z \in \partial B(c, r]$:

$$|f(z)-w| = |f(z)-f(c)+(f(c)-w)| \geq |f(z)-f(c)|-|f(c)-w| \geq \mu-|f(c)-w| > \mu-\frac{\mu}{2} = \frac{\mu}{2} > |f(c)-w|,$$

and 2.5.7 says then that $f - w$ has a zero in $B(c, r[$, that is, for some $z \in B(c, r[$ we have $f(z) = w$. □

As a corollary

. Open mapping theorem. *Let D be open in $\mathbb{C}$, and $f : D \to \mathbb{C}$ holomorphic and nowhere locally constant in D. Then f is an open mapping.*

Proof. Immediate from the local statement. □

2.8.2. *An equivalent statement.* Another way of stating the open mapping theorem is the following:

. *Let D be a region, and let $f : D \to \mathbb{C}$ be holomorphic. Assume that $E \subseteq D$ contains the image $f(D)$ of f, that is, $f(D) \subseteq E$. If for some $c \in D$ the point $f(c)$ belongs to the boundary ∂E of E, then f is a constant.*

Proof. If f is non constant then it is an open mapping, in particular $f(D)$ is an open subset of $\mathbb{C}$, hence $f(D)$, being contained in E by hypothesis, is contained in the topological interior $\mathrm{int}(E)$ of E, and contains no boundary point of E. □

Remark. As said before, the preceding statement, which we derived from the open mapping theorem, is actually equivalent to it. In fact if $f(D)$ is not open in $\mathbb{C}$, then there exists an element of $f(D)$, say $f(c)$, which is in the boundary of $f(D)$. This implies $f(z) = f(c)$ for every $z \in D$. We have proved that if D is a region and $f : D \to \mathbb{C}$ is holomorphic, then either $f(D)$ is open, or f is constant. Since on a region constancy ad local constancy are equivalent for holomorphic functions, the open mapping theorem is proved.

2.8.3. *An observation.* We have seen that a holomorphic map with zero imaginary or real part, or with constant modulus, is locally constant (1.3.2.2,1.3.2.4). The open mapping theorem clarifies the question: the real and imaginary parts of a non locally constant holomorphic map cannot verify a non–trivial relation of functional dependence, as the following explains.

Proposition. *Let D be a region, $f : D \to \mathbb{C}$ holomorphic; let E be an open set containing $f(D)$, and assume that $h : E \to \mathbb{R}$ is differentiable (in the real sense) and that its set of critical points has empty interior. Then $h \circ f$ is constant if and only if f is constant.*

Proof. Exercise (of course, the critical points of h are the points where the gradient of h is zero): simply note that $h \circ f$ is constant if and only if h is constant on $f(D)$, and that this set is open if f is non–constant . . . ; and a differentiable function constant on an open set has gradient identically zero on this open set. □

In particular, if $h : \mathbb{R}^2 \to \mathbb{R}$ is a non–constant polynomial function of two real variables, then $h \circ f$ is constant on a region D if and only if f is constant on D.

2.9. The maximum modulus property. For holomorphic functions the open mapping theorem gives an immediate proof of:

2.9.1.

. The maximum modulus theorem. *Let D be open in $\mathbb{C}$, $f : D \to \mathbb{C}$ be holomorphic. If $c \in D$ is a local maximum point for $|f|$, then f is locally constant at c.*

Proof. We recall that c is of local maximum for $|f|$ if there is a neighborhood U of c in D such that $|f(z)| \le |f(c)|$, for all $z \in U$. But if f is not locally constant at c then $f(U)$ must be a neighborhood of $f(c)$, by the open mapping theorem; and obviously any neighborhood of $f(c)$ contains some w with $|w| > |f(c)|$. □

Remark. Of course if f is locally constant at c then by the identity theorem f will be constant on all of the connected component of D containing c.

The same result holds for harmonic functions (of n real variables, here we prove it for two variables):

Exercise 2.9.1.1. Let D be a region, $f : D \to \mathbb{C}$ holomorphic. If $c \in D$ is a local maximum (or a local minimum) for the real part $u = \mathrm{Re}\, f$ of f, then f is constant in D.

Solution. If c is of local maximum for $u = \mathrm{Re}\, f$, pick an open disc u centered at c contained in D such that $u(z) \le u(c)$ for every $z \in U$; if E is the closed half–plane $E = \{w \in \mathbb{C} : \mathrm{Re}\, w \le u(c)\}$ we have $f(U) \subseteq E$, and $f(c)$ is on the boundary of E (which of course is the line $\mathrm{Re}\, w = u(c)$); thus f is constant on U, by 2.8.2, and by the identity theorem also on D. For the minimum argue in the same way, or use $-f$. □

EXERCISE 2.9.1.2. Let $D \subseteq \mathbb{C}$ be simply connected, and let $u : D \to \mathbb{R}$ be harmonic. By using the result of the previous exercise prove that if u has a local maximum $c \in D$ then u is constant in D (connectedness alone suffices, but requires a longer argument).

2.9.2. *Minimum modulus theorem.*

. MINIMUM MODULUS THEOREM *Let D be open in $\mathbb{C}$, and let $f : D \to \mathbb{C}$ be holomorphic. If $|f|$ has a local minimum at $c \in D$, then either $f(c) = 0$, or f is locally constant at c.*

Proof. Exercise: apply the open mapping theorem. □

EXERCISE 2.9.2.1. Let D be a bounded region of $\mathbb{C}$, and let $K = \bar{D} = D \cup \partial D$ be the closure of D in $\mathbb{C}$, and assume that $f : K \to \mathbb{C}$ is continuous on K and holomorphic on D.

(i) Prove that $\|f\|_K = \|f\|_{\partial K}$ (the maximum of the modulus of f on K coincides with the maximum of the modulus of f on the boundary of K)

(ii) Assume now that f has constant modulus on the boundary ∂D of D. Prove that either f has a zero in D, or f is constant in D.

Solution. (i) Since K is compact, and $|f|$ is continuous on K, both $\max\{|f(z)| : z \in K\} = \|f\|_K$ and $\max\{|f(z)| : z \in \partial K\} = \|f\|_{\partial K}$ exist; moreover K is closed, being compact, so $\partial K \subseteq K$, hence $\|f\|_{\partial K} \leq \|f\|_K$. Assume that $\|f\|_K = |f(c)|$ with $c \in D$. By the maximum modulus theorem f is then constant on the connected open set D, hence also on its closure $K = \bar{D}$; in particular $|f(z)| = |f(c)| = \|f\|_K$ for *every* $z \in K \supseteq \partial K$.

(ii) Assume that f is non constant. We prove that $Z_D(f) \neq \emptyset$. Let μ be the constant value of $|f(z)|$ for $z \in \partial K$. Then for every $z \in D$ we have $|f(z)| \leq \mu$, by what we just proved. It follows that there exists $c \in D$ such that $|f(c)| = \min\{|f(z)| : z \in K\}$. Of course c is also a local minimum for $|f|$ on D; by the minimum modulus theorem, since f is non constant, we have $f(c) = 0$.

□

REMARK. Statement (i) of the preceding exercise can be strengthened in this way: let K be a compact subset of $\mathbb{C}$, and assume that $f : K \to \mathbb{C}$ is continuous on K and holomorphic on $D = \operatorname{int}(K)$. Then $\|f\|_K = \|f\|_{\partial K}$ (i.e. no connectedness of D is needed; argue exactly as before; the only new thing to be proved is that if A is a connected component of D, then the boundary of A intersects the boundary of K).

EXERCISE 2.9.2.2. Let $f : D \to \mathbb{C}$ be holomorphic and non–constant on a region D containing the origin. For $0 < r < \operatorname{dist}(0, \mathbb{C} \setminus D)$ we can define the function $r \mapsto \|f\|_r = \|f\|_{\partial(rB)}$. Prove that it is strictly increasing. What can be said about the function $r \mapsto \min\{|f(z)| : |z| = r\}$? (distinguish the cases $Z(f) = \emptyset$ and $Z(f) \neq \emptyset$).

2.10. Laurent series. Functions which are holomorphic on discs have power series developments; those holomorphic on open annuli can be developed into two–sided power series, power series which involve also negative integral exponents.

