# Selected topics of Complex Analysis A.A.2011–2012

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#### 1. The Riemann mapping theorem

#### 1.1. Holomorphic isomorphisms.

1.1.1. Injectivity of holomorphic maps. We recall (one of the possible statements of) the:

. OPEN MAPPING THEOREM. Let D be a region of  $\mathbb{C}$ , and let  $f: D \to \mathbb{C}$  be holomorphic and nonconstant. Given  $c \in D$ , denote by m the multiplicity of c as a zero of f - f(c). Then there exist open neighborhoods U of c and V of f(c) such that f(U) = V, and for every  $w \in V \setminus \{f(c)\}$  the equation w = f(z) has m distinct solutions  $z_1, \ldots, z_m \in U \setminus \{c\}$ .

In particular f is an open mapping, that is, f(A) is open for every open subset of D.

We remark that the proof is a nice application of the argument principle (see, e.g. [Rudin]), and we reproduce it here for completeness.

Proof. Since f is non constant the identity theorem says that c is isolated in  $f^{\leftarrow}(f(c))$ , and moreover Z(f') is a discrete subset of D, closed in D. Thus there exists  $\delta > 0$  such that  $B(c, \delta] \subseteq D$ , and  $B(c, \delta] \cap f^{\leftarrow}(f(c)) = \{c\}$ , whereas  $B(c, \delta] \cap Z(f')$  contains at most c. Then  $f(c) \notin f(\partial B(c, \delta])$ , and  $Z(f') \cap \partial B(c, \delta] = \emptyset$ . Let V be the connected component of  $\mathbb{C} \setminus f(\partial B(c, \delta])$  that contains f(c); remember that V is open. Set  $U = B(c, \delta[\cap f^{\leftarrow}(V),$ so that U is open, contains c, and  $f(U) \subseteq V$ . If  $\gamma(t) = c + \delta e^{it}$ ,  $t \in [0, 2\pi]$  is a parametrization of the boundary of  $B(c, \delta]$ , and  $\Gamma = f \circ \gamma$ , then we have

$$m = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z) - f(c)} dz = \frac{1}{2\pi i} \int_{\Gamma} \frac{d\zeta}{\zeta - f(c)} = \operatorname{ind}_{\Gamma}(f(c)).$$

But we have  $\operatorname{ind}_{\Gamma}(w) = m$  for every  $w \in V$ , that is

$$m = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z) - w} \, dz \quad \text{for every } w \in V;$$

By the argument principle the equation f(z) = w has then m solutions  $z \in B(c, \delta[$ , counting multiplicities; since  $Z(f') \cap B(c, \delta[$  is either empty, or contains only c, it follows that if  $w \in V$  and  $w \neq f(c)$  then the equation w = f(z) has m distinct solutions  $z_1, \ldots, z_m \in B(c, \delta[$ ; clearly  $z_1, \ldots, z_m$  belong also to U, so that  $V \subseteq f(U)$ .

COROLLARY. Let  $f: D \to \mathbb{C}$ , with D open subset of  $\mathbb{C}$ , be holomorphic. If  $c \in D$ , and f'(c) = 0, then there is no neighborhood U of c such that  $f_{|U}$  is injective. If, conversely,  $f'(c) \neq 0$ , then there are open neighborhoods U of c and V of f(c) such that f induces a homeomorphism of U onto V.

*Proof.* For, c has in the first case multiplicity at least two as a zero of f - f(c). The second case is the above theorem if  $f'(c) \neq 0$ .

1.1.2. Local injectivity at an isolated singularity. The preceding corollary says that a holomorphic mapping is locally injective at a point c of its domain iff  $f'(c) \neq 0$ , in which case it is a local homeomorphism at c. What about isolated singularities? that is, if D is open,  $c \in D$ , and f is holomorphic in  $D \setminus \{c\}$ , what can be said about injectivity of f in some (punctured) neighborhood of c? Assuming that c is a pole, then f is nonzero in every sufficiently small neighborhood of c, and it is locally injective iff so is 1/f in some (perhaps smaller) neighborhood of c. Since 1/f has a zero at c, whose multiplicity is the order of c as a pole for f, it follows that f is locally injective near a pole iff that pole has order 1. If c is an essential singularity the Casorati–Weierstrass theorem says that for every r the set  $f(B(c, r[\cap D \setminus \{c\}))$  is (open) and dense in  $\mathbb{C}$ , and this clearly implies that f is not one–to–one in any punctured neighborhood of c (the precise verification of this statement is left as an exercise; see 1.1.3 for a solution). We have proved:

. A holomorphic function is injective near an isolated non removable singularity if and only if that singularity is a simple pole.

Of course this works also at infinity: if a function is holomorphic in a (punctured) neighborhood of  $\infty$ , say a set like  $\mathbb{C} \setminus B(0, r] = \{z \in \mathbb{C} : |z| > r\}$ , then it is injective in some neighborhood of infinity if and only if its Laurent development in this neighborhood is of the type

$$f(z) = \sum_{n=1}^{\infty} \frac{c_{-n}}{z^n} + c_0 + c_1 z \quad (c_1 \neq 0) \quad \text{or} \quad f(z) = \sum_{n=2}^{\infty} \frac{c_{-n}}{z^n} + \frac{c_{-1}}{z} + c_0 \quad (c_{-1} \neq 0)$$

this is immediate, simply by considering the function g(z) = f(1/z) near z = 0. In particular, an entire function is injective in some neighborhood of  $\infty$  iff it is of the form  $z \mapsto c_0 + c_1 z$ , with  $c_1 \neq 0$ , that is, a polynomial of degree 1. The following statement, whose proof is left as an exercise, subsumes all that has been said in this section, taking account also of removable singularities:

. Let D be open in  $\mathbb{C}$ ,  $c \in D$ ,  $f: D \setminus \{c\} \to \mathbb{C}$  be holomorphic, and let

$$\sum_{n=0}^{\infty} c_n \, (z-c)^n + \sum_{n=1}^{\infty} \frac{c_{-n}}{(z-c)^n}$$

be the Laurent development of f at c. Then f is injective in some neighborhood of c iff either  $c_{-n} = 0$ for n > 1 and  $c_{-1} \neq 0$ , or  $c_{-n} = 0$  for n > 0, and  $c_1 \neq 0$ . The same holds true if  $c = \infty$  (in this case the development is written as if c = 0).

1.1.3. On the Casorati–Weierstrass theorem. For the reader's comfort we sketch here a proof of the Casorati–Weierstrass theorem: if  $f: B(c, r[\backslash \{c\} \to \mathbb{C} \text{ is holomorphic with an essential singularity at } c,$  then  $f(B(c, \delta[\backslash \{c\}) \text{ is dense in } \mathbb{C}, \text{ for every } \delta \text{ with } 0 < \delta \leq r$ . In fact, if  $f(B(c, \delta[\backslash \{c\}) \text{ is not dense in } \mathbb{C}, \text{ then there exists } w \in \mathbb{C} \text{ and } \rho > 0$  such that  $|f(z) - w| \geq \rho$  for every  $z \in B(c, \delta[\backslash \{c\})$ . Then g(z) = 1/(f(z) - w) is holomorphic and bounded in  $B(c, \delta[\backslash \{c\}, \text{ and hence has a removable singularity at } c;$  but then f(z) = w + 1/g(z) has at worst a polar singularity at c. This concludes the proof.

Since f is an open map, f maps the open annulus  $B(c, ]s, t[= \{z \in \mathbb{C} : s < |z - c| < t\}$ , where  $0 < s < t \leq r$ , onto an open set; and since  $f(B(c, s[\smallsetminus \{c\}) \text{ is dense in } \mathbb{C} \text{ the intersection } f(B(c, s[\smallsetminus \{c\}) \cap f(B(c, ]s, t]) \text{ is non empty: this clearly implies the non-injectivity of } f \text{ on } B(c, t[\smallsetminus \{c\}).$ 

1.1.4. *Holomorphic isomorphisms*. The inverse of a holomorphic mapping, whenever it exists, is holomorphic.

PROPOSITION. Le D be open in  $\mathbb{C}$ , and let  $f: D \to \mathbb{C}$  be holomorphic and injective; denote by E the image f(D) of f. Then E is also open, f is a homeomorphism of D onto E, and the inverse  $g: E \to D$  of f is holomorphic; moreover

$$g'(w) = \frac{1}{f'(g(w))}$$
 for every  $w \in E$ .

*Proof.* That E is open follows from the open mapping theorem; and the same theorem shows that the inverse g of f is continuous (if A is an open subset of D, then  $g^{\leftarrow}(A) = f(A)$  is open). It remains to prove that g is complex-differentiable. Given  $w = f(z) \in E$  let's compute the limit

$$\lim_{\zeta \to w} \frac{g(\zeta) - g(w)}{\zeta - w};$$

since  $f : D \to E$  is a homeomorphism, we can use it as a change of variable  $\zeta = f(\xi)$ ; that is, the preceding limit exists in  $\mathbb{C}$  iff the limit

$$\lim_{\xi \to z} \frac{g(f(\xi)) - g(w)}{f(\xi) - f(z)}$$

exists in  $\mathbb{C}$ , and in that case the two limits coincide. But since f is injective, we have  $f'(z) \neq 0$  (1.1.1); hence

$$\lim_{\xi \to z} \frac{g(f(\xi)) - g(f(z))}{f(\xi) - f(z)} = \lim_{\xi \to z} \frac{\xi - z}{f(\xi) - f(z)} = \lim_{\xi \to z} \left(\frac{f(\xi) - f(z)}{\xi - z}\right)^{-1} = (f'(z))^{-1},$$
ing the proof.

thus concluding the proof.

Given two regions  $D, E \subseteq \mathbb{C}$ , an holomorphic isomorphism, or simply an isomorphism of D onto E is a bijective holomorphic map  $f: D \to E$ . By what we have just proved the inverse map  $f^{-1}: E \to D$  is then also a holomorphic isomorphism of E onto D. The self-isomorphisms of a region D are called automorphisms of D; their set Aut(D) is clearly a group under map composition, the group of holomorphic automorphisms of D.

EXAMPLE 1.1.4.1. The principal (branch of the) logarithm is the function log :  $\mathbb{C}_{-} \to \mathbb{C}$ , where  $\mathbb{C}_{-} = \mathbb{C} \setminus \mathbb{R}_{-}$ , is the slit plane ( $\mathbb{R}_{-} = \{x \in \mathbb{R} : x \leq 0\}$  is the negative half–line); it may be defined as the inverse of  $\exp_{|S|}$ , where S is the open strip  $S = \{z \in \mathbb{C} : \operatorname{Im} z \in ] - \pi, \pi[\}$ . It is then an isomorphism of the slit plane onto the strip.

EXAMPLE 1.1.4.2. The mapping  $f(z) = z^2$  is an isomorphism of the right half-plane  $S = \{z \in \mathbb{C} : \text{Re } z > 0\}$  onto the slit plane  $\mathbb{C}_- = \mathbb{C} \setminus \mathbb{R}_-$ . It is immediate to check that  $z \mapsto \sqrt{z} = |z|^{1/2} e^{i \arg z/2} = e^{\log z/2}$ , the principal branch of the square root, is the inverse of  $z \mapsto z^2$  between these regions.