2.10.1. *Preliminaries on two–sided power series.* Such a series, with initial point $c \in \mathbb{C}$ and coefficient sequence $(c_n)_{n\in\mathbb{Z}}$ (two–sided!) is written

$$\text{(Two sided power series)} \qquad \sum_{n=-\infty,\, n\in\mathbb{Z}}^{n=\infty} c_n\,(z-c)^n = \sum_{n=0}^{\infty} c_n\,(z-c)^n + \sum_{n=1}^{\infty} \frac{c_{-n}}{(z-c)^n} = R(z) + Q(z),$$

where $R(z)$, $Q(z)$ are respectively the power series

$$R(z) = \sum_{n=0}^{\infty} c_n\,(z-c)^n; \quad Q(z) = \sum_{n=1}^{\infty} \frac{c_{-n}}{(z-c)^n}.$$

Convergence (simple, absolute, uniform, normal) of the two–sided series means, by definition, convergence (simple, absolute, uniform, normal) of *both* the two series $R(z)$ and $Q(z)$. Notice that the second series is meaningless for $z = c$; if we set $w = 1/(z-c)$ we have the power series $\sum_{n=1}^{\infty} c_{-n}\, w^n$; if we denote by ρ the radius of convergence of this series, and put $\alpha = 1/\rho$ (with the customary convention $\alpha = 0$ if $\rho = +\infty$, $\alpha = +\infty$ if $\rho = 0$), then for the series $Q(z)$ we can say that the series converges absolutely if $|z-c| > \alpha$, and converges normally on every set $\mathbb{C} \smallsetminus B(c, r[$, for every $r > \alpha$. If β denotes the radius of convergence of the series R, we immediately have that:

- If $\alpha > \beta$ then the two sided power series never converges.
- If $\alpha = \beta = r$, with $0 < r < +\infty$, then the two sided power series may converge for some $z \in \partial B(c, r]$.
- If $\alpha < \beta$ then the the two sided power series converges absolutely for every z in the open annulus $B(c,]\alpha, \beta[) = \{z \in \mathbb{C} : \alpha < |z-c| < \beta\}$, and converges normally in every compact annulus $B(c, [r, s]) = \{z \in \mathbb{C} : r \le |z-c| \le s\}$ with $\alpha < r < s < \beta$.

Remark. In this case we have defined convergence of the two–sided series by requiring separate convergence of the two series indexed by the positive and the negative integers. This is not always the case: for instance, when discussing pointwise convergence of trigonometric series the definition of convergence is not equivalent to the one given here.

2.10.2. *The Laurent series development.*

Theorem. *Let D be open in $\mathbb{C}$, and let $f : D \to \mathbb{C}$ be holomorphic. Assume that D contains an open annulus $B(c,]\alpha, \beta[)$, where $0 \le \alpha < \beta \le +\infty$. Then there exists a two sided sequence $(c_n)_{n\in\mathbb{Z}}$ such that, for every $z \in B(c,]\alpha, \beta[)$ we have*

$$f(z) = \sum_{n=-\infty,\, n\in\mathbb{Z}}^{+\infty} c_n\,(z-c)^n \qquad \alpha < |z-c| < \beta.$$

Moreover the coefficients c_n are unique and we have

$$c_n = \frac{1}{2\pi i} \int_{\partial B(c,r]} \frac{f(\zeta)}{(\zeta-c)^{n+1}}\, d\zeta,$$

where r is any real number with $\alpha < r < \beta$.

Proof. First of all observe that if the function $\zeta \mapsto g(\zeta)$ is holomorphic in the annulus, then

$$\int_{\gamma_r} g(\zeta)\, d\zeta = \int_{\gamma_s} g(\zeta)\, d\zeta \quad \text{for all } r, s \text{ with } \alpha < r < s < \beta,$$

since clearly γ_r and γ_s are homotopic in the annulus (here $\gamma_r = \partial B(c, r])$. Given $z \in B(c,]\alpha, \beta[$ we pick r, s such that $\alpha < r < |z-c| < s < \beta$, and we consider the function $g : D \to \mathbb{C}$ defined by:

$$g(\zeta) = \frac{f(\zeta) - f(z)}{\zeta - z} \quad \text{if} \quad \zeta \in D \smallsetminus \{z\};\ g(z) = f'(z).$$

Then g is holomorphic on D: it is clearly holomorphic on $D \smallsetminus \{z\}$, and on every disk $B(z, \delta[\subseteq D$ the function $g(\zeta)$ coincides with the sum of the power series:

$$f'(z) + \frac{f''(z)}{2!}\,(\zeta - z) + \cdots + \frac{f^{(n)}(z)}{n!}\,(\zeta - z)^{n-1} + \ldots,$$

so that g is also holomorphic on this disk.

Thus we have

$$\int_{\gamma_r} \frac{f(\zeta)-f(z)}{\zeta - z}\, d\zeta = \int_{\gamma_s} \frac{f(\zeta)-f(z)}{\zeta - z}\, d\zeta,$$

whence

$$\int_{\gamma_r} \frac{f(\zeta)}{\zeta - z}\, d\zeta - 2\pi i \operatorname{ind}_{\gamma_r}(z)\, f(z) = \int_{\gamma_s} \frac{f(\zeta)}{\zeta - z}\, d\zeta - 2\pi i \operatorname{ind}_{\gamma_s}(z)\, f(z),$$

and $\operatorname{ind}_{\gamma_r}(z) = 0$, while $\operatorname{ind}_{\gamma_s}(z) = 1$ so that

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_s} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i}\int_{\gamma_r} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

We expand the function $1/(\zeta - z)$ in the first of these integrals into a geometric series:

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - c) - (z - c)} = \frac{1}{\zeta - c}\frac{1}{1 - (z-c)/(\zeta - c)} = \sum_{n=0}^{\infty} \frac{(z-c)^n}{(\zeta - c)^{n+1}} \quad (|z-c| < s)$$

and exactly as in 2.3.5 and 2.5.1 we get:

$$\frac{1}{2\pi i}\int_{\gamma_s} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \sum_{n=0}^{\infty} c_n (z-c)^n \quad \text{where} \quad c_n = \frac{1}{2\pi i}\int_{\gamma_s} \frac{f(\zeta)}{(\zeta - c)^{n+1}}\, d\zeta \quad (|z-c| < s).$$

If $\zeta \in [\gamma_r]$ then $|\zeta - c| = r < |z - c|$ and we have to collect $|z-c|$:

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - c) - (z - c)} = \frac{-1}{z - c}\,\frac{1}{1 - (\zeta-c)/(z - c)} = -\sum_{k=0}^{\infty} \frac{(\zeta-c)^k}{(z - c)^{k+1}} \quad (|z-c| > r);$$

multiplying by $-f(\zeta)/(2\pi i)$ and integrating over γ_r we get:

$$-\frac{1}{2\pi i}\int_{\gamma_r} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \int_{\gamma_r}\left(\sum_{k=0}^{\infty} \frac{1}{2\pi i}\, f(\zeta)\, \frac{(\zeta-c)^k}{(z-c)^{k+1}}\right) d\zeta =$$

(we can exchange the integral and the series since the series converges normally on $\partial B(c, r]$: in fact $|f(\zeta)||(\zeta - c)^k/(z-c)^{k+1}| \le (\|f\|_{\partial B(c,r]}/|z-c|)(r/|z-c|)^k$ for $\zeta \in \partial B(c, r]$, and $r/|z-c| < 1$)

$$\sum_{k=0}^{\infty}\left(\frac{1}{2\pi i}\int_{\gamma_r} f(\zeta)\,(\zeta - c)^k\, d\zeta\right)\frac{1}{(z-c)^{k+1}} \quad (|z-c| > r).$$

If in this sum we set $n = k+1$ and

$$c_{-n} = \frac{1}{2\pi i}\int_{\gamma_r} f(\zeta)\,(\zeta - c)^{n-1}\, d\zeta \left(= \frac{1}{2\pi i}\int_{\gamma_r} \frac{f(\zeta)}{(\zeta - c)^{-n+1}}\, d\zeta\right),$$

then

$$-\frac{1}{2\pi i}\int_{\gamma_r} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \sum_{n=1}^{\infty} \frac{c_{-n}}{(z-c)^n} \quad (|z-c| > r).$$

We then obtain

$$f(z) = \sum_{n=0}^{\infty} c_n (z-c)^n + \sum_{n=1}^{\infty} \frac{c_{-n}}{(z-c)^n} \quad (-r < |z-c| < s),$$

where for every $n \in \mathbb{Z}$:

$$c_n = \frac{1}{2\pi i}\int_{\gamma_t} \frac{f(\zeta)}{(\zeta - c)^{n+1}}\, d\zeta, \quad \text{here } \gamma_t = \partial B(c, t], \text{ for any } t \in]\alpha, \beta[.$$

Thus

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-c)^n \quad z \in B(c,]\alpha, \beta[.$$

This proves the existence of the Laurent development, as asserted. Assume now that there exists a two–sided sequence $(d_n)_{n\in\mathbb{Z}}$ of complex numbers such that we have:

$$f(z) = \sum_{n=-\infty}^{\infty} d_n (z-c)^n \quad z \in B(c,]\alpha, \beta[.$$

Given $m \in \mathbb{Z}$, multiply both sides for $1/(z-c)^{m+1}$, obtaining

$$\frac{f(z)}{(z-c)^{m+1}} = \sum_{n=-\infty}^{\infty} d_n (z-c)^{n-m-1} \quad z \in B(c,]\alpha, \beta[.$$

The two sided series converges normally on every compact subset of the annulus, in particular on $\partial B(c,t]$ if $\alpha < t < \beta$. Integrating both sides of the preceding equality, the right–hand side may then be integrated term by term; and we have

$$\int_{\partial B(c,t]} (z-c)^{n-m-1}\, dz = 0 \quad \text{unless} \quad n-m-1=-1,$$

i.e. unless $n = m$, in which case

$$\int_{\partial B(c,t]} (z-c)^{-1}\, dz = 2\pi i.$$

Thus we get

$$\int_{\partial B(c,t]} \frac{f(z)}{(z-c)^{m+1}}\, dz = d_m\, 2\pi i,$$

and uniqueness of the Laurent development is proved. (see Analisi Due, 10.13.1). □

2.11. Isolated singularities. Assume that D is open in $\mathbb{C}$ and that $f : D \to \mathbb{C}$ is holomorphic.

Definition. A point $c \in \mathbb{C} \setminus D$ is said to be an *isolated singularity* for the function f if f is holomorphic on a punctured neighborhood $U \setminus \{c\}$ of c: that is, the domain D of f contains $U \setminus \{c\}$ for some neighborhood U of c (Analisi Due, 10.14)

Being an isolated singularity is only a property of the domain D of the function. We invite the reader to prove that:

. *Let D be open in $\mathbb{C}$. Then the following are equivalent, for every $c \in \mathbb{C}$:*

(i) *c is an isolated point of the boundary ∂D of D.*
(ii) *$c \notin D$, and there exists a neighborhood U of c in $\mathbb{C}$ such that $U \setminus \{c\}$ is contained in D.*
(iii) *$c \notin D$, and $D \cup \{c\}$ is open in $\mathbb{C}$.*

(the easy proof is at the end of the section).