EXAMPLE 1.1.4.3. The Cayley map  $h_C(z) := (z - i)/(z + i)$  is an isomorphism of the (open) upper half-plane  $H = \{z \in \mathbb{C} : \text{Im } z > 0\}$  onto the unit disc  $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ . In fact the formula defines an isomorphism of  $\mathbb{C} \setminus \{-i\}$  onto  $\mathbb{C} \setminus \{1\}$ , whose inverse is  $k_C(z) = i(1+z)/(1-z)$ ; geometrically it is easy to see that  $h_C$  maps the upper half plane onto the unit disc: the real axis is the axis of the segment [-i, i], that is  $\mathbb{R}$  is the locus of points which have equal distance from -i, i; clearly the upper half-plane is the set of points whose distance from i is less than the distance from -i, i.e.  $H = \{z \in \mathbb{C} : |z - i| < |z + i|\}$ , equivalently,  $H = \{z \in \mathbb{C} : |h_C(z)| < 1\}$ . This proves that  $h_C$  maps H into  $\Delta$ , while  $\mathbb{C} \setminus \overline{H}$  is mapped in the outside of  $\Delta$ ; since  $h_C$  (as a map of  $\mathbb{C} \setminus \{-i\}$  onto  $\mathbb{C} \setminus \{1\}$ ) has an inverse  $k_C$  the map  $h_C$  actually maps H onto  $\Delta$ .

EXAMPLE 1.1.4.4. If  $D = \mathbb{C}$ , then clearly the maps  $z \mapsto az + b$ , with  $a \in \mathbb{C}_*$  are automorphisms of  $\mathbb{C}$  (the *affine* automorphisms); in fact the inverse is  $z \mapsto a^{-1}z - a^{-1}b$ ; and 1.1.2 says that these are *all* the automorphisms of  $\mathbb{C}$ . In other words, Aut( $\mathbb{C}$ ) is the affine group.

EXAMPLE 1.1.4.5. Given the punctured plane  $\mathbb{C}_* = \mathbb{C} \setminus \{0\}$ , let us determine the injective maps with domain  $\mathbb{C}_*$ . Injectivity near 0 requires the singular part of such a map to be either zero or a/z, at infinity to be of the type cz, or zero (1.1.2). An injective holomorphic map  $f: \mathbb{C}_* \to \mathbb{C}$  must then be of the type f(z) = a/z + b + cz; but it is clear that if both a and c are nonzero then the map is not injective (the equation w = a/z + b + cz has two solutions in z if  $a, c \neq 0$ , for most values of w). Then f is either of the form f(z) = a/z + b with  $a \neq 0$ , or of the form f(z) = b + cz with  $c \neq 0$ , this last an affine map injective on all of  $\mathbb{C}$  considered in the previous example. The automorphisms of the punctured plane are then the maps  $z \mapsto az$  or the maps  $z \mapsto a/z$ , with  $a \in \mathbb{C}$  nonzero.

EXERCISE 1.1.4.6. Write an explicit isomorphism of the strip  $S = \{z \in \mathbb{C} : \text{Im } z \in ] -\pi, \pi[\}$  onto the unit disc (look at some of the above ...).

1.1.5. *Möbius transformations.* We have seen that entire holomorphic maps are injective iff they are affine maps, whereas maps on a punctured plane  $\mathbb{C} \setminus \{k\}$  are injective if and only if they are of the form  $z \mapsto a/(z-k) + b$  or  $z \mapsto a + b(z-k)$ . We are led to the study of the maps of the form

$$f(z) = \frac{az+b}{cz+d}$$

Excluding the trivial case c = d = 0, which gives an empty domain of definition, such a map is everywhere defined, and is in fact an affine map, if c = 0 and  $d \neq 0$ , whereas if  $c \neq 0$  its domain is the punctured

plane  $\mathbb{C} \setminus \{-d/c\}$ . Next, observe that if

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc = 0,$$

but c, d not both zero, then the map is constant on its domain: in fact the rows (a, b) and (c, d) are linearly dependent, and hence  $a = \lambda c$ ,  $b = \lambda d$  for some  $\lambda \in \mathbb{C}$ , which implies  $f(z) = \lambda$  for every  $z \in \text{domain}(f)$ . From this point on we assume nonsingularity of the matrix  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , and we extend the map f to the *Riemann sphere*  $\mathbb{C}_{\infty} = \mathbb{C} \cup \{\infty\}$ , the one-point compactification of the complex plane, by setting  $f(-d/c) = \infty$  and  $f(\infty) = a/c$  if  $c \neq 0$ , and  $f(\infty) = \infty$  if c = 0 (when f is an affine map). In this way f is a self-homeomorphism of  $\mathbb{C}_{\infty}$ , whose inverse is again a Möbius transformation:

**PROPOSITION.** Let  $G = \operatorname{Gl}_2(\mathbb{C})$  be the group of nonsingular  $2 \times 2$  complex matrices; for

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \operatorname{Gl}_2(\mathbb{C})$$

let  $f_A$  be the Möbius transformation given by

$$f_A(z) = \frac{az+b}{cz+d}.$$

Then  $A \mapsto f_A$  is a surjective group homomorphism of  $\operatorname{Gl}_2(\mathbb{C})$  onto the group of Möbius transformations of the Riemann sphere  $\mathbb{C}_{\infty}$ ; the kernel of this homomorphism is the subgroup  $\left\{ \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} : \lambda \in \mathbb{C}^{\times} \right\}$  of all scalar matrices.

*Proof.* The proof, an easy computation, is left to the reader.

EXERCISE 1.1.5.1. Denote by  $Sl_2(\mathbb{C})$  the subgroup of  $Gl_2(\mathbb{C})$  consisting of all matrices with determinant 1. Prove that the restriction to  $Sl_2(\mathbb{C})$  of the preceding homomorphism is still onto, and has  $\{I_2, -I_2\}$  as kernel, where  $I_2$  is the 2 × 2 identity matrix.

We note that the Riemann sphere  $\mathbb{C}_{\infty}$  can be canonically identified with  $\mathrm{P}^1\mathbb{C}$ , the one dimensional complex projective space; in this context the Möbius transformations are exactly the projective transformations of  $\mathrm{P}^1\mathbb{C}$ . Let us recall that the space  $\mathrm{P}^1\mathbb{C}$  is the quotient of  $\mathbb{C}^2_* = \mathbb{C}^2 \setminus \{(0,0)\}$ , punctured two dimensional complex space, by the equivalence relation  $(x, y) \sim (u, v)$  iff there exists  $\lambda \in \mathbb{C}$  such that  $x = \lambda u$  and  $y = \lambda v$ , that is, the equivalence relation whose equivalence classes are the punctured onedimensional complex subspaces of  $\mathbb{C}^2$ . If we consider the map  $k : \mathbb{C}^2_* \to \mathbb{C}_{\infty}$  given by  $k(z_1, z_2) = z_1/z_2$ when  $z_2 \neq 0$  and  $k(z_1, 0) = \infty$ , the map k is continuous and onto, and its fibers are exactly the equivalence classes which give the points of the projective space  $\mathrm{P}^1\mathbb{C}$ ; since  $\mathrm{P}^1\mathbb{C}$  is a compact space under the quotient topology we have that k induces an homeomorphism h of  $\mathrm{P}^1\mathbb{C}$  onto  $\mathbb{C}_{\infty}$ . The linear automorphisms of  $\mathbb{C}^2$  clearly induce bijective maps of  $\mathrm{P}^1\mathbb{C}$ ; these maps transferred to  $\mathbb{C}_{\infty}$  by the conjugation with h are exactly the Möbius transformations. We refer the reader to [Conway] for more about Möbius transformations "an amazing class of maps".

1.1.6. Involutions. We address the following problem: given  $a \in \mathbb{C}$ ,  $a \neq 0$ , find a Möbius transformation g which is an involution (i.e., equal to its inverse) and maps 0 to a. Since g(a) = 0 by the involutive requirement, we can certainly write g(z) = (z-a)/(cz+d); and since g(0) = a, we have -a/d = a, hence d = -1 (since  $a \neq 0$ ). So far we have

$$g(z) = \frac{z-a}{c\,z-1}$$
; but the determinant det  $\begin{bmatrix} 1 & -a \\ c & -1 \end{bmatrix} = -1 + ac$ 

must be non zero, hence  $c \neq 1/a$ . Next, a computation gives

$$g(g(z)) = \frac{(1-ac)z}{(1-ac)} = z;$$

so that g is indeed an involution. It follows that all maps g(z) = (z - a)/(cz - 1) are the required involutions, provided that  $c \neq 1/a$ . For future use, we impose the further restriction that the involution preserves the unit circle  $\mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}$ ; this of course implies  $a \notin \mathbb{U}$  and holds if and only if we have |z - a|/|cz - 1| = 1 for every  $z \in \mathbb{U}$ , i.e. if and only if

$$|cz-1| = |z-a|$$
 for all  $z \in \mathbb{U}$ , i.e.  $|cz-1|^2 = |z-a|^2$  for all  $z \in \mathbb{U}$ ,

equivalently

$$|c|^{2}|z|^{2} + 1 - 2\operatorname{Re}(cz) = |z|^{2} + |a|^{2} - 2\operatorname{Re}(z\bar{a}) \iff |c|^{2} - |a|^{2} = 2\operatorname{Re}(z(c-\bar{a})) \text{ for all } z \in \mathbb{U}.$$

This is impossible if  $|c| \neq |a|$ : in fact, changing z in -z the right-hand side changes sign; then we have  $\operatorname{Re}(z(c-\bar{a})) = 0$  for every  $z \in \mathbb{U}$ , from which we immediately deduce that  $c - \bar{a} = 0$ , i.e.  $c = \bar{a}$ .

We have obtained:

. For every  $a \neq 0$ , with  $|a| \neq 1$ , there exists exactly one Möbius involution  $g_a$  which exchanges 0 and a and preserves the unit circle  $\partial \Delta = \mathbb{U}$ . We have

$$g_a(z) = \frac{z-a}{\bar{a}\,z-1}.$$

If |a| < 1 then  $g_a$  induces a holomorphic automorphism of the unit disc.

Proof. The only statement not already checked is the last; since  $g_a(\Delta) \cap g_a(\partial\Delta) \subseteq g_a(\Delta) \cap \partial\Delta = \emptyset$ and  $g_a(\Delta)$  is connected we have either  $g_a(\Delta) \subseteq \Delta$  or  $g_a(\Delta) \subseteq \mathbb{C} \setminus \overline{\Delta}$ ; but  $a = g_a(0) \in \Delta$ , so  $g_a(\Delta) \subseteq \Delta$ ; by applying  $g_a$  to both sides we get  $g_a(g_a(\Delta)) \subseteq g_a(\Delta)$ , but  $g_a(g_a(\Delta)) = \Delta$ , hence  $g_a(\Delta) = \Delta$  (at any rate, an easy computation proves directly that  $g_a(\Delta) \subseteq \Delta$ ).