We do not discuss singularities in general: tentatively, given $f : D \to \mathbb{C}$ holomophic, with D open in $\mathbb{C}$, the points of the boundary ∂D of D might be called singular for f. But let's leave the subject.

2.11.1. *Classification of isolated singularities.* If c is an isolated singularity for f the domain D of f contains a punctured disk $B(c,]0, \beta[$, for some $\beta > 0$. We can write, for z this punctured disk:

$$f(z) = \sum_{n=0}^{\infty} c_n (z-c)^n + \sum_{n=1}^{\infty} \frac{c_{-n}}{(z-c)^n} = R_c(z) + Q_c(z),$$

where $R_c(z)$ is the *regular part* of f at c, and Q_c is the *singular part* of f at c. The series defining the singular part converges for values of $|z-c|$ nonzero, but as small as desired; thus the series $\sum_{n=1}^{\infty} c_{-n}\, w^n$ has an infinite radius of convergence; the formula

$$Q_c(z) = \sum_{n=1}^{\infty} \frac{c_{-n}}{(z-c)^n},$$

actually makes sense for every $z \in \mathbb{C} \setminus \{c\}$, and defines a function holomorphic there. The formula $R_c(z) = \sum_{n=0}^{\infty} c_n (z-c)^n$ defines a function holomorphic on a disk of radius at least $\beta > 0$ centered at c, and the formula $z \mapsto f(z) - Q_c(z)$ extends R_c to a function holomorphic on $D \cup \{c\}$. The singular part is used to give a classification of the isolated singularities:

- If Q_c is identically zero, that is, if $c_{-n} = 0$ for every $n = 1, 2, 3, \ldots$ then c is said to be a *removable singularity* (italiano: singolarità eliminabile).
- If Q_c is non zero, but has only a finite number of nonzero terms, that is there exists a maximum $m \in \mathbb{N}$ such that $c_{-m} \neq 0$, so that

  $$Q_c(z) = \frac{c_{-1}}{z-c} + \cdots + \frac{c_{-m}}{(z-c)^m},$$

  then c is said to be a *pole of order m* for f.

- In the last remaining case, in which infinitely many terms of the singular part are nonzero, c is called *essential singularity.*

EXAMPLE 2.11.1.1. The following functions $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ have $z = 0$ as isolated singularity:

(i) $f(z) = \sin z/z$; removable singularity; in fact

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!}$$

has only the regular part around 0.

(ii) $f(z) = \cos z/z^m$, with $m \geq 1$ integer, has a pole of order m at 0; in fact we have

$$\frac{\cos z}{z^m} = \frac{1}{z^m} - \frac{1/2!}{z^m - 2} + \cdots + \sum_{n \geq m/2,\, n \in \mathbb{N}} \frac{z^{2n-m}}{(2n)!};$$

(iii) $f(z) = \cos(1/z)$ has an essential singularity at $z = 0$:

$$\cos(1/z) = 1 - \frac{1/2!}{z^2} + \cdots = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{1/(2n!)}{z^{2n}}.$$

2.11.2. *Removable singularities.* When c is a removable singularity there is a holomorphic function which extends f to c, the regular part $R_c(z)$, which coincides in this case with f on a punctured neighborhood of c. The condition that $\lim_{z \to c} f(z)$ exists in $\mathbb{C}$ is clearly necessary; we shall see that it is also sufficient (and actually much less is needed).

PROPOSITION. *Let D be open in $\mathbb{C}$, and let c be an isolated singularity for the holomorphic function $f : D \to \mathbb{C}$. The following are then equivalent:*

(i) *c is a removable singularity.*
(ii) *$\lim_{z \to c} f(z)$ exists in $\mathbb{C}$.*
(iii) *f is bounded on a neighborhood of c.*
(iv) *$\lim_{z \to c} (z - c) f(z) = 0$.*

Proof. (i) implies (ii), (ii) implies (iii), (iii) implies (iv) are trivial. Let us prove that (iv) implies (i). We have to prove that $c_{-n} = 0$ for all $n \geq 1$. We have, for every $r > 0$ small enough:

$$c_{-n} = \frac{1}{2\pi i} \int_{\partial B(c,r]} (\zeta - c)^{n-1} f(\zeta)\, d\zeta;$$

By the hypothesis, $\lim_{\zeta \to c} (\zeta - c)((\zeta - c)^{n-1} f(\zeta)) = 0$ if $n \geq 1$. By the small circle lemma we conclude that

$$\lim_{r \to 0^+} \int_{\partial B(c,r]} (\zeta - c)^{n-1} f(\zeta)\, d\zeta = 0;$$

thus $c_{-n} = 0$, for $n = 1, 2, 3, \ldots$.

□

2.11.3. *Polar singularities.*

PROPOSITION. *Let D be open in $\mathbb{C}$, and let c be an isolated singularity for the holomorphic function $f : D \to \mathbb{C}$. The following are then equivalent:*

(i) *c is a polar singularity.*
(ii) *$\lim_{z \to c} f(z) = \infty$ ($= \infty_{\mathbb{C}}$, the point at infinity added to $\mathbb{C}$)*
(iii) *There exist an integer $m \geq 1$ such that $\lim_{z \to c} (z - c)^m f(z)$ exists and is non zero in $\mathbb{C}$ (i.e., f is infinite of the same order of $1/(z - c)^m$).*

Moreover the integer m in (iii) is the order of c as a pole for f.

Proof. (i) implies (ii): we have $f(z) = R_c(z) + (c_{-1}/(z - c) + \cdots + c_{-m}/(z - c)^m)$ with $c_{-m} \neq 0$ and $m \geq 1$; then

$$f(z) = \frac{c_{-m}}{(z - c)^m}(1 + u(z)) \quad \text{where } u(c) = 0,\ u \text{ holomorphic near } c,$$

from which it easily follows that f diverges to ∞ as z tends to c.

(ii) implies (iii). The function f clearly is non zero in an open punctured neighborhood of c; then $1/f(z)$ can be defined in this set and has c as an isolated singularity, removable because $\lim_{z \to c} 1/f(z) = 0$; if g extends $1/f$ to c, clearly we have $g(c) = 0$, but g is zero only on c, thus g has finite order at c; if

$\operatorname{ord}(g, c) = m$ we have then $m \geq 1$, and $g(z) = (z-c)^m k(z)$, with k holomorphic and $k(c) \neq 0$; from this we deduce, in a punctured neigborhood of c:

$$1 = (z-c)^m k(z)\, f(z) \iff (z-c)^m f(z) = \frac{1}{k(z)} = h(z),$$

which immediately implies that $\lim_{z \to c}(z-c)^m f(z) = h(c) \neq 0$, and proves that (ii) implies (iii).

(iii) implies (i): the function $h(z) = (z-c)^m\, f(z)$ has a removable singularity at c, with $h(c) = \lim_{z\to c}(z-c)^m\, f(z) \neq 0$. Writing the Taylor series for h we deduce the Laurent development of f at c:

$$\begin{aligned}&(z-c)^m\, f(z) = h_0 + h_1\,(z-c) + \cdots + h_{m-1}\,(z-c)^{m-1} + h_m\,(z-c)^m + \cdots \Rightarrow \\ &f(z) = \frac{h_0}{(z-c)^m} + \cdots + \frac{h_{m-1}}{z-c} + h_m + \dots,\end{aligned}$$

and since $h_0 = h(0) \neq 0$, this representation of f says that c is for f a pole of order m. □

Remark. For future use, notice that the coefficient of $1/(z-c)$ in the Laurent expansion of f is the coefficient $h_{m-1} = h^{(m-1)}(c)/(m-1)!$ in the Taylor expansion of $h(z) = (z-c)^m\, f(z)$.