1.1.7. The group of automorphisms of the unit disc. We want to explicitly determine every automorphism of the unit disc  $\Delta = \{z \in \mathbb{C} : |z| < 1\}.$ 

LEMMA. (SCHWARZ'S LEMMA) Let  $f : \Delta \to \Delta$  be a holomorphic self-map of the unit disc  $\Delta$ . Assume that f(0) = 0. Then  $|f(z)| \leq |z|$  for every  $z \in \Delta$ , and  $|f'(0)| \leq 1$ . If |f(z)| = |z| occurs for some nonzero  $z \in \Delta$ , or if |f'(0)| = 1, then f is a rotation, that is, there exists  $u \in \mathbb{U}(=\partial\Delta)$  such that f(z) = uz for every  $z \in \Delta$ .

Proof. Since f(0) = 0, the function g(z) := f(z)/z has a removable singularity at z = 0. For every r, with 0 < r < 1, if  $|z| \le r$  we have  $|g(z)| \le \max\{|g(\zeta)| : |\zeta| = r\}$  by the maximum modulus theorem. But  $|g(\zeta)| = |f(\zeta)|/|\zeta| = |f(\zeta)|/r < 1/r$  if  $|\zeta| = r$ , hence |g(z)| < 1/r if  $|z| \le r$ ; letting  $r \to 1^-$  we get  $|g(z)| \le 1$  for all  $z \in \Delta$ , equivalently,  $|f(z)| \le |z|$  for all  $z \in \Delta$ . Since g(0) = f'(0), we also have  $|f'(0)| \le 1$ . If for any  $z \in \Delta$  we have |g(z)| = 1, then |g| attains its maximum in  $\Delta$ , and again by the maximum modulus theorem g is then constant on  $\Delta$ , of course of absolute value 1. This concludes the proof.

#### **PROPOSITION.** An automorphism of the unit disc $\Delta$ which leaves 0 fixed is a rotation.

Proof. Let f be such an automorphism; by hypothesis f(0) = 0, hence also  $f^{-1}(0) = 0$ . Then  $|f(z)| \leq |z|$ , and also  $|f^{-1}(z)| \leq |z|$ , for all  $z \in \Delta$ , by the Schwarz lemma applied to f and  $f^{-1}$ . Substitution of f(z) in place of z in the second inequality yields  $|z| \leq |f(z)|$ , for all  $z \in \Delta$ . But then |f(z)| = |z| for every  $z \in \Delta$ , and again by the lemma there exists  $u \in \mathbb{U}$  such that f(z) = uz for every  $z \in \Delta$ .

We can now describe all the elements of  $Aut(\Delta)$  when  $\Delta$  is the unit disc.

THEOREM. Every automorphism h of the unit disc  $\Delta$  is of the form

$$h(z) = e^{i\alpha} \frac{z-a}{\bar{a}\,z-1} \quad a \in \Delta, \, \alpha \in ]-\pi,\pi] \quad (the involution \, g_a \text{ followed by a rotation of angle } \alpha).$$

*Proof.* Let  $h \in \operatorname{Aut}(\Delta)$  be given; if  $h(0) = b \neq 0$  then  $g_b \circ h(=g_b^{-1} \circ h)$  is an automorphism of  $\Delta$  which keeps 0 fixed hence there exists  $\alpha \in ]-\pi,\pi]$  such that  $g_b \circ h(z) = e^{i\alpha} z$ . Then  $h(z) = g_b(e^{i\alpha} z)$  for every  $z \in \Delta$ ; but

$$h(z) = g_b(e^{i\alpha} z) = \frac{e^{i\alpha} z - b}{(\bar{b})(e^{i\alpha} z) - 1} = e^{i\alpha} \frac{z - (e^{-i\alpha}b)}{(\bar{e}^{-i\alpha}b) z - 1} = e^{i\alpha} \frac{z - a}{\bar{a} z - 1},$$

if  $a = e^{-i\alpha}b$ .

REMARK. In passing, we have also obtained the commutation formula:

$$g_b \circ (e^{i\alpha} \cdot) = e^{i\alpha} g_{e^{-i\alpha}b}(\cdot)$$

and seen that every automorphism of  $\Delta$  may be written in the form

$$h(z) = g_b(e^{i\alpha} z) = \frac{e^{i\alpha} z - b}{(\overline{b})(e^{i\alpha} z) - 1}.$$

**1.2. The Riemann mapping theorem.** There is a very deep result due to Riemann concerning isomorphism classes of simply connected regions in  $\mathbb{C}$ , which we now state. The proof will be completed later; here we pave the way for it. There clearly can be no holomorphic isomorphism of  $\mathbb{C}$  onto the unit disc: every entire function which takes values only in the unit disc is constant, by Liouville's theorem. But this is the only exception! a simply connected region of  $\mathbb{C}$  is indeed holomorphically isomorphic to the unit disc, if it is not all of  $\mathbb{C}$ ! Among simply connected subregions of  $\mathbb{C}$  there are thus only two isomorphism classes, one containing  $\mathbb{C}$  alone, the other containing all simply connected open proper subsets of  $\mathbb{C}$ .

. RIEMANN MAPPING THEOREM. Let G be a proper simply connected subregion of  $\mathbb{C}$ , and let  $\Delta$  denote the unit disc. For every  $c \in G$  there exists a holomorphic bijection  $f: G \to \Delta$  with f(c) = 0.

1.2.1. Regions with the square root property. We say that a region G has the square root property if every function holomorphic on G which never vanishes has a holomorphic square root, that is, for every  $u \in \mathcal{O}(G)$  with  $Z_G(u) = \emptyset$ , there exists  $v \in \mathcal{O}(G)$  such that  $v^2 = u$ . A simply connected region has the square root property:

PROPOSITION. Let G be a simply connected region, and let  $f : G \to \mathbb{C}$  be holomorphic and zero-free in G. Then f has logarithms on G (i.e., there exists  $g \in \mathcal{O}(G)$  such that  $f = \exp g$ ), and also roots of all orders (in particular square roots).

Proof. Observe that  $g \in \mathcal{O}(G)$  is a logarithm of f iff for some  $c \in G$  we have  $\exp g(c) = f(c)$ , and moreover g'(z) = f'(z)/f(z) for all  $z \in G$ . This last assertion says that g is a primitive of the logarithmic derivative f'/f of f; and since G is simply connected, every holomorphic function has a primitive in G; thus f has logarithms. Given a logarithm g of f, for every natural number m the function  $v = \exp(g/m)$ is an m-th root of f; in fact  $(v(z))^m = \exp(m(g(z)/m)) = \exp g(z) = f(z)$ , for every  $z \in G$ .

We also observe that the square root property is invariant by holomorphic isomorphisms: if E has the square root property and G is isomorphic to E, then G has the square root property: if  $f: G \to \mathbb{C}$ is holomorphic and never 0, and  $\eta: E \to G$  is an holomorphic isomorphism, then  $f \circ \eta$  is a nowherevanishing holomorphic function on E; if  $v: E \to \mathbb{C}$  is a square root of  $f \circ \eta$ , then  $v \circ \eta^{-1}$  is a square root of f, as is easy to check.

After the Riemann mapping theorem is proved, it will be obvious that the square root property is equivalent to simple connectedness; but this property is the only tool needed for the proof of the theorem.

REMARK. Even if not relevant to the present aim, it is interesting to observe that the square root property for a region E implies directly that every nowhere–vanishing  $f \in \mathcal{O}(E)$  has a logarithm in  $\mathcal{O}(E)$ . In fact, f has a logarithm if and only if f'/f has a primitive on E, and this happens if and only if for every loop  $\gamma$  in E the integral  $\int_{\gamma} (f'/f) dz$  vanishes. We know that

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz = m$$
, an integer.

But if f has roots of arbitrarily high order, then m = 0. In fact, if v is a  $k^{\text{th}}$  root of f, that is  $f = v^k$ , then m is divisible by k:

$$m = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \int_{\gamma} \frac{k \, (v(z))^{k-1} \, v'(z)}{(v(z))^k} \, dz = \frac{k}{2\pi i} \int_{\gamma} \frac{v'(z)}{v(z)} \, dz = k \, p_k,$$

with  $p_k$  an integer (the winding number of the loop  $v \circ \gamma$  around 0). But the only integer divisible by infinitely many different integers is of course 0.

1.2.2. Injective mappings into the unit disc. We start by proving that for every proper region G with the square root property there exist injective holomorphic mappings  $f: G \to \Delta$ , which map a prescribed point  $c \in G$  to 0. We need only to prove that there is at least one injective map  $f: G \to \Delta$ ; in fact, if  $f(c) = a \neq 0$  we consider  $g_a \circ f$ , where  $g_a(z) = (z-a)/(\bar{a} z - 1)$  is the involution of  $\Delta$  which sends a to 0. And if  $\mathbb{C} \setminus G$  has non-empty interior (equivalently, if G is not dense in  $\mathbb{C}$ ) it is trivial to find an injective holomorphic map  $f: G \to \Delta$ ; simply take c in the interior of  $\mathbb{C} \setminus G$ , so that for some  $\delta > 0$  we have  $|z-c| > \delta$  for all  $z \in G$ ; then  $f(z) = \delta/(z-c)$  maps injectively G into  $\Delta$ . Of course  $\mathbb{C} \setminus G$  may have empty interior, but if it is non empty some square root function will establish an isomorphism between G and a non dense open region E, as we now prove. If  $a \in \mathbb{C} \setminus G$ , then z-a is never 0 in G, hence there exists  $v \in \mathcal{O}(G)$  such that  $(v(z))^2 = z - a$  for all  $z \in G$ , by the square root property. Observe that v is injective, since  $z \mapsto z - a$  is injective; moreover  $v(G) \cap (-v(G)) = \emptyset$ : in fact, if v(p) = -v(q) for some  $p,q \in \mathbb{C}$  then  $p-a = (v(p))^2 = (-v(q))^2 = q-a$ , hence p = q, but then v(p) = -v(p), hence v(p) = 0, impossible. Thus -v(G) is an open set disjoint from v(G); hence E = v(G) is isomorphic to G and not dense in  $\mathbb{C}$ , and as seen above it can be mapped injectively into  $\Delta$ . We have proved:

. If G is a region with the square root property which is not all of  $\mathbb{C}$ , then for every  $c \in G$  there exists an injective holomorphic mapping  $f: G \to \Delta$  such that f(c) = 0.

1.2.3. Tentative isomorphisms. If we want the map f to be also bijective, the Schwarz lemma says that we have to pick the map f "as large as possible". This is in some sense to be expected, since we want the image of f to fill the entire unit disc, and is made more precise by the following observation:

. Let G be a region,  $c \in G$ , and let  $f, h : G \to \Delta$  be injective maps of G into  $\Delta$ , with f(c) = h(c) = 0. If h is bijective, then  $|f(p)| \leq |h(p)|$  for every  $p \in G$ ; and if equality holds, for some  $p \in G \setminus \{c\}$ , then f is also bijective.