2.11.4. *Essential singularities.* The only remaining case, in which the singular part $Q_c(z)$ is not a rational function, but it has infinitely many non zero coefficients c_{-n}, is then the case in which the limit $\lim_{z\to c} f(z)$ does not exist, neither finite nor infinite. The behavior is very strange:

. The Casorati–Weierstrass theorem *If the holomorphic function f has an essential singularity at c, then $f(U \setminus \{c\})$ is dense in $\mathbb{C}$, for every neighborhood U of c.*

Proof. If not, there exist $w \in \mathbb{C}$ and $\delta > 0$ such that we have $|f(z) - w| \geq \delta$ for every z in some punctured neighborhood $U \setminus \{c\}$ of c. It follows that $g(z) = 1/(f(z) - w)$ is bounded (by $1/\delta$) in $U \setminus \{c\}$, hence it has a removable singularity at c. But then $f = w + 1/g$ has either a pole or a removable singularity at c, a contradiction. □

Remark. A much stronger result can be proved. Picard's theorem asserts that in every (punctured) neighborhood of an essential singularity a holomorphic function assumes all complex values, but at most one. The proof is however much more difficult than that of the preceding theorem.

2.11.5. *Proof of 2.11.*

Proof. (i) implies (ii) Assume that $c \in \partial D$ is isolated in ∂D. First, $c \notin D$, because D is open, hence disjoint from its boundary; and there is an open disc $U = B(c, \delta[$ about c such that $U \cap \partial D = \{c\}$. Then the punctured disc $U \setminus \{c\}$ is contained in D: in fact it intersects D because $U \cap D$ cannot be empty, hence also $(U \setminus \{c\}) \cap D \neq \emptyset$. And since $U \setminus \{c\}$ is path–connected, if it contained some point not in D it would contain some point of the boundary of D ("custom's passage" theorem), contrary to assumptions. So $U \setminus \{c\} \subseteq D$. (ii) implies (iii). Trivial: $D \cup \{c\} = D \cup (U \setminus \{c\}) \cup \{c\} = D \cup U$, which is a union ot two open sets, hence open.

(iii) implies (i). Being open, $D \cup \{c\}$ is a neighborhood of c, and contains no point of ∂D other than c. □

2.12. Order at isolated singularities. The preceding section suggests the convenience of extending the definition of $\operatorname{ord}(f, c)$ to the case in which c is a removable or a polar singularity. In general, we say that $\operatorname{ord}(f, c) = m \in \mathbb{Z}$ means that c is a regular point, or an isolated singularity, and

$$\lim_{z\to c} \frac{f(z)}{(z-c)^m} \quad \text{exists in } \mathbb{C} \text{ and is non zero.}$$

Equivalently:

. *If c is a regular point, or an isolated singularity for the holomorphic function f, then*

$$\operatorname{ord}(f, c) = m \in \mathbb{Z} \quad \textit{is equivalent to the assertion:}$$

there exists g, holomorphic in a neighborhood of c and non–zero in c, such that

$$f(z) = (z-c)^m\, g(z) \quad \textit{for all } z \textit{ in a punctured neighborhood of } c\textit{; } g(c) \neq 0.$$

The function $(z-c)^m\, g(c)$ is the principal part *of f at c.*

Thus $\operatorname{ord}(f,c) \geq 0$ exactly for regular points and removable singularities; and once the singularity (when there) has been removed: $\operatorname{ord}(f,c) = 0$ means $f(c) \neq 0$; $\operatorname{ord}(f,c) = m > 0$ means that $f(c) = f'(c) = \cdots = f^{(m-1)}(c) = 0$, but $f^{(m)}(c) \neq 0$; $\operatorname{ord}(f,c) = m < 0$ means that c is a pole for f of order $-m$. It is an easy exercise left to the reader the verification of the following facts:

$$\operatorname{ord}(fg,c) = \operatorname{ord}(f,c) + \operatorname{ord}(g,c); \quad \operatorname{ord}(1/f,c) = -\operatorname{ord}(f,c).$$

And if $\operatorname{ord}(f,c) \neq 0$, then $\operatorname{ord}(f',c) = \operatorname{ord}(f,c) - 1$.

We can extend the notion of order to the function which is 0 on a neighborhood of c, by declaring $\operatorname{ord}(0,c) = +\infty$. With this definition we have, additively:

$$\operatorname{ord}(f+g,c) \geq \min\{\operatorname{ord}(f,c), \operatorname{ord}(g,c)\} \quad \text{with equality if } \operatorname{ord}(f,c) \neq \operatorname{ord}(g,c).$$

Thus, the only isolated singularities for which the order is not defined are the essential singularities.

Very often a holomorphic function f is given as the quotient of two holomorphic functions, $f = g/h$, with h not identically 0. Then the zeroes of h are isolated, and are isolated singularities of f. To determine $\operatorname{ord}(f,c)$ with $c \in Z(h)$ we simply compute $\operatorname{ord}(g,c) - \operatorname{ord}(h,c)$.

EXERCISE 2.12.0.1. Let D be a region in $\mathbb{C}$. Prove that if $f, g \in \mathcal{O}(D)$ then f divides g in $\mathcal{O}(D)$ if and only if $\operatorname{ord}(f,z) \leq \operatorname{ord}(g,z)$ for every $z \in D$.

Solution. Recall that f divides g in $\mathcal{O}(D)$ means that there exists $h \in \mathcal{O}(D)$ such that $g = f\,h$. Since $\operatorname{ord}(g,z) = \operatorname{ord}(f,z) + \operatorname{ord}(h,z) \geq \operatorname{ord}(f,z)$ the condition is clearly necessary. If f is the zero function then it divides only itself; and if f is not identically zero then $Z(f)$ is a discrete subset of the region D, so that $g/f : D \setminus Z(f) \to \mathbb{C}$ has every point of $Z(f)$ as an isolated singularity; the hypothesis $\operatorname{ord}(f,z) \leq \operatorname{ord}(g,z)$ shows that all these singularities are removable, so that g/f extends to $h \in \mathcal{O}(D)$. □

2.13. Residues. At an isolated singularity c for the holomorphic function $f : D \to \mathbb{C}$ the coefficient c_{-1} of the singular part is particularly important for computational purposes; we write $\operatorname{Res}(f,c)$ for it, and we call it the *residue*, sometimes the *integral residue*, of f at c. Recall that we have:

$$\operatorname{Res}(f,c) = \frac{1}{2\pi i} \int_{\partial B(c,r]} f(z)\, dz,$$

for every $r > 0$ such that $B(c,r] \setminus \{c\} \subseteq D$. In general we do *not* compute the integral to compute the residue; the computation is done by other means, and used to compute integrals related to f.

2.13.1. *First order poles.* The computation is of course simpler for first order poles: in this case we have

$$f(z) = R_c(z) + \frac{\operatorname{Res}(f,c)}{z-c} \quad \text{thus} \quad (z-c)\,f(z) = (z-c)\,R_c(z) + \operatorname{Res}(f,c)$$

so that $\operatorname{Res}(f,c) = \lim_{z\to c}(z-c)\,f(z)$ (notice that $(z-c)\,R_c(z)$ tends to $(c-c)\,R_c(c) = 0$). Consequently:

. *If g, h are holomorphic in a neighborhood of c, with $g(c) \neq 0$, $h(c) = 0$, $h'(c) \neq 0$, then f has a first order pole at c, with residue*

$$\operatorname{Res}(f,c) = \frac{g(c)}{h'(c)}.$$

Proof. That c is a first order pole is clear; then, by the above observation:

$$\lim_{z\to c}(z-c)\,f(z) = \lim_{z\to c} \frac{g(z)}{h(z)/(z-c)} = \lim_{z\to c} \frac{g(z)}{(h(z)-h(c))/(z-c)} = \frac{g(c)}{h'(c)}.$$

□

2.13.2. *Other singularities.* Recall that if c is a pole of order m for f, then $h(z) = (z-c)^m f(z)$ is holomorphic around c, with $h(c) \neq 0$, and the singular part of f around c is

$$Q_c(z) = \frac{h(c)}{(z-c)^m} + \cdots + \frac{h^{(m-1)}(c)/(m-1)!}{z-c}, \quad \text{so that} \quad \operatorname{Res}(f,c) = \frac{h^{(m-1)}(c)}{(m-1)!}$$

(see 2.11.3).

There are no simple general methods for essential singularities.

2.13.3. *Examples and exercises.*

EXERCISE 2.13.3.1. Given the function

$$f(z) = \pi \frac{\operatorname{cotan}(\pi z)}{z^2 - c^2} = \pi \frac{\cos(\pi z)}{\sin(\pi z)(z^2 - c^2)} \qquad \text{(where } c \in \mathbb{C} \smallsetminus \mathbb{Z} \text{ is a constant)},$$

find and classify the singularities. Determine residues.

Solution. The denominator ha simple zeroes at $\pm c$ and at every integer. At the integers the numerator is nonzero; then the integers are simple poles and:

$$\operatorname{Res}(f, n) = \pi \frac{\cos(\pi n)/(n^2 - c^2)}{\pi \cos(\pi n)} = \frac{1}{n^2 - c^2}.$$

The numerator is zero at $\pm c$ if and only if $c = \pm 1/2 + n$, with $n \in \mathbb{Z}$. Excluding this case, $\pm c$ are simple poles and

$$\operatorname{Res}(f, \pm c) = \frac{\pi \operatorname{cotan}(\pi(\pm c)}{2(\pm c)} = \frac{\pi}{c} \operatorname{cotan}(\pi c);$$

If $c = \pm 1/2 + n$ then $\pm c$ are removable singularities, so that the residue at them is 0. □

EXERCISE 2.13.3.2. If $\alpha, c \in \mathbb{C}$ are constant, and $p \geq 1$ is an integer, find singularities and residues of $f(z) = e^{\alpha z}/(z - c)^p$.

Solution. One pole of order p at c. Since $h(z) = (z - c)^p f(z) = e^{\alpha z}$, the derivatives of h at c are easy to find, being $h^{(n)}(c) = \alpha^n e^{\alpha c}$. The residue is then

$$\operatorname{Res}(e^{\alpha z}/(z - c)^p, c) = \frac{\alpha^{p-1}}{(p - 1)!} e^{\alpha c}.$$

□

But at poles of order larger than one things are not always so simple, as the next example shows.

EXERCISE 2.13.3.3. Find the singularities of the function

$$f_\alpha(z) = \frac{e^{\alpha z}}{\cosh^2 z} \qquad \alpha \in \mathbb{C} \text{ constant},$$

and find the residue at the singularity with the smallest positive imaginary part.

Solution. The function cosh has simple zeroes at $z_k = i(\pi/2 + k\pi)$ (use the identity $\cosh z = \cos(iz)$ to find the zeroes); then the derivative $\sinh z$ at these points is

$$\sinh(i(\pi/2 + k\pi)) = i \sin(\pi/2 + k\pi) = i(-1)^k;$$

thus f_α has two–order poles at $z_k = i(\pi/2 + k\pi)$. We find the residue at $i\pi/2$; simply consider the function

$$h(z) = (z - i\pi/2)^2 \frac{e^{\alpha z}}{\cosh^2 z},$$

which has a removable singularity at $i\pi/2$ and we compute $h'(i\pi/2) = \operatorname{Res}(f_\alpha, i\pi/2)$. We have

$$h'(z) = \alpha\, e^{\alpha z} \frac{(z - i\pi/2)^2}{\cosh^2 z} + e^{\alpha z} \frac{2(z - i\pi/2)\cosh^2 z - (z - i\pi/2)^2 2 \cosh z \sinh z}{\cosh^4 z};$$

we have to compute the limit of this expression as z tends to $i\pi/2$. Notice first that

$$\lim_{z \to i\pi/2} \frac{\cosh z}{z - i\pi/2} = \sinh(i\pi/2) = i,$$

so that the first addend tends to $-\alpha e^{i\alpha\pi/2}$; the second, without the factor $e^{\alpha z}$ is

$$\frac{2(z - i\pi/2)\cosh z}{\cosh^2 z} \, \frac{\cosh z - (z - i\pi/2)\sinh z}{\cosh^2 z};$$

the factor $\dfrac{2(z - i\pi/2)\cosh z}{\cosh^2 z} = \dfrac{2(z - i\pi/2)}{\cosh z}$ tends to $2/i = -2i$; we are left to compute (we substitute $(i(z - i\pi/2))^2 = -(z - i\pi/2)^2$ to $\cosh^2 z$):

$$\lim_{z \to i\pi/2} \frac{\cosh z - (z - i\pi/2)\sinh z}{-(z - i\pi/2)^2} = (\text{Hôpital}) = \lim_{z \to i\pi/2} \frac{\sinh z - \sinh z - (z - i\pi/2)\cosh z}{-2(z - i\pi/2)} =$$
$$\lim_{z \to i\pi/2} (-2 \cosh z) = 0$$

We find then:

$$\operatorname{Res}(f_\alpha, i\pi/2) = -\alpha e^{i\alpha\pi/2}.$$

□

EXERCISE 2.13.3.4. Let c be an isolated singularity for f holomorphic in $D\smallsetminus\{c\}$ (with $c\in D$). Prove that $\operatorname{Res}(f,c)=0$ if and only if the function f admits a primitive in some punctured neighborhood of c.

Solution. If f admits a primitive in $B(c,r[\smallsetminus\{c\}$ then clearly $\int_{\partial B(c,\delta]} f(z)\,dz = 0$ for every δ with $0<\delta<r$, so $\operatorname{Res}(f,c)=0$. And if $\operatorname{Res}(f,c)=0$ then the Laurent development of f in a punctured neighborhood of c is

$$f(z) = \sum_{n=0}^{\infty} c_n(z-c)^n + \sum_{n=2}^{\infty}\frac{c_{-n}}{(z-c)^n},$$

so that

$$F(z) = \sum_{n=0}^{\infty}\frac{c_n}{n+1}\,(z-c)^{n+1} + \sum_{n=2}^{\infty}\frac{c_{-n}}{(1-n)(z-c)^{n-1}},$$

clearly convergent in the same annulus, by well–known theorems on power series, is the Laurent development of a primitive of f in that annulus, a punctured neighborhood of c. □

2.13.4. *Change of variables for residues.*

EXERCISE 2.13.4.1. Let D be open in $\mathbb{C}$, $c\in D$, $f,g : D\smallsetminus\{c\}\to\mathbb{C}$ holomorphic. Prove that $\operatorname{Res}(f+g,c)=\operatorname{Res}(f,c)+\operatorname{Res}(g,c)$. Prove that $\operatorname{Res}(f,c)=k$ if and only if there exists a function F holomorphic on some punctured neigborhood $U\smallsetminus\{c\}$ of c such that $f(z)=F'(z)+k/(z-c)$ for $z\in U$ (cfr. 2.13.3.4). Assume now that $\varphi : E\to D$ is holomorphic, with $\varphi(a)=c$ for some $a\in E$, and $\varphi'(a)\neq 0$. Prove the formula

CHANGE OF VARIABLE FOR RESIDUES $\qquad \operatorname{Res}(f,c)=\operatorname{Res}((f\circ\varphi)\,\varphi',a)$

(use the preceding representation of f at c; this formula shows that, like the integral on a curve, the notion of residue should be given for differential forms, rather than functions).

2.14. Nullhomologous loops. Given a region D, a loop γ in D (recall that this means: $\gamma : [a,b]\to D$ is continuous, piecewise C^1, and $\gamma(a)=\gamma(b)$) is said to be *nullhomologous in* D if for every f holomorphic on D we have

$$\int_\gamma f(z)\,dz = 0.$$

A loop nullhomotopic in D, i.e. homotopically equivalent to a constant loop is then also nullhomologous in D (by theorem 10.6.6 of Analisi Due); the converse is not true. An example is given by the following circuit in $D=\mathbb{C}\smallsetminus\{-1,1\}$: let $\alpha(t)=-1+e^{2\pi it}$ and $\beta(t)=1-e^{2\pi it}$, with $t\in[0,1]$ (so that α, β are the circles of center ±1 and radius 1, described counterclockwise, both starting from 0). The circuit γ is obtained first taking α, then β, then the opposite $\tilde\alpha$ of α, finally the opposite $\tilde\beta$ of β, i.e. $\gamma=\alpha\,\beta\,\tilde\alpha\,\tilde\beta$. Trivially γ is nullhomologous: if f is holomorphic in $\mathbb{C}\smallsetminus\{-1,1\}$ then

$$\begin{aligned}\int_\gamma f(z)\,dz &= \int_\alpha f(z)\,dz + \int_\beta f(z)\,dz + \int_{\tilde\alpha} f(z)\,dz + \int_{\tilde\beta} f(z)\,dz = \\ &\int_\alpha f(z)\,dz + \int_\beta f(z)\,dz - \int_\alpha f(z)\,dz - \int_\beta f(z)\,dz = 0,\end{aligned}$$

but it can be proved (it is quite difficult!) that γ is *not* nullhomotopic in $\mathbb{C}\smallsetminus\{-1,1\}$.

Nevertheless the following is true:

. *Let D be a region. The following are equivalent:*

(i) *D is simply connected.*
(ii) *Every loop of D is nullhomotopic in D.*
(iii) *Every loop of D is nullhomologous in D.*

Proof. (i) is equivalent to (ii) by definition of simple connectedness; that (ii) implies (iii) has been noted above, and is theorem 10.6.6 of Analisi Due.

We omit the proof of (iii) implies (i). □

2.14.1. *Nullhomology and winding number.* We can understand then that for a loop γ, the condition of being nullhomologous in D, although weaker than the condition of being nullhomotopic in D, still has to do with γ not going around any hole of D. In fact one can prove:

PROPOSITION. *Let D be a region of $\mathbb{C}$ and let γ be a loop of D. Then γ is nullhomologous in D if and only if we have* $\operatorname{ind}_\gamma(z) = 0$ *for every $z \in \mathbb{C} \setminus D$.*

That is, γ is nullhomologous in D iff it does not wind around any point not in D. We shan't prove this proposition; one half is immediate by the very definition of nullhomology and the fact that $\zeta \mapsto 1/(\zeta - z)$ is holomorphic on D if $z \in \mathbb{C} \setminus D$; the other half, that $\operatorname{ind}_\gamma(\mathbb{C} \setminus D) = \{0\}$ implies nullhomology of γ, will not be proved here. We refer the reader to [Remmert] or [Conway] for a proof of this fact.