*Proof.* Take  $g: \Delta \to \Delta$  defined by  $g(z) = f \circ h^{-1}(z)$ . Then  $g(\Delta) \subseteq \Delta$  and g(0) = 0. By the Schwarz lemma we have  $|g(z)| \leq |z|$  for all  $z \in \Delta$ , hence, if z = h(p), we have  $|f(p)| \leq |h(p)|$ ; and still by the Schwarz lemma, if for some nonzero  $w \in \Delta$  we have |g(w)| = |w| then there exists  $u \in \mathbb{U}$  such that g(z) = uz for every  $z \in \Delta$ , so that f(p) = uh(p) for every  $p \in G$ ; but then f is h rotated, and hence f is also bijective.

1.2.4. *Maximality of isomorphisms*. Next we prove that for regions with the square root property the preceding maximality is also sufficient for isomorphisms. That is

. Let G be a region with the square root property,  $c \in G$  and  $f: G \to \Delta$  injective and holomorphic with f(c) = 0. If f is not bijective then there exist  $g: G \to \Delta$  injective and holomorphic, with g(c) = 0, and |f(p)| < |g(p)| for every  $p \in G \setminus \{c\}$ .

Proof. Set E = f(G); then E is isomorphic to G via f, hence E has the square root property, too. Assuming  $\Delta \setminus E$  non empty, we manifacture a holomorphic injection  $\eta : E \to \Delta$  with  $\eta(0) = 0$  and  $|\eta(z)| > |z|$  for every  $z \in E \setminus \{0\}$ ; then  $g = \eta \circ f$  is as required. Take  $b \in \Delta$  such that  $a = b^2 \notin E$ ; consider the involution  $g_a$  (as usual,  $g_a(z) = (z-a)/(\bar{a}z-1)$ ) which then is never 0 on E, and take that square root  $v : E \to \Delta$  of  $g_a$  such that v(0) = b. We define  $\eta : E \to \Delta$  by  $\eta(z) = g_b \circ v(z)$ . Then  $\eta(0) = 0$ . To prove that  $\eta$  has the "dilation" property  $|\eta(z)| > |z|$  for  $z \in E \setminus \{0\}$ , we construct  $\psi : \Delta \to \Delta$  which is a left inverse for  $\eta$ , i.e. such that  $\psi(\eta(z)) = z$  for every  $z \in E$ . First use the involution  $g_b$  to map 0 to b, then use the squaring map  $p_2(z) = z^2$ ; under  $p_2 \circ g_b$  the origin 0 is mapped to  $b^2 = a$ ; apply the involution  $g_a$  to send a back to 0; define then  $\psi = g_a \circ p_2 \circ g_b$ . We prove that  $\psi \circ \eta = id_E$ : in fact  $\psi \circ \eta = g_a \circ p_2 \circ g_b \circ g_b \circ v = g_a \circ p_2 \circ v$ ; and  $g_a \circ p_2 \circ v(z) = g_a((v(z))^2)$ ; but for  $z \in E$  we have  $(v(z))^2 = g_a(z)$ , by definition of v; hence  $g_a((v(z))^2) = g_a(g_a(z)) = z$ . Moreover  $\psi(0) = 0$ , and  $\psi$  cannot be an automorphism of  $\Delta$  (it is factored through  $p_2$ , which is not injective), in particular  $\psi$  is not a rotation; by the Schwarz lemma we then have  $|\psi(z)| < |z|$  for  $z \in \Delta \setminus \{0\}$ ; if in that inequality we put  $\eta(z)$  in place of z, we obtain  $|z| < |\eta(z)|$  for  $z \in E \setminus \{0\}$ , as required.

1.2.5. Characterization of isomorphisms.

. Let G be a proper region with the square root property. Given  $c \in G$ , let  $I_c(G)$  denote the set of injective holomorphic mappings  $g: G \to \Delta$  such that g(c) = 0. The following are then equivalent, for every  $h \in I_c(G)$ :

(i) h is an isomorphism (i.e.,  $h(G) = \Delta$ ).

(ii) For every  $p \in G \setminus \{c\}$  we have  $|h(p)| = \sup\{|g(p)| : g \in I_c(G)\}$ .

(iii) For some  $p \in G \setminus \{c\}$  we have  $|h(p)| = \sup\{|g(p)| : g \in I_c(G)\}$ .

*Proof.* (i) implies (ii) see 1.2.3; (ii) implies (iii) trivial; (iii) implies (i) see 1.2.4.

REMARK. Observe that  $I_c(G)$  is non empty by 1.2.2.

1.2.6. Existence and uniqueness.

. Let G be a proper simply connected region. For every  $c \in G$  there exists a unique isomorphism  $f: G \to \Delta$  such that f(c) = 0, and f'(c) > 0.

*Proof.* We shall use only the square root property of G.

Uniqueness: two isomorphisms  $f, g: G \to \Delta$  which coincide in c differ from one another by a rotation of  $\Delta$ , as it is clear from the Schwarz lemma; that is, there exists  $u \in \mathbb{U}$  such that g(z) = u f(z), for every  $z \in G$ . Then g'(z) = u f'(z), and in particular g'(c) = u f'(c). If g'(c), f'(c) are both real and positive, then g'(c)/f'(c) is also real and positive, and hence u = 1.

Existence: the proof of existence is deferred to the next section (2.7.2). But we can anticipate here how it will be done: we shall introduce a topology on the set  $\mathcal{O}(G, \mathbb{C})$  of all holomorphic functions on G; in this topology, the evaluation maps  $f \mapsto f(p)$  are continuous from  $\mathcal{O}(G, \mathbb{C})$  to  $\mathbb{C}$ , for every  $p \in G$ ; moreover, the set  $I_c(G) \cup \{0\}$  is compact, so that the map  $f \mapsto |f(p)|$  has an absolute maximum on this set; this maximum is clearly not attained on the 0 map; any map  $f \in I_c(G)$  which realizes this maximum is an isomorphism of G onto  $\Delta$ , by the preceding result.

EXERCISE 1.2.6.1. The mapping  $z \mapsto z/\sqrt{1+|z|^2}$  is the standard diffeomorphism of  $\mathbb{C} = \mathbb{R}^2$  onto the unit disc; it is a  $C^{\infty}$  diffeomorphism (in the real sense). Prove it, finding the inverse.

- EXERCISE 1.2.6.2. (i) Let A and B be open subsets of  $\mathbb{C}$ , and let  $\varphi : A \to B$  be a homeomorphism of A onto B. Assume that  $z_j$  is a sequence in A which converges to a point  $a \in \partial A$ , and that  $\varphi(z_j)$  converges to a point  $b \in \mathbb{C}$ . Prove that  $b \in \partial B$ .
- (ii) Let  $A = \Delta \setminus \{0\} = \{z \in \mathbb{C} : 0 < |z| < 1\}$  be the punctured unit disc, and let B be the annulus  $B = \{z \in \mathbb{C} : 1 < |z| < 2\}$ . Prove that there exists no holomorphic isomorphism of A onto B (Hint: for a holomorphic mapping  $f : A \to B$ , 0 is a removable singularity ... use then (i) ...).
- (iii) Define explicitly a diffeomorphism of A onto B, with A, B as in (ii).

1.2.7. Local normal form. Very often Theorem 1.1.1 is stated in the following way:

. Let D be open in  $\mathbb{C}$ , let  $f: D \to \mathbb{C}$  be holomorphic and not locally constant at  $c \in D$ ; let  $m = \operatorname{ord}(f - f(c), c)$ . Then

- (i) existence There exist an open disc  $B \subseteq D$  centered at c, and an holomorphic isomorphism  $h: B \to h(B)$  such that  $f(z) = f(c) + (h(z))^m$  for every  $z \in B$ .
- (ii) uniqueness If  $\tilde{B} \subseteq D$  is another disc centered at c, and  $\tilde{h} : \tilde{B} \to \tilde{h}(\tilde{B})$  another isomorphism such that  $f(z) = f(c) + (\tilde{h}(z))^p$  for every  $z \in \tilde{B}$  and an integer p, then p = m and there is an m-th root of unity  $\xi$  such that  $h(z) = \xi \tilde{h}(z)$  for  $z \in B \cap \tilde{B}$ .

*Proof.* We assume that for m = 1 the theorem is known: that is we know that if  $f'(c) \neq 0$  then f is locally biholomorphic at c.

Proof of existence: we have  $f(z) = f(c) + (z - c)^m g(z)$ , with g holomorphic on D and  $g(c) \neq 0$ . On every disc  $B \subseteq D$  centered at B on which g(z) is non-zero, there is an m-th root of g(z), say v(z). Put h(z) = (z-c)v(z). Then  $(h(z))^m = (z-c)^m g(z)$ , and h'(z) = v(z) + (z-c)v'(z), so that  $h'(c) = v(c) \neq 0$ (recall that  $(v(c))^m = g(c) \neq 0$ ). Thus h is locally biholomorphic at c, and by restricting B if necessary we can make h an isomorphism of B onto h(B).

Proof of uniqueness: We have  $(h(z))^m = (\dot{h}(z))^p$  for every  $z \in B \cap \dot{B} = C$ . Since  $h(c) = \dot{h}(c) = 0$ , and ord $(h, c) = \operatorname{ord}(\tilde{h}, c) = 1$  (recall that h and  $\tilde{h}$  are both bi-holomorphic at c), we have  $m = \operatorname{ord}(h^m, c) =$ ord $(\tilde{h}^p, c) = p$  so that m = p. And  $(h(z))^m = (\tilde{h}(z))^m$  for every  $z \in C$  is equivalent to  $(h(z)/\tilde{h}(z))^m = 1$ for every  $z \in C$ , whence  $h(z)/\tilde{h}(z)$  maps continuously C into the group of m-th roots of unity; but C is connected and this group is discrete, so the map is constant, say constantly  $\xi$ .

EXERCISE 1.2.7.1. An expansion on a region E of  $\mathbb{C}$  containing 0 is a holomorphic map  $\eta : E \to \mathbb{C}$  such that  $\eta(0) = 0$  and  $|\eta(z)| > |z|$  for every  $z \in E \setminus \{0\}$ . Prove that if  $\eta : E \to \mathbb{C}$  is an expansion then  $|\eta'(0)| > 1$  (use the minimum modulus theorem).

EXERCISE 1.2.7.2. Using the preceding exercise prove that if G is a region with the square root property and  $c \in G$ , then  $f \in I_c(G, \Delta)$  is an isomorphism if and only if  $|f'(c)| = \max\{|g'(c)| : g \in I_c(G, \Delta)\}$  (as before,  $I_c(G, \Delta) := \{f \in \mathcal{O}(G) : f(G) \subseteq \Delta, f(c) = 0, f \text{ injective}\}$ .

PROBLEM 1.2.7.3. Let  $\Delta = \{z \in \mathbb{C} : |z| < 1\}$  be the open unit disc. Recall that for every  $c \in \Delta \setminus \{0\}$  the mapping  $g_c(z) = (z - c)/(\bar{c}z - 1)$  is an involution of  $\Delta$  which exchanges 0 and c.