REMARK. In algebraic topology the concept of nullhomology of a loop is defined in a quite different way, for loops which are only continuous, in general topological spaces, and of course without recurring at all to holomorphic functions, absent in this more general context. Roughly speaking, nullhomology of a loop in a space means that the loop is in some sense the "boundary" of a formal combination of parametric surfaces of the space. Happily this notion turns out to be equivalent, for regions of the plane, to the one given here.

EXERCISE 2.14.1.1. ODD LOOPS AND ODD WINDING NUMBER Let $\gamma : [0, 1] \to \mathbb{C}_* = \mathbb{C} \setminus \{0\}$ be an "odd" loop: by this we mean that for $t \in [0, 1/2]$ we have $\gamma(t + 1/2) = -\gamma(t)$. Prove that then $\operatorname{ind}_\gamma(0)$ is odd.

Solution. We have

$$2\pi i \operatorname{ind}_\gamma(0) = \int_0^1 \frac{\gamma'(t)}{\gamma(t)}\, dt = \int_0^{1/2} \frac{\gamma'(t)}{\gamma(t)}\, dt + \int_{1/2}^1 \frac{\gamma'(t)}{\gamma(t)}\, dt;$$

In the second integral we put $t = \theta + 1/2$ and we get

$$\int_{1/2}^1 \frac{\gamma'(t)}{\gamma(t)}\, dt = \int_0^{1/2} \frac{\gamma'(\theta + 1/2)}{\gamma(\theta + 1/2)}\, d\theta = \int_0^{1/2} \frac{\gamma'(\theta)}{\gamma(\theta)}\, d\theta;$$

(by $\gamma(\theta + 1/2) = -\gamma(\theta)$ we also get $\gamma'(\theta + 1/2) = -\gamma'(\theta)$, so that the logarithmic derivative does not change by the translation of $1/2$). We have got:

$$2\pi i \operatorname{ind}_\gamma(0) = 2 \int_0^{1/2} \frac{\gamma'(t)}{\gamma(t)}\, dt;$$

by the lemma 2.4.1 we have

$$\exp\left(\int_0^{1/2} \frac{\gamma'(t)}{\gamma(t)}\, dt\right) = \frac{\gamma(1/2)}{\gamma(0)}; \quad \text{but } \gamma(1/2) = -\gamma(0), \text{ so that } \quad \frac{\gamma(1/2)}{\gamma(0)} = -1,$$

which implies that the integral is an odd integral multiple of πi. Thus the integral $\int_0^1 (\gamma'(t)/\gamma(t))dt$, which is twice that, is an odd integral multiple of $2\pi i$.

REMARK. A loop as above is called odd because if loops are considered as functions on the unit circle $\mathbb{U} = \mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ then the above loops correspond exactly to odd functions on the circle, i.e. functions $\varphi : \mathbb{U} \to \mathbb{C}$ such that $\varphi(-u) = -\varphi(u)$, for every $u \in \mathbb{U}$. The preceding exercise is the $1-$dimensional version of the famous Borsuk antipodal theorem.

$\square$

2.14.2. *A more general Cauchy formula.* We proved this formula only for a circle, since this form was enough for the development of the theory. But for any nulhomologous circuit we have a Cauchy formula, albeit a little more complicated by the winding number:

PROPOSITION. *Let D be open in $\mathbb{C}$, $f : D \to \mathbb{C}$ be holomorphic, and let γ be a loop in D, nullhomologous in D. Then, for every $z \in D \setminus [\gamma]$ we have*

$$\operatorname{ind}_\gamma(z)\, f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

Proof. Given $z \in D \setminus [\gamma]$ consider the function $g : D \to \mathbb{C}$ given by $g(\zeta) = (f(\zeta) - f(z))/(\zeta - z)$ for $\zeta \in D \setminus \{z\}$, $g(z) = f'(z)$. It has been observed, in the proof of the existence of the Laurent development, see 2.10.2, that g is holomorphic on D. Then, since γ is nullhomologous in D we have $\int_\gamma g(\zeta)\, d\zeta = 0$, that is

$$\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta - \int_\gamma \frac{f(z)}{\zeta - z}\, d\zeta = 0,$$

and the conclusion is immediate. □

2.15. The residue theorem. All sorts of definite integrals can be computed with the help of the *residue theorem.*

. RESIDUE THEOREM *Let D be a region of $\mathbb{C}$, and be γ be a loop nullhomologous in D. Let S be a finite subset of $D \setminus [\gamma]$. If f is holomorphic on $D \setminus S$, then*

$$\int_\gamma f(z)\, dz = 2\pi i \left(\sum_{c \in S} \mathrm{Res}(f, c)\ \mathrm{ind}_\gamma(c) \right).$$

Proof. Let

$$Q_c(z) = \frac{\mathrm{Res}(f,c)}{z - c} + \sum_{n=2}^{\infty} \frac{c_{-n}(c)}{(z - c)^n}$$

be the singular part of f at c, for every $c \in S$; remember that Q_c is holomorphic on $\mathbb{C} \setminus \{c\}$. The formula

$$g(z) = f(z) - \sum_{c \in S} Q_c(z)$$

defines then g as a function holomorphic in $D \setminus S$, which has removable singularities at every $c \in S$, i.e., is extendable to a function holomorphic on D. Then, since γ is nullhomologous in D:

$$0 = \int_\gamma g(z)\, dz = \int_\gamma f(z)\, dz - \sum_{c \in S} \int_\gamma Q_c(z)\, dz, \text{ equivalently } \int_\gamma f(z)\, dz = \sum_{c \in S} \int_\gamma Q_c(z)\, dz.$$

To compute this last sum: since $c \notin [\gamma]$ and $[\gamma]$ is compact, we have that $\mathrm{dist}(c, [\gamma]) = \delta(c) > 0$; then the series defining Q_c normally converges on $\mathbb{C} \setminus B(c, \delta(c)[$, hence also on $[\gamma]$, and termwise integration on γ is then possible:

$$\int_\gamma Q_c(z)\, dz = \int_\gamma \frac{\mathrm{Res}(f,c)}{z - c}\, dz + \sum_{n=2}^{\infty} c_{-n}(c) \int_\gamma \frac{dz}{(z - c)^n};$$

the first term is $2\pi i\ \mathrm{ind}_\gamma(c)\ \mathrm{Res}(f, c)$, while all terms $\int_\gamma dz/(z - c)^n$, for $n \geq 2$, are 0. □

2.16. Consequences of the residue theorem.

2.16.1. *Singularities of the logarithmic derivative.* We already know that a logarithmic derivative, when integrated over a loop, gives an integral multiple of $2\pi i$ (2.4.2). With the help of the residue theorem, this integral can now be related to other significant elements of f. If D is a region of $\mathbb{C}$, $P = P(f)$ is a discrete subset of D and $f : D \setminus P \to \mathbb{C}$ is holomorphic on D, with only polar singularities on P (we abbreviate it by saying that f is *meromorphic on* D) then the logarithmic derivative of f (which by definition is f'/f) is holomorphic on $D \setminus (Z \cup P)$, where $Z = Z(f)$ is the zero–set of f, and points of $Z \cup P$ are *first order poles* for f'/f. In fact, if $c \in D$ is a point such that $\mathrm{ord}(f, c) = m \neq 0$ we have $f(z) = (z - c)^m g(z)$ with $g(c) \neq 0$ and g holomorphic on D (see 2.12). Then we have

$$f'(z) = m(z - c)^{m-1}\, g(z) + (z - c)^m\, g'(z) \quad \text{whence} \quad \frac{f'(z)}{f(z)} = \frac{m}{z - c} + \frac{g'(z)}{g(z)},$$

with g'/g holomorphic near c. We have proved:

LEMMA. *If* $\mathrm{ord}(f, c) = m \neq 0$*, then* $\mathrm{ord}(f'/f, c) = -1$*, and the singular part of f'/f at c is $m/(z - c)$, that is, c is a first–order pole for f'/f, with residue $m = \mathrm{ord}(f, c)$.*

2.16.2. *The argument principle.* (it is called "teorema dell'indicatore logaritmico" in italian)

Theorem. *Let f be holomorphic on a region D except for a finite set P of polar singularities, and with a finite set of zeroes Z. If γ is a loop in $D \setminus (Z \cup P)$, nullhomologous in D, then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{c\in Z\cup P} \mathrm{ind}_\gamma(c)\,\mathrm{ord}(f,c) = \sum_{c\in Z}\mathrm{ind}_\gamma(c)\,m_c - \sum_{c\in P}\mathrm{ind}_\gamma(c)\,n_c,$$

where if $c \in Z$ then $m_c = \mathrm{ord}(f,c)$ is the multiplicity of c as a zero for f, while if $c \in P$ then $n_c = -\mathrm{ord}(f,c)$ is the order of c as a pole for f.

Proof. It is an immediate application of the residue theorem and the preceding lemma. □

Let us state in more elementary terms what the preceding theorem means: if we take a loop γ which does not pass through any zero or any pole of f, which does not wind around any hole of D (meaning that γ is nullhomologous in D), and (to simplify matters) it has exactly two winding numbers: 1 (points "inside γ") or 0 (points "outside γ"), loops which from now on we shall call *simply closed*, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \text{number of zeroes inside } \gamma - \text{number of poles inside } \gamma,$$

counting zeroes with their multiplicity, and poles with their order.

2.16.3. *Simply closed loops and Jordan loops.* We repeat the definition of simply closed loop: it is a loop γ which has exactly two winding numbers, 1 (the set of points inside γ) and 0 (points outside γ). In general, the boundary of a convex geometrical figure, or even only a simply connected one (if not too complicated), when suitably parametrized, is a simply closed loops. For every given figure (a rectangle, a half–disc, a sector of an annulus or of a disc, etc.) this can be easily proved, but we shall not in general take the trouble of doing it. We recall that a Jordan loop, called also a *simple closed curve*, is a loop like $\gamma : [a,b] \to \mathbb{C}$ such that $\gamma(a) = \gamma(b)$, as required for loops, but this is the only exception to injectivity: that is, $\gamma(t_1) = \gamma(t_2)$, with $a \le t_1 < t_2 \le b$ implies $t_1 = a$ and $t_2 = b$. The Jordan theorem, quite hard to prove in such general terms, says that if γ is a Jordan loop then $\mathbb{C} \setminus [\gamma]$ has exactly two connected components, and moreover $\mathrm{ind}_\gamma(z) = \pm 1$ on the bounded component. Thus, up to orientation, a Jordan loop is a simply closed loop, and the "inside" is exactly the bounded connected component of the complement of the trace, as our intuition says it ought to be.

Exercise 2.16.3.1. Give an example of a simply closed loop that is not a Jordan loop.

2.16.4. *The fundamental theorem of algebra.* We have already proved the fundamental theorem of algebra as a corollary of Liouville's theorem. But the argument principle gives a very direct proof of the fact that a polynomial of degree m has m zeroes on $\mathbb{C}$, counting multiplicities. Let $p(z) = a_0 + a_1\,z + \cdots + a_m\,z^m$, with $a_m \ne 0$ be such a polynomial. Then:

$$\lim_{z\to\infty} z\frac{p'(z)}{p(z)} = \lim_{z\to\infty} z\,\frac{\sum_{n=1}^m n\,a_n\,z^{n-1}}{\sum_{n=0}^m a_n\,z^n} = \lim_{z\to\infty}\frac{\sum_{n=1}^m n\,a_n\,z^n}{\sum_{n=0}^m a_n\,z^n} =$$

(we divide by $a_m\,z^m$ numerator and denominator)

$$\lim_{z\to\infty}\frac{\sum_{n=1}^{m-1} n(a_n/a_m)(1/z^{m-n}) + m}{\sum_{n=1}^{m-1}(a_n/a_m)(1/z^{m-n}) + 1} = m.$$

From the great arc of circle lemma (2.2) applied to p'/p, with a full circle, we get

$$\lim_{r\to\infty}\frac{1}{2\pi i}\int_{\partial B(c,r]}\frac{p'(z)}{p(z)}\,dz = m;$$

but clearly also $\lim_{z\to\infty} p(z) = \infty$, which means that for $|z|$ large enough, say $|z| > R$, we have $|p(z)| > 0$. Thus all the zeroes of p are in the disk $\{|z| \le R\}$, and thus they must be a finite set (we know this also from purely algebraic reasons, of course), since $Z(p)$ would otherwise have an accumulation point, and then p would be identically zero. Thus the integral is constantly equal to the number of zeroes of p for $r > R$; and its limit as r tends to ∞ is m.

Exercise 2.16.4.1. Let D be open and simply connected, $S \subseteq D$ a finite subset, $f : D \setminus S \to \mathbb{C}$ holomorphic. Prove that f admits a primitive on $D \setminus S$ if and only if $\mathrm{Res}(f,c) = 0$ for every $c \in S$.

2.17. Another proof of the open mapping theorem. The argument principle yields another proof of the open mapping theorem, with a more precise statement on the local structure of a holomorphic map near a point of multiplicity $m \geq 0$.

PROPOSITION. *Let D be open in $\mathbb{C}$, let $f : D \to \mathbb{C}$ be holomorphic, and let c be a point of D. Assume that f is not locally constant at c, and that the multiplicity of f at c is $m \geq 1$ (that is $\nu(f,c) = \mathrm{ord}(f - f(c), c) = m$). Then there exist open neighborhoods U of c and V of $f(c)$ with the following property:*

$V = f(U)$, and moreover for every $w \in V \smallsetminus \{f(c)\}$ the equation $w = f(z)$ has m distinct simple solutions $z_1, \dots, z_m \in U$.

Proof. Since f is not locally constant at c, c is an isolated point of $Z(f - f(c))$, and for the same reason either $f'(c) \neq 0$ (when $m = 1$), or c is an isolated zero for f' (when $m > 1$). We can then find a closed disc $B(c, r] \subseteq D$ such that $Z(f - f(c)) \cap B(c, r] = \{c\}$, and $Z(f') \cap B(c, r] \subseteq \{c\}$. Consider $\gamma(t) = c + r\, e^{it}$ and let $\Gamma(t) = f(\gamma(t))$ $(t \in [0, 2\pi])$. Then $f - f(c)$ has only one zero "inside" γ, i.e. c, with multiplicity m; of course this function has no poles inside γ. By the argument principle we then have

$$\mathrm{ind}_\Gamma(f(c)) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z) - f(c)}\, dz = m.$$

Let V be that connected component of $\mathbb{C} \smallsetminus [\Gamma]$ which contains $f(c)$, and put $U = f^{\leftarrow}(V) \cap B(c, r[$; observe that U is open, contains c, and $f(U) \subseteq V$. Since $w \mapsto \mathrm{ind}_\Gamma(w)$ is locally constant, we have $\mathrm{ind}_\Gamma(w) = m$ for every $w \in V$. Then

$$\mathrm{ind}_\Gamma(w) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z) - w}\, dz = m \quad \text{for every } w \in V.$$

Counting multiplicities, the equation $w = f(z)$ has then m solutions in $B(c, r[$, for every $w \in V$; clearly these solutions are also in U. But if $w \neq f(c)$ a solution $z \in U$ of this equation cannot have multiplicity larger than 1: in fact then $z \neq c$, hence $f'(z) \neq 0$ (recall that $Z(f') \cap U \subseteq \{c\}$. Then the equation has m distinct solutions $z_1, \dots, z_m \in U$; and in particular $V \subseteq f(U)$, so that $V = f(U)$.

□

2.18. Biholomorphic mappings. If $f : D \to \mathbb{C}$ is holomorphic non–locally constant at c, but $f'(c) = 0$, then there is no neighborhood U of c such that the restriction of f to U is injective, that is, f is not locally injective at c: this follows immediately from the precise version of the open mapping theorem (contrast again with real functions: the mapping $x \mapsto x^3$ is a bijective self–map of $\mathbb{R}$, with zero derivative at 0). If, conversely, $f'(c) \neq 0$, then f is a local homeomorphism at c, with holomorphic local inverse, as we are going to see in the next section.

2.18.1. *Inverse function theorem.* The inverse of a holomorphic mapping, whenever it exists, is holomorphic.

PROPOSITION. *Le D be open in $\mathbb{C}$, and let $f : D \to \mathbb{C}$ be holomorphic and injective; denote by E the image $f(D)$ of f. Then E is also open, f is a homeomorphism of D onto E, and the inverse $g : E \to D$ of f is holomorphic; moreover*

$$g'(w) = \frac{1}{f'(g(w))} \qquad \textit{for every} \quad w \in E.$$

Proof. That E is open follows from the open mapping theorem; and the same theorem shows that the inverse g of f is continuous (if A is an open subset of D, then $g^{\leftarrow}(A) = f(A)$ is open). It remains to prove that g is complex–differentiable. Given $w = f(z) \in E$ let's compute the limit

$$\lim_{\zeta \to w} \frac{g(\zeta) - g(w)}{\zeta - w};$$

since $f : D \to E$ is a homeomorhism, we can use it as a change of variable $\zeta = f(\xi)$; that is, the preceding limit exists in $\mathbb{C}$ iff the limit

$$\lim_{\xi \to z} \frac{g(f(\xi)) - g(w)}{f(\xi) - f(z)}$$

exists in $\mathbb{C}$, and in that case the two limits coincide. But since f is injective, we have $f'(z) \neq 0$ as observed above (2.18); hence

$$\lim_{\xi \to z} \frac{g(f(\xi)) - g(f(z))}{f(\xi) - f(z)} = \lim_{\xi \to z} \frac{\xi - z}{f(\xi) - f(z)} = \lim_{\xi \to z} \left(\frac{f(\xi) - f(z)}{\xi - z} \right)^{-1} = (f'(z))^{-1},$$

thus concluding the proof. □

Given two regions $D, E \subseteq \mathbb{C}$, a *biholomorphic mapping, or holomorphic isomorphism*, of D onto E is a bijective holomorphic map $f : D \to E$. By what we have just proved the inverse map $f^{-1} : E \to D$ is then also a holomorphic isomorphism of E onto D.