(i) Are there other involutions in  $Aut(\Delta)$  which exchange 0 and c? For c = 0, find all involutions of  $Aut(\Delta)$  which keep 0 fixed.

Let now  $f: \Delta \to \Delta$  be holomorphic.

(ii) Prove that for every  $w \in \Delta$  we have

$$|f'(w)| \le \frac{1 - |f(w)|^2}{1 - |w|^2},$$

and that equality holds for some  $w \in \Delta$  if and only if f is an automorphism of  $\Delta$ , in which case equality holds for every  $w \in \Delta$  (consider  $g = g_{f(w)} \circ f \circ g_w \ldots$ ).

PROBLEM 1.2.7.4. Recall that for every open set D of  $\mathbb{C}$  we denote by  $\operatorname{Aut}(D)$  the group of holomorphic automorphisms of D. For every subset  $S \subseteq D$  we denote by  $\operatorname{Aut}_S(D)$  the subset of  $\operatorname{Aut}(D)$  consisting of all automorphisms under which S is invariant, i.e.  $\operatorname{Aut}_S(D) = \{f \in \operatorname{Aut}(D) : f(S) = S\}$ ; plainly  $\operatorname{Aut}_S(D)$  is a subgroup of  $\operatorname{Aut}(D)$ . We shorten  $\operatorname{Aut}_{\{c\}}(D)$  to  $\operatorname{Aut}_c(D)$  when  $\{c\} \subseteq D$  is a singleton.

- (i) Assume that D is a region and that S has empty interior and is closed in D. Prove that the restriction map  $f \mapsto f_{|D \setminus S}$  is an injective group homomorphism of  $\operatorname{Aut}_S(D)$  into  $\operatorname{Aut}(D \setminus S)$ .
- (ii) With  $D = \mathbb{C}$  and  $S = \{0\}$ , observe that in general the preceding restriction monomorphism is not an isomorphism.
- (iii) However, with  $D = \Delta$ , open unit disc, and  $S = \{0\}$ ,  $f \mapsto f_{|\Delta \setminus \{0\}}$  is an isomorphism of  $\operatorname{Aut}_0(\Delta)$  onto  $\operatorname{Aut}(\Delta_*)$ , where  $\Delta_* = \Delta \setminus \{0\}$  is the punctured disc (hint: for every holomorphic self-mapping g of  $\Delta_*$ , 0 is a removable singularity ... (why?))
- (iv) In fact, the preceding result can be substantially generalized. Prove that:

. Let D be a bounded region, and let S be a discrete subset of D, closed in D. Assume also that  $\partial D$  has no isolated points. Then  $f \mapsto f_{|D \setminus S}$  is an isomorphism of  $\operatorname{Aut}_S(D)$  onto  $\operatorname{Aut}(D \setminus S)$ .

#### 2. The topology of compact convergence

**2.1. Definitions.** Let D be open in  $\mathbb{C}$  and let  $C(D) = C(D, \mathbb{C})$  denote the set of all continuous complex-valued functions on D; C(D) has a natural structure of  $\mathbb{C}$ -algebra under pointwise operations. It is customary to topologize C(D) with the compact-open topology, also called topology of uniform convergence on compacta, or simply topology of compact convergence.

Given a compact subset K of D, and an open subset U of  $\mathbb{C}$ , we define

$$[K, U] = \{ f \in C(D); f(K) \subseteq U \},\$$

the set of all continuous functions on D which map K inside U. The compact-open topology has as a subbase all the sets [K, U], with K varying in the set of all compact subsets of D, and U in the topology of  $\mathbb{C}$ . A base for the open sets of this topology is then the family of all finite intersections

 $[K_1, U_1] \cap \cdots \cap [K_m, U_m],$  with  $K_j \subseteq D$  compact,  $U_j \subseteq \mathbb{C}$  open, for  $j = 1, \ldots, m$ .

This description of the compact-open topology works also in very abstract settings; here, since  $\mathbb{C}$  is metrizable, there is an alternate description, perhaps more illuminating, that we shall hereafter use. For every compact subset K of D and every  $f \in C(D)$  we set

$$||f||_K = \max\{|f(z)| : z \in K\} \qquad (\text{sup-norm of } f \text{ on } K).$$

Clearly this maximum exists by Weierstrass theorem. The function  $f \mapsto ||f||_K$  is a *seminorm* on C(D): by this we mean that it has the following properties, easily checked by the reader:

- (i)  $||f||_K \ge 0$ , for every  $f \in C(D)$ , and  $||0||_K = 0$  (positivity);
- (ii)  $||f + g||_K \le ||f||_K + ||g||_K$ , for every  $f, g \in C(D)$  (subadditivity);
- (iii)  $\|\alpha f\| = |\alpha| \|f\|_K$ , for every  $f \in C(D)$  and every  $\alpha \in \mathbb{C}$  (absolute homogeneity).

This family of seminorms, with K describing the set of all compact subsets of D, is used to give C(D) the topology of uniform convergence on compacta: given  $f, g \in C(D)$  the number  $||f - g||_K$  is the (uniform) distance of f, g over K, and given  $f \in C(D)$  a neighborhood of f in this topology is any subset of C(D) containing a subset of the form

$$B_K(f,\varepsilon) = \{g \in C(D) : \|g - f\|_K < \varepsilon\}$$
 for some compact  $K \subseteq D$  and some  $\varepsilon > 0$ ;

 $B_K(f,\varepsilon)$  is the set of continuous functions that everywhere on K are  $\varepsilon$ -close to f, the open K-ball of radius  $\varepsilon$  and center f. It is a topology because if K, L are compact of D and  $\varepsilon > 0$  then:

$$B_K(f,\varepsilon[\cap B_L(f,\varepsilon[=B_{K\cup L}(f,\varepsilon[,$$

and  $K \cup L$  is of course compact (is this enough? see 2.2.1). Notice also that if K and L are compact and  $K \subseteq L$  then  $||f||_K \leq ||f||_L$ , for every  $f \in C(D)$ , since actually  $||f||_K \vee ||f||_L = ||f||_{K \cup L}$ .

**2.2.** A metric for the compact–open topology. It is not difficult to prove that the two topologies previously introduced coincide (see 2.7.5). We observe instead that the topology in discussion is metrizable. First, observe that

LEMMA. For every open set D of  $\mathbb{C}$  there exists a sequence  $(K_n)_{n\in\mathbb{N}}$  of compact subsets of D such that for every  $n \in \mathbb{N}$  we have  $K_n \subseteq int(K_{n+1})$ , and  $D = \bigcup_{n\in\mathbb{N}} K_n$ .

*Proof.* Simply take  $K_n = \{z \in \mathbb{C} : |z| \leq 2^n, \operatorname{dist}(z, \mathbb{C} \setminus D) \geq 1/2^n)\}$ . Every  $K_n$  is closed and bounded, hence compact, it is contained in D, and

$$K_n \subseteq A_n = \{ z \in \mathbb{C} : |z| < 2^{n+1}, \operatorname{dist}(z, \mathbb{C} \setminus D) > 1/2^{n+1} \} \subseteq K_{n+1},$$

and  $A_n$  is open.

The usefulness of this sequence is due to the fact that given any compact subset K of D there exists  $m \in \mathbb{N}$  such that  $K \subseteq int(K_m)$  (since  $K \subseteq \bigcup_{n \in \mathbb{N}} int(K_n)$ , and the sequence  $int(K_n)$  is monotone increasing). In what follows we write  $||f||_n$  in place of  $||f||_{K_n}$ , and we define first the distance between 0 and a function  $f \in C(D)$ : it is

$$[f] = \max\{\|f\|_n \land 2^{-n}; n \in \mathbb{N}\};\$$

by definition,  $||f||_n \wedge 2^{-n} = \min\{||f||_n, 2^{-n}\}$ ; the maximum exists because every sequence of non negative numbers with limit 0 has a maximum (prove it!). The function  $f \mapsto [f]$  has the following properties:

- (i)  $[f] \ge 0, [f] = 0$  iff f = 0;
- (ii)  $[f+g] \le [f] + [g];$
- (iii) If  $|\alpha| \leq 1$ , then  $[\alpha f] \leq [f]$ , for every  $f \in C(D)$ ; moreover  $[\alpha f] = [f]$  if  $|\alpha| = 1$ .

(warning:  $f \mapsto [f]$  is not a norm). The verification of these properties is left to the reader; we suggest to prove first that if the function  $\rho: [0, +\infty[ \rightarrow [0, +\infty[$  is defined by  $\rho(t) = t \wedge c$ , where c > 0 is a given constant, then  $\rho(t+s) \leq \rho(t) + \rho(s)$  (see 2.2.1).

We then define the distance between  $f, g \in C(D)$  as d(f,g) = [f-g], and it is immediate to prove that this definition makes C(D) into a metric space; moreover d is also translation invariant, in the sense that d(f,g) = d(f+h,g+h), for every  $f,g,h \in C(D)$ . We prove that the topology of this metric d is that of the seminorms  $\|\cdot\|_K$ .

PROPOSITION. For every  $\varepsilon > 0$  there exists a compact subset  $K = K_{\varepsilon}$  of D, and a  $\delta = \delta_{\varepsilon} > 0$  such that  $||f - g||_K < \delta$  implies  $[f - g] < \varepsilon$ .

Conversely, given a compact subset K of D, and  $\varepsilon > 0$ , there is  $\delta = \delta_{\varepsilon} > 0$  such that  $[f - g] < \delta$ implies  $||f - g||_K < \varepsilon$ .

*Proof.* Simply observe that we have, for every  $f \in C(D)$  and every  $m \in \mathbb{N}$ :

$$[f] < 2^{-m} \iff ||f||_m < 2^{-m}.$$

In fact,  $[f] < 2^{-m}$  iff for every  $n \in \mathbb{N}$  we have  $||f||_n \wedge 2^{-n} < 2^{-m}$ ; for n > m this poses no restriction, for n = m implies that  $||f||_m < 2^{-m}$ ; and since  $||f||_n \le ||f||_m$ , and  $2^{-m} \le 2^{-n}$  for  $n \le m$ , we have that  $||f||_m < 2^{-m}$  implies also that  $[f] < 2^{-m}$ .

Moreover every compact subset K of D is contained in some  $K_m$ , so that  $||f||_K \leq ||f||_m$ . The conclusion is easy.  $\square$ 

The preceding proposition clearly proves that the two topologies, that of the metric d, and that of the seminorms  $\|\cdot\|_K$  are the same; and also gives the following:

COROLLARY. A sequence  $(f_k)_{k\in\mathbb{N}}$  of functions of C(D) converges to  $f \in C(D)$  in the metric d (respectively, is a Cauchy sequence for the metric d) if and only if it converges to f in every seminorm  $\|\cdot\|_{K}$  (respectively, is a Cauchy sequence in every seminorm  $\|\cdot\|_{K}$ ).

From the above it is also clear that although the metric depends on the particular sequence  $(K_n)_{n\in\mathbb{N}}$ of compacta used for its definition, any other such sequence will yield a uniformly equivalent metric, not only with the same topology, but also with the same Cauchy sequences.

It is important to know that the compact-open topology is metrizable; but the metric [f-g] is rather cumbersome, and we shall never use it; it is much more comfortable to work with the seminorms  $\|\cdot\|_{K}$ . 2.2.1. *Minutiae*. We collect here some minor punctualizations of the preceding proofs.

First, in section 2.1: we are essentially declaring that by defining a subset U of C(D) open if it coincides with the union of the  $B_K(f,\varepsilon)$  it contains, we get a topology  $\tau_c$  on C(D). Closedness of  $\tau_c$  under arbitrary unions is trivial by the definition; we have to verify closedness under (finite) intersection. For this it is convenient to observe that if  $A \in \tau_c$ , then for every  $f \in A$  there is  $\delta > 0$  and a compact  $K \subseteq D$  such that  $B_K(f, \delta) \subseteq A$ : in other words, a set is in  $\tau_c$  iff it is also a union of balls centered at its elements. This can be proved directly for a ball  $B_K(f, r]$  as in Analisi Due, 2.3.26, keeping K fixed. Then if  $A, B \in \tau_c$  and  $f \in A \cap B$  for some r, s > 0 and some compact sets K, L we have  $B_K(f, r \subseteq A$  and  $B_L(f, s \subseteq B; and since B_{K \cup L}(f, r \land s \subseteq B_K(f, r \cap B_L(f, s \subseteq A \cap B)))$ we are done. Thus  $\mathcal{B} = \{B_K(f, \varepsilon : f \in C(D), \varepsilon > 0, K \in \mathcal{K}\}$ , where  $\mathcal{K}$  is the set of all compact subsets of D is a basis for the topology  $\tau_c$ .

Second: if c > 0 then  $\rho(t) = t \wedge c$  is subadditive on  $[0, +\infty]$  and  $\rho(\lambda t) \leq \rho(t)$  if  $0 \leq \lambda \leq 1$  is quite trivial; the second fact follows from the fact that  $\rho$  is increasing; and if t, s > 0 then if s + t < c we also have s, t < c so that  $\rho(s) = s, \ \rho(t) = t \text{ and } \rho(s+t) = s+t, \text{ i.e. } \rho(s+t) = \rho(s) + \rho(t); \text{ if } c \ge s+t \text{ then } \rho(s+t) = c. \text{ and } \rho(s) \le s,$  $\rho(t) \leq t$ , so that  $\rho(s) + \rho(t) \leq s + t \leq c$ . This readily implies  $[f+g] \leq [f] + [g]$ : in fact  $||f+g||_n \leq ||f||_n + ||g||_n$ , so that  $||f+g||_n \wedge 2^{-n} \leq (||f||_n + ||g||_n) \wedge 2^{-n}$ , and  $(||f||_n + ||g||_n) \wedge 2^{-n} \leq ||f||_n \wedge 2^{-n} + ||g||_n \wedge 2^{-n}$  as just observed; this clearly implies  $[f + g] \leq [f] + [g]$ . All other requirements for [f] are trivial to prove.

#### 2.3. Completeness.

**PROPOSITION.** The space C(D) is complete with respect to the metric of compact convergence.

*Proof.* Let  $(f_i)_{i \in \mathbb{N}}$  be a sequence of C(D), Cauchy with respect to every seminorm  $\|\cdot\|_K$ . In particular (take  $K = \{z\}$ ), for every point  $z \in D$  the sequence  $f_j(z)$  is Cauchy in  $\mathbb{C}$ , hence converges in  $\mathbb{C}$  to a limit which we call f(z). We claim that  $f \in C(D)$ , and that  $(f_j)_{j \in \mathbb{N}}$  compactly converges to f. We first prove the second claim: given a compact  $K \subseteq D$  and  $\varepsilon > 0$ , let  $n_{\varepsilon} \in \mathbb{N}$  be such that  $||f_j - f_k||_K \le \varepsilon$  for  $j, k \geq n_{\varepsilon}$ ; this is equivalent to say that

keeping z and k fixed in the preceding inequality we let j tend to infinity, and we get

$$|f(z) - f_k(z)| \le \varepsilon$$
 for every  $z \in K, k \ge n_{\varepsilon}$ ,

which shows that the sequence converges to f uniformly on K. To see that f is continuous, given  $z \in D$  take K to be a compact neighborhood of z in D; given  $\varepsilon > 0$  and  $k \ge n_{\varepsilon}$  such that the above inequality holds, choose a neighborhood  $B(z, \delta] \subseteq K$  of z such that  $|f_k(w) - f_k(z)| \le \varepsilon$  for  $w \in B(z, \delta]$ . Then, for  $w \in B(z, \delta]$ :

$$|f(w) - f(z)| \le |f(w) - f_k(w)| + |f_k(w) - f_k(z)| + |f_k(z) - f(z)| \le 2\varepsilon + |f_k(w) - f_k(z)| \le 3\varepsilon,$$

and since  $\varepsilon > 0$  is arbitrary we have proved the continuity of f in z.

**2.4. Continuity of the algebraic operations.** With the compact–open topology C(D) is a topological algebra. That is, the operations that make C(D) into a  $\mathbb{C}$ –algebra are continuous:

PROPOSITION. The addition  $(f,g) \mapsto f + g$  and the multiplication  $(f,g) \mapsto f g$  are continuous (as maps of  $C(D) \times C(D)$ , with the product topology, into C(D)). The multiplication  $(\alpha, f) \mapsto \alpha f$  is continuous (as a map of  $\mathbb{C} \times C(D)$ , with the product topology, into C(D)).

Proof. Given a compact  $K \subseteq D$  and  $\varepsilon > 0$ , it is immediate to observe that  $B_K(f, \varepsilon/2[+B_K(g, \varepsilon/2] \subseteq B_K(f + g, \varepsilon])$ . This proves continuity of addition. Multiplication is a little more delicate; let us prove that there exists  $\delta > 0$  such that  $B_K(f, \delta[\cdot B_K(g, \delta] \subseteq B_K(fg, \varepsilon])$ . Writing elements of  $B_K(f, \delta[, B_K(g, \delta] \subseteq S_K(fg, \varepsilon)]$  as f + u and g + v respectively, with  $\|u\|_K$ ,  $\|v\|_K < \delta$  we have

$$\begin{aligned} \|(f+u)(g+v) - fg\|_{K} &= \|fv + gu + uv\|_{K} \le \|fv\|_{K} + \|gu\|_{K} + \|uv\|_{K} \le \\ \|f\|_{K} \|v\|_{K} + \|g\|_{K} \|u\|_{K} + \|u\|_{K} \|v\|_{K} \le (\|f\|_{K} + \|g\|_{K} + \delta)\delta; \end{aligned}$$

we can take  $\delta < \min\{\varepsilon/(1+\|f\|_K+\|g\|_K), 1\}$ . Continuity of multiplication implies also the last statement (in fact, on the constant functions the topology is that of  $\mathbb{C}$ )  $\Box$ 

REMARK. In the language of functional analysis: C(D) with the topology of compact convergence is a *Fréchet space*, that is, a locally convex metrizable and complete topological vector space.

EXERCISE 2.4.0.1. The set of units of the algebra C(D) is obviously the set  $U(D) = \{f \in C(D) : Z_D(f) = \emptyset\}$  of functions which do not assume the value 0 (here  $Z_D(f) = \{z \in D : f(z) = 0\}$  is the zero-set of f in D). As in every associative algebra with an identity element U(D) is a group under multiplication. Prove that  $f \mapsto 1/f$  is a continuous self-map of U(D). Prove that U(D) is never open in C(D).

2.4.1.  $\mathcal{O}(D)$  is closed in C(D). When one has a linear topological space, continuous linear functionals, that is continuous linear maps from the space to the field of scalars, are often significant. On the space C(D) integration over a path is a very useful linear functional. Recall that a *path* in the open subset D of  $\mathbb{C}$  is a continuous piecewise  $C^1$  function  $\alpha : [a, b] \to D$ , where [a, b] is a compact interval of  $\mathbb{R}$ ; if  $f \in C(D)$  we define the integral of f over  $\alpha$ :

$$\int_{\alpha} f(z) \, dz := \int_{a}^{b} f(\alpha(t)) \, \alpha'(t) \, dt$$

(notice that the right-hand side is the integral of a function with at most a finite set of jump discontinuities on the compact interval [a, b]; hence the integral exists, in the Riemann sense). The set  $[\alpha] = \alpha([a, b])$ is called *trace* of the path  $\alpha$ ; it is clearly a compact subset of D; we shorten  $||f||_{[\alpha]}$  to  $||f||_{\alpha}$ ; recall the fundamental inequality:

$$\left| \int_{\alpha} f(z) \, dz \right| \leq \int_{\alpha} |f(z)| \, |dz| \leq \|f\|_{\alpha} \, V(\alpha),$$

where  $|dz| = |\alpha'(t)| dt$  is the "length element" of  $\alpha$  and  $V(\alpha) = \int_{\alpha} |dz| = \int_{a}^{b} |\alpha'(t)| dt$  is the length of the path  $\alpha$ . We leave it to the reader to prove that this inequality immediately implies

. For every path  $\alpha$  in D the mapping  $\int_{\alpha} # : C(D) \to \mathbb{C}$  defined by  $f \mapsto \int_{\alpha} f(z) dz$  is a continuous linear functional on C(D).

Very important is the corollary:

COROLLARY. For every open subset D of  $\mathbb{C}$  the set  $\mathcal{O}(D)$  of holomorphic functions is a closed subalgebra of C(D).

Proof. Recall that a continuous function  $f \in C(D)$  is holomorphic on D if and only if  $\int_{\gamma} f(z) dz = 0$ for every nullhomologous circuit  $\gamma$  of D. Much less also suffices: since the derivative of a holomorphic function is also holomorphic, we know that a continuous function is holomorphic iff it has a complex primitive on every open disc contained in D; and this happens if and only if  $\int_{\gamma} f(z) dz = 0$  for every circuit  $\gamma$  whose trace  $[\gamma]$  is contained in a disc contained in D. Calling  $\Gamma$  the set of these circuits we then have

$$\mathcal{O}(D) = \bigcap_{\gamma \in \Gamma} \operatorname{Ker}\left(\int_{\gamma} \#\right),$$

and Ker  $\left(\int_{\gamma} \#\right)$  is closed in C(D) for every  $\gamma \in \Gamma$ , being the nullspace of a continuous linear form.  $\Box$ 

REMARK. Of course it is equivalent to observe that if  $f_j$  is a sequence of holomorphic functions which converges to the continuous function f, then  $\int_{\gamma} f_j$  tends to  $\int_{\gamma} f$  for every path  $\gamma$  of D; if  $\gamma \in \Gamma$  then  $\int_{\gamma} f_j = 0$ for every j, so that also  $\int_{\gamma} f = 0$ . And we don't even need all of  $\Gamma$ : remember Morera's theorem: a continuous function  $f: D \to \mathbb{C}$  is holomorphic if and only if for every triangle  $T \subseteq D$  we have  $\int_{\partial T} f(z) dz = 0$ . In any case the closedness of  $\mathcal{O}(D)$  in C(D) appears to be a consequence of continuity of integrals with respect to the compact-open topology.