Example 2.18.1.1. The *principal (branch of the) logarithm* is the function $\log : \mathbb{C}_- \to \mathbb{C}$, where $\mathbb{C}_- = \mathbb{C} \setminus \mathbb{R}_-$ is the *slit plane* ($\mathbb{R}_- = \{x \in \mathbb{R} : x \le 0\}$ is the negative real half–line); it may be defined as the inverse of $\exp_{|S}$, where S is the open strip $S = \{z \in \mathbb{C} : \operatorname{Im} z \in]-\pi, \pi[\}$. It is then an isomorphism of the slit plane onto the strip.

Example 2.18.1.2. The mapping $p_2(z) = z^2$ is an isomorphism of the right half–plane $T = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ onto the slit plane $\mathbb{C}_- = \mathbb{C} \setminus \mathbb{R}_-$. It is immediate to check that

$$z \mapsto \sqrt{z} = |z|^{1/2} e^{i \arg z/2} = e^{\log z/2},$$

the principal (branch of the) square root, is the inverse of $z \mapsto z^2$ between these regions.

Exercise 2.18.1.3. Recall that the function $\tan : \mathbb{C} \setminus (\pi/2 + \mathbb{Z}\,\pi) \to \mathbb{C}$ is defined by the formula

$$\tan z = \frac{\sin z}{\cos z}.$$

(i) Solve the equation $\tan z = w$, in the unknown z. Prove that tan assumes all complex values, except $\pm i$.
(ii) Prove that tan induces a holomorphic isomorphism of the strip $S = \{z \in \mathbb{C} : -\pi/2 < \operatorname{Re} z < \pi/2\}$ onto a " doubly slit plane" to be described. Find an explicit expression of the inverse, which by definition is the principal branch of the function arctan. Compute the derivative of this inverse, and verify that it is $1/(1 + z^2)$. (Solution: $\arctan z = (1/2i) \log((1 + iz)/(1 - iz))$)

2.19. Rouché's theorem. The integral in the argument principle is stable under small modification of either the loop or the mapping f. It is this second aspect that is discussed by the next theorem.

. Rouché's theorem *Let D be a region of $\mathbb{C}$, let $f, g : D \to \mathbb{C}$ be holomorphic with a finite set of zeroes, and let γ be a simply closed loop of D, nullhomologous in D. Assume that we have*

$$|f(\zeta) - g(\zeta)| < |f(\zeta)| + |g(\zeta)| \qquad \textit{for every } \zeta \in [\gamma].$$

Then

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = \frac{1}{2\pi i} \int_\gamma \frac{g'(z)}{g(z)}\, dz,$$

so that f and g have the same number of zeroes inside γ.

Proof. Recall first the triangle inequality: if w, z are complex numbers then $|w + z| \le |w| + |z|$, and equality holds if and only if one of the two numbers is zero, or $w = \lambda\, z$, with $\lambda > 0$ positive and real, equivalently, $w/z \in]0, +\infty[$. The strict inequality of the hypothesis forbids then to both $g(\zeta)$ and $f(\zeta)$ to be 0 on $[\gamma]$, and moreover says that $f(\zeta)/(-g(\zeta))$ is never real and strictly positive, equivalently that $f(\zeta)/g(\zeta) \notin \mathbb{R}_-$, for every $\zeta \in [\gamma]$. Then we can consider, on the open set $E = \{z \in D \setminus Z(g) : f(z)/g(z) \in \mathbb{C} \setminus \mathbb{R}_-\}$, which contains $[\gamma]$, the function $\log(f/g)$, where $\log : \mathbb{C} \setminus \mathbb{R}_- \to \mathbb{C}$ is the principal logarithm. This function is a primitive of $f'/f - g'/g$ on E:

$$D \log(f/g)(z) = \frac{g(z)}{f(z)} \frac{f'(z)g(z) - f(z)g'(z)}{(g(z))^2} = \frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)}.$$

Then we have

$$\int_\gamma \left(\frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)} \right) dz = 0 \quad \text{which implies} \quad \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = \frac{1}{2\pi i} \int_\gamma \frac{g'(z)}{g(z)}\, dz.$$

□

It is not easy in general to verify the inequality $|f(\zeta) - g(\zeta)| < |f(\zeta)| + |g(\zeta)|$ for $\zeta \in [\gamma]$. Very often what we have is that g is obtained from f by adding some term which on $[\gamma]$ is small with respect to f (or vice–versa). If $g(\zeta) = f(\zeta) + \varepsilon(\zeta)$, with $|\varepsilon(\zeta)| < |f(\zeta)|$ on $[\gamma]$ then the hypotheses of the theorem are verified. In fact , if $f(\zeta) = -\lambda(f(\zeta) + \varepsilon(z))$ for some $\lambda > 0$ and some $\zeta \in [\gamma]$ then $(1 + \lambda)f(\zeta) = -\lambda\varepsilon(\zeta)$, which implies $|\varepsilon(\zeta)| = (1 + 1/\lambda)|f(\zeta)| \ge |f(\zeta)|$, contradicting the hypothesis $|\varepsilon(\zeta)| < |f(\zeta)|$ on $[\gamma]$.

EXAMPLE 2.19.0.4. Determine the number of zeroes in the unit disc of the function $z^5+z^3/3+z^2/4+1/3$.

We take $f(z)=z^5$; $g(z)=z^5+\varepsilon(z)$ is the given function, where $\varepsilon(z)=z^3/3+z^2/4+1/3$; then f has five zeroes in the disc. If $|z|=1$ then $|f(z)|=1$ while $|\varepsilon(z)|\le 1/3+1/4+1/3=11/12<1$, so that g has also five zeroes in the unit disc (and hence it has all its zeroes in the disc).

EXERCISE 2.19.0.5. Let $p(z)=z^n+a_{n-1}\,z^{n-1}+\cdots+a_0$ be a monic polynomial of degree $n\ge 1$. Prove that there exists $c\in\mathbb{U}$ (i.e., $|c|=1$) such that $|p(c)|\ge 1$.

EXERCISE 2.19.0.6. $\odot\odot$ Prove that if $\lambda\in\mathbb{R}$ and $\lambda>1$ then the function $f(z)=\lambda-z-e^{-z}$ has exactly one zero in the closed right half plane $\{z\in\mathbb{C}:\ \operatorname{Re} z\ge 0\}$, that this zero is real, and that it lies in $B(1,\lambda]$ (not easy: take $g(z)=\lambda-z$ in Rouché's theorem and use $\partial B(1,\lambda]\ \ldots$).

2.19.1. *Logarithms of a function.*

EXERCISE 2.19.1.1. Let D be open in $\mathbb{C}$, and let $f,g:D\to\mathbb{C}$ be holomorphic and non identically zero. Compute the logarithmic derivatives of fg and f/g in terms of the logarithmic derivatives of f and g. Prove then that if D is a region the following are equivalent:

(i) f and g have the same logarithmic derivative on D.
(ii) The logarithmic derivatives of f and g agree on a set with an accumulation point in D.
(iii) There is a nonzero constant $k\in\mathbb{C}$ such that $f=k\,g$.

EXERCISE 2.19.1.2. Let D be a region, and let $f:D\to\mathbb{C}$ be holomorphic and nowhere vanishing in D (i.e., $Z_D(f)=\emptyset$). We say that $g:D\to\mathbb{C}$ is a *logarithm* of f on D if $e^{g(z)}=f(z)$, for every $z\in D$.

(i) Prove that if g is a logarithm of f on D, then $h\in\mathcal{O}(D)$ is another logarithm of f on D if and only if $g=h+2k\pi i$, where $k\in\mathbb{Z}$ (if $e^g=e^h$ then $e^{g(z)-h(z)}=1$, for every $z\in D$; but the set $2\pi i\mathbb{Z}$ is discrete ...).
(ii) Prove that if g is a logarithm of f, then g is a primitive of the logarithmic derivative f'/f of f.
(iii) Conversely, prove that if $g\in\mathcal{O}(D)$ is a primitive of f'/f, and $e^{g(c)}=f(c)$ for at least one $c\in D$, then g is a logarithm of f on D.
(iv) Conclude that f has logarithms on D if and only if f'/f admits a primitive on D.

EXERCISE 2.19.1.3. Let $\alpha,\beta:[0,1]\to\mathbb{C}_*$ be loops in the punctured plane $\mathbb{C}_*=\mathbb{C}\setminus\{0\}$. If α and β are homotopic in $\mathbb{C}^*$, then $\operatorname{ind}_\alpha(0)=\operatorname{ind}_\beta(0)$ (easy).

Conversely (and less easily): if $\operatorname{ind}_\alpha(0)=\operatorname{ind}_\beta(0)$ then α and β are homotopic in $\mathbb{C}_*$: take $a,b\in\mathbb{C}$ such that $e^a=\alpha(0)$ and $e^b=\beta(0)$; consider the functions $A,B:[0,1]\to\mathbb{C}$ given by:

$$A(t)=a+\int_0^t\frac{\alpha'(s)}{\alpha(s)}\,ds;\qquad B(t)=b+\int_0^t\frac{\beta'(s)}{\beta(s)}\,ds.$$

Prove that under the given hypothesis on winding numbers the function $h:[0,1]\times[0,1]\to\mathbb{C}_*$ given by:

$$h(t,\lambda)=\exp((1-\lambda)\,A(t)+\lambda\,B(t))$$

is a loop homotopy between α and β.

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