Other important linear functionals on C(D) are the point evaluations; for every  $c \in D$  we define  $\delta_c : C(D) \to \mathbb{C}$  by  $\delta_c(f) = f(c)$ ; we leave it to the reader to prove that  $\delta_c$  is continuous from C(D) to  $\mathbb{C}$ , for every  $c \in D$ .

**2.5. Bounded and totally bounded subsets of** C(D). A subset F of C(D) is said to be *bounded* if for every neighborhood V of the zero function of C(D) there exists t > 0 such that  $F \subseteq tV$ , that is F is *absorbed* by every neighborhood of 0 in C(D). It is easy to see that

PROPOSITION. A subset F of C(D) is bounded if and only if it is uniformly bounded on every compact subset of D; that is, for every compact  $K \subseteq D$  there exists  $M_K > 0$  such that  $||f||_K \leq M_K$ , for every  $f \in F$ .

Proof. The sets  $B_K(0, \varepsilon[(=\{g \in C(D) : \|g\|_K < \varepsilon\}))$  are neighborhoods of 0, for every compact  $K \subseteq D$  and every  $\varepsilon > 0$ ; since  $tB_K(0, \varepsilon[=B_K(0, t\varepsilon[$  for every t > 0 we see that  $F \subseteq tB_K(0, \varepsilon[$  if and only if  $\|f\|_K < t\varepsilon$  for every  $f \in F$ . And since every neighborhood of 0 in C(D) contains  $B_K(0, \varepsilon[$  for some compact K and some  $\varepsilon > 0$ , the proof is concluded.

In other words, a subset F of C(D) is bounded if it is bounded for every seminorm  $\|\cdot\|_K$ , for every compact subset K of D.

In finite dimensional normed linear spaces a subset is compact iff it is closed and bounded; in complete metric spaces, a subset S is compact iff it is closed and *totally bounded*: this means that for every  $\varepsilon > 0$  there exist  $f_1, \ldots, f_m \in S$  such that  $S \subseteq B(f_1, \varepsilon \cup \cdots \cup B(f_m, \varepsilon)$  (see, e.g. the Chapter 0 of G.Folland, Real Analysis (Wiley & Sons) for a proof of this theorem).

It is easy to see that in the metric d(f, g) = [f - g] introduced on C(D) a subset S is totally bounded if and only if it is totally bounded in every seminorm  $\|\cdot\|_K$  for every compact subset K of D: in other words, a subset S of C(D) is totally bounded iff for every compact  $K \subseteq D$  and every  $\varepsilon > 0$  there exists a finite subset  $F = \{f_1, \ldots, f_m\}$  of C(D) such that  $S \subseteq \bigcup_{j=1}^m B_K(f_j, \varepsilon]$ .

We leave as an exercise to the reader the proof of the fact that a totally bounded subset of C(D) is also bounded (see at the end of the section). In general bounded subsets of C(D) are not totally bounded, as the following example shows.

EXAMPLE 2.5.0.1. Let  $\Delta = \{z \in \mathbb{C} : |z| < 1\}$  be the open unit disc; let  $f_n : \Delta \to \mathbb{C}$  be defined by  $f_n(z) = (nz)/(1+n|z|)$ , for  $n = 0, 1, 2, 3, \ldots$  and  $z \in \Delta$ . This sequence is clearly bounded (we have  $|f_n(z)| < 1$  for all  $z \in \Delta$ ) but it is not totally bounded in  $C(\Delta)$ . This will be clear from subsequent results; however we can directly prove that if K = B(0, 1/2] then the sequence cannot be covered by a finite set of balls  $B_K(f_j, 1/8[$ . In fact, we have

$$\left|\frac{mz}{1+m\,|z|} - \frac{nz}{1+n\,|z|}\right| = \frac{|m-n|\,|z|}{(1+m\,|z|)(1+n\,|z|)};$$

if  $n \ge 2$  then  $1/n \in K$ ; we compute the above for z = 1/n, and take m = kn with k > 1 an integer, obtaining (k-1)/(2(k+1)); we have

$$\frac{k-1}{2(1+k)} \ge \frac{1}{4} \iff 2(k-1) \ge k+1 \iff k \ge 3.$$



FIGURE 1. Some functions  $f_n$ , on the interval [-1, 1].

It follows that

$$||f_{3^j} - f_{3^k}||_K \ge \frac{1}{4}$$
 for all  $j, k \ge 1, j \ne k$ ,

and no  $B_K(f, 1/8)$  can contain two different members of the sequence  $f_{3^k}$ .

EXERCISE 2.5.0.2. Prove that a set  $S \subseteq C(D)$  is totally bounded if and only if for every compact Kand every  $\varepsilon > 0$  there exist  $f_1, \ldots, f_m \in C(D)$  such that  $S \subseteq B_K(f_1, \varepsilon[\cup \cdots \cup B_K(f_m, \varepsilon[$  (i.e., it is not necessary to assume that  $f_j$  belongs to S). Show that every subset of a totally bounded set is also totally bounded, and that the closure of a totally bounded set is also totally bounded.

Prove that a totally bounded subset of C(D) is bounded (if  $S \subseteq B_K(f_1, \varepsilon[\cup \cdots \cup B_K(f_m, \varepsilon[$  take  $M_K = \max\{||f_j||_K\} + \varepsilon)$ ).

EXAMPLE 2.5.0.3. The following is an example of a sequence bounded in  $\mathcal{O}(\Delta)$ , i.e. uniformly bounded on every compact subset of  $\Delta$ , not uniformly bounded on all of  $\Delta$ ; the sequence is  $f_n(z) = \sum_{k=0}^{n} z^k$ , sequence of partial sums of the series  $\sum_{n=0}^{\infty} z^n$ . Since, for  $|z| \leq r < 1$  we have

$$f_n(z) = \frac{1 - z^{n+1}}{1 - z} \implies |f_n(z)| \le \frac{1 + |z|^{n+1}}{1 - |z|} \le \frac{2}{1 - r}$$

and every compact subset of  $\Delta$  is contained in some disc rB = B(0, r] the first statement is proved; since

$$\lim_{x \to 1^-} f_n(x) = n + 1$$

we have  $||f_n||_{\Delta} \ge n+1$ , so that there is no common bound to all the  $f_n$ 's on  $\Delta$ .

EXERCISE 2.5.0.4. Observe that a subset F of C(D) is bounded iff it is *locally uniformly bounded* in D, that is, for every  $c \in D$  there exists a neighborhood U of c and a constant  $M_U$  such that  $|f(z)| \leq M_U$  for every  $z \in U$  and every  $f \in F$ .

**2.6. Equicontinuity and Ascoli's theorem.** Let us recall the definition of continuity of a function  $f: D \to \mathbb{C}$  at a point  $z \in D$ : for every  $\varepsilon > 0$  there exists a  $\delta = \delta_{\varepsilon} > 0$  such that if  $w \in B(z, \delta]$  then  $|f(w) - f(z)| < \varepsilon$ . Here  $\delta$  depends on  $\varepsilon$  but also on f, it is really a  $\delta(f, \varepsilon)$  if various functions f are considered; clearly different functions f will have in general different  $\delta$  for the same  $\varepsilon$ ; equicontinuity is the requirement that  $\delta$  can be chosen to depend on  $\varepsilon$  only. That is:

DEFINITIONS. Given a set  $F \subseteq C(D)$  and  $z \in D$ , we say that F is equicontinuous at z if for every  $\varepsilon > 0$  there is a  $\delta_{\varepsilon} > 0$  such that

$$|f(w) - f(z)| < \varepsilon$$
 for every  $w \in B(z, \delta_{\varepsilon})$ , and every  $f \in F$ .

A set  $F \subseteq C(D)$  is said to be *equicontinuous* in D, or simply equicontinuous, if it is equicontinuous at every  $z \in D$ .

Of course a finite set F of continuous functions is always equicontinuous; but the set of functions  $\{f_n = nz/(1 + n |z|) : n = 0, 1, 2, 3...\}$  considered in the previous example is not equicontinuous at z = 0; it can be easily shown that it is however equicontinuous at every nonzero z.

The following is a very neat characterization of total boundedness in C(D).

. THEOREM OF ASCOLI–ARZELÀ. A subset of C(D) is totally bounded if and only if it is bounded and equicontinuous. Proof. Let F be totally bounded in C(D). Every totally bounded set is bounded, as remarked above. To show equicontinuity, given  $z \in D$  and  $\varepsilon > 0$  fix a compact neighborhood K of z in D, say  $K = B(z,r] \subseteq D$ , and pick  $f_1, \ldots, f_m \in F$  such that  $F \subseteq B_K(f_1, \varepsilon[\cup \cdots \cup B_K(f_m, \varepsilon[$ . Since every  $f_j$  is continuous at z there exist  $\delta_j > 0$  such that  $|f_j(w) - f_j(z)| < \varepsilon$  for  $w \in B(z, \delta_j]$ . Let  $\delta = \min\{\delta_1, \ldots, \delta_m, r\}$ . Given  $f \in F$  there exists  $j \in \{1, \ldots, m\}$  such that  $f \in B_K(f_j, \varepsilon[$ ; if  $w \in B(z, \delta]$ we have, since  $w, z \in K$ :

$$|f(w) - f(z)| \le |f(w) - f_j(w)| + |f_j(w) - f_j(z)| + |f_j(z) - f(z)| \le 3\varepsilon,$$

proving equicontinuity of F at z.

Conversely, assume that F is equicontinuous and bounded. Take  $K \subseteq D$  compact, and  $\varepsilon > 0$ . We have to prove that F is contained in the union of some finite family of K-balls of radius small with  $\varepsilon$ . By equicontinuity, for every  $z \in K$  there exists  $\delta(z) > 0$  such that  $f(B(z, \delta(z)[) \subseteq B(f(z), \varepsilon[$ , for every  $f \in F$ . By compactness of K there exist  $z_1, \ldots, z_p \in K$  such that  $K \subseteq B(z_1, \delta(z_1)[\cup \cdots \cup B(z_p, \delta(z_p)[$ . Let us consider the mapping  $\rho : f \mapsto (f(z_1), \ldots, f(z_p))$  of C(D) into  $\mathbb{C}^p$ . Since F is bounded,  $\rho(F)$ is a bounded subset of  $\mathbb{C}^p$  (if  $||f||_K \leq M_K$  for every  $f \in F$ , then  $\rho(F)$  is contained in the polydisc  $(M_K B)^p$ ). And since in finite dimensional spaces bounded sets are totally bounded there exists a finite set  $\{f_1, \ldots, f_m\} \subseteq F$  such that every  $\rho(f) \in \rho(F)$  has distance less than  $\varepsilon$  from some  $\rho(f_j)$ , equivalently, for every  $f \in F$  there exists  $f_j$  in this finite set such that

$$|f(z_k) - f_j(z_k)| \le \varepsilon$$
 for every  $k = 1, \dots, p$ .

With this same  $f_j$  we have  $||f - f_j||_K \leq 3\varepsilon$ : in fact, if  $z \in K$  then  $z \in B(z_k, \delta(z_k))$  for some  $k \in \{1, \ldots, p\}$ , and then

$$|f(z) - f_j(z)| \le |f(z) - f(z_k)| + |f(z_k) - f_j(z_k)| + |f_j(z_k) - f_j(z)| \le 3\varepsilon$$

(the first and the third term are smaller than  $\varepsilon$  for equicontinuity, the middle term because of the choice of  $f_i$ ). Thus F is totally bounded.

**2.7.** Compact convergence and holomorphic mappings. The topology of compact convergence is eminently suited to the subalgebra of holomorphic mappings, as we now see. We first prove the following:

(CAUCHY ESTIMATE FOR A COMPACT SET). Let D be open in  $\mathbb{C}$ , and let K be a compact subset of D. If r > 0 is strictly smaller that the distance of K from the complement of D, we have, for every  $f \in \mathcal{O}(D)$ :

$$||f'||_K \le \frac{||f||_{K+rB}}{r}.$$

*Proof.* For every  $c \in K$  the disc B(c, r] is contained in K + rB and we have

$$f'(c) = \frac{1}{2\pi i} \int_{\partial B(c,r]} \frac{f(\zeta)}{(\zeta - c)^2} \, d\zeta,$$

by the Cauchy formula for the derivative; thus (Cauchy estimate of the derivative):

$$|f'(c)| \le \frac{1}{2\pi} \int_{\partial B(c,r]} \frac{|f(\zeta)|}{|\zeta - c|^2} |d\zeta| \le \frac{1}{2\pi} \int_{\partial B(c,r]} \frac{\|f\|_{\partial B(c,r]}}{r^2} |d\zeta| = \frac{1}{r} \|f\|_{\partial B(c,r]} \le \frac{\|f\|_{K+rB}}{r}$$

Since this holds for every  $c \in K$  we get

$$||f'||_K \le \frac{||f||_{K+rB}}{r}$$
 for every  $f \in \mathcal{O}(D)$ .

THEOREM. Let D be an open subset of  $\mathbb{C}$ . Then

- (i) The complex differentiation operator f → f' is continuous from O(D) into itself: in other words, if a sequence of holomorphic mappings compactly converges to a limit function, then the sequence of the derivatives of these mappings compactly converges to the derivative of the limit.
- (ii) Every bounded subset of  $\mathcal{O}(D)$  is totally bounded.
- (iii) MONTEL'S THEOREM The compact subsets of  $\mathcal{O}(D)$  are exactly the closed and bounded subsets of  $\mathcal{O}(D)$ .

*Proof.* (i) What was proved in the preceding Cauchy's formula for compact sets clearly implies the continuity of the map  $f \mapsto f'$  at the 0 function, and hence at every function by linearity of the derivative: explicitly, given  $f \in \mathcal{O}(D)$ ,  $\varepsilon > 0$  and a compact subset K of D, pick r > 0 such that  $K + r B \subseteq D$ ; for every  $g \in \mathcal{O}(D)$  we then have

$$||g' - f'||_K \le \frac{||g - f||_{K+rB}}{r},$$

so that if  $g \in B_{K+rB}(f, r\varepsilon)$  we have  $g' \in B_K(f', \varepsilon)$ .

(ii) We need to show that in  $\mathcal{O}(D)$  every bounded set F is equicontinuous. We first observe that the set  $F' = \{f' : f \in F\}$  of the derivatives of the elements of F is also bounded. Take in fact a neighborhood V of 0 in  $\mathcal{O}(D)$ ; by continuity of the derivation operator there exists a neighborhood U of 0 in  $\mathcal{O}(D)$  such that  $f \in U$  implies  $f' \in V$ . Let t > 0 be such that  $tU \supseteq F$ ; then  $tV \supseteq F'$ .

Given  $c \in D$  take r > 0 such that  $B(c, r] \subseteq D$ ; there exists L > 0 such that  $||f'||_{B(c,r]} \leq L$  for every  $f \in F$ , by boundedness of F'; then for every  $z \in B(c, r]$  we have, by the mean value theorem:

$$|f(z) - f(c)| \le ||f'||_{[c,z]} |z - c| \le L |z - c|,$$

proving equicontinuity of F at c.

(iii) is now immediate, since the compact subsets of C(D) are the closed totally bounded ones; recall that  $\mathcal{O}(D)$  is closed in C(D) (see Corollary 2.4.1), so closed subsets of  $\mathcal{O}(D)$  are closed also in C(D).  $\Box$ 

REMARK. An old-fashioned but still very common terminology calls normal families the bounded sets of holomorphic functions. So: the relatively compact subsets of  $\mathcal{O}(D)$  are exactly the normal families (by a very ill-chosen terminology relatively compact means: having compact closure).

#### EXERCISE 2.7.0.5. Prove the following

. THEOREM OF VITALI Let D be a region of  $\mathbb{C}$ , and let  $f_n$  be a bounded sequence of  $\mathcal{O}(D)$ . Assume that there is a subset  $C \subseteq D$ , with an accumulation point belonging to D, such that  $\lim_{n\to\infty} f_n(z)$  exists in  $\mathbb{C}$ , for every  $z \in C$ . Then there exists a holomorphic function  $f \in \mathcal{O}(D)$  such that  $f_n$  converges compactly to f in D.

(a solution is given at the end of the section).

2.7.1. Injective mappings and Hurwitz's theorem. Recall a lemma on existence of zeroes on a disc, which may be used to prove the open mapping theorem: if f is holomorphic on an open set containing the closed disc B(c, r], and |f(z)| > |f(c)| for every  $z \in \partial B(c, r]$ , then f(w) = 0 for some  $w \in B(c, r[$ . This simple fact readily implies the following result:

PROPOSITION. (HURWITZ) Let D be a region of  $\mathbb{C}$ .

- (i) If  $f_n : D \to \mathbb{C}$  is a sequence of zero-free holomorphic functions which converges compactly on D to a function  $f : D \to \mathbb{C}$ , then either f is identically zero, or it is zero-free.
- (ii) If  $f_n : D \to \mathbb{C}$  is a sequence of injective holomorphic functions which converges compactly on D to a function  $f : D \to \mathbb{C}$ , then either f is constant, or it is injective.
- (iii) If  $f_n : D \to \mathbb{C}$  is a sequence of holomorphic functions which converges compactly on D to a non-constant function  $f : D \to \mathbb{C}$ , and E is a set such that for every n we have  $f_n(D) \subseteq E$ , then also  $f(D) \subseteq E$

Proof. (i) Assume that f(c) = 0 for some  $c \in D$ , with f not identically zero in D; then we can pick a disc  $B(c,r] \subseteq D$  such that  $Z_D(f) \cap B(c,r] = \{c\}$ ; then  $\min\{|f(z)| : z \in \partial B(c,r]\} = \mu > 0$  Take  $\bar{n} \in \mathbb{N}$ such that  $||f - f_n||_{B(c,r]} \leq \mu/3$  for  $n \geq \bar{n}$ . Then  $|f_n(c)| \leq \mu/3 < 2\mu/3 \leq \min\{|f_n(z)| : z \in \partial B(c,r]\}$ , so that  $f_n$  has a zero in B(c,r], a contradiction. (ii) Given  $c \in D$ , each  $f_n - f_n(c)$  is zero-free in  $D \setminus \{c\}$ , which is also connected; and if f were constant on  $D \setminus \{c\}$  then it would be constant on D; then f - f(c)is zero free on  $D \setminus \{c\}$ , by (i). Since c is arbitrary in D, this shows that f is injective.

(iii) If  $w \in \mathbb{C} \setminus E$  then  $f_n - w$  is zero-free and converges compactly on D to the non-constant function f - w, which by (i) is then zero-free.

2.7.2. Completion of the proof of the Riemann mapping theorem. Given a region G with the square root property, and  $c \in G$ , recall that we considered in 1.2.5 the set

 $I_c(G) = \{ f \in \mathcal{O}(G) : f \text{ injective, } f(G) \subseteq \Delta, f(c) = 0 \}.$ 

It is easy to show, by means of the above, that the closure of  $I_c(G)$  in the compact-open topology is  $I_c(G) \cup \{0\}$ . Given  $p \in G \setminus \{c\}$ , the mapping  $f \mapsto |f(p)|$  is continuous and hence it takes its absolute

maximum on the compact set  $I_c(G) \cup \{0\}$ . If  $g \in I_c(G)$  is a maximum point for this function, by 1.2.5 the function g is an isomorphism. And if f = (|g'(c)|/g'(c)) g then f also an isomorphism, and is derivative at c is f'(c) = |g'(c)| > 0.

EXERCISE 2.7.2.1. Describe the closure in  $\mathcal{O}(D)$  of the set of injective mappings, when D is a region.

Solution. Calling  $\operatorname{In}(D)$  this set, the closure is  $\operatorname{In}(D) \cup \kappa(\mathbb{C})$ , where  $\kappa(\mathbb{C})$  is the set of all constant mappings. In fact, clearly every mapping with constant value k is in the closure of  $\operatorname{In}(D)$ , as limit of the sequence  $f_n(z) = k + z/n$ . And by (ii) of Hurwitz's theorem, a non constant limit of mappings in  $\operatorname{In}(D)$  is still in  $\operatorname{In}(D)$ .

2.7.3. A more precise statement of Hurwitz's theorem. (This fact is not needed in the sequel)

If a holomorphic function  $f: D \to \mathbb{C}$  is not locally constant at a point  $c \in D$  then the multiplicity  $\nu(f, c)$  of f at c is defined as the order of f - f(c) at c, i.e. the smallest natural number  $m \ge 1$  such that  $f^{(m)}(c) \ne 0$ . We have the following:

. THEOREM OF HURWITZ Let D be a region of  $\mathbb{C}$  and let  $f_n$  be a sequence in  $\mathcal{O}(D)$ , which compactly converges on D to a non-constant function f on D. Let U be an open bounded set whose closure is contained in D. Assume that  $w \in \mathbb{C} \setminus f(\partial U)$ . Then there is an index  $n(U, w) \in \mathbb{N}$  such that for  $n \ge n(U, w)$  the equations w = f(z) and  $w = f_n(z)$  have in U the same number of solutions, counting multiplicities: i.e. for  $n \ge n(U, w)$  we have

$$\sum \{\nu(f,z) : z \in U, f(z) = w\} = \sum \{\nu(f_n,z) : z \in U, f_n(z) = w\}.$$

*Proof.* The set  $f^{\leftarrow}(w) \cap U$  is finite, since f is by hypothesis not constant; put  $f^{\leftarrow}(w) \cap U = \{z_1, \ldots, z_p\}$ , and choose  $\delta > 0$  so small that the discs  $B(z_k, \delta]$ ,  $k = 1, \ldots, p$  are contained in U and pairwise disjoint. By the argument principle we have

$$I = \sum \{\nu(f, z) : z \in U, f(z) = w\} = \sum_{k=1}^{p} \frac{1}{2\pi i} \int_{\partial B(z_k, \delta]} \frac{f'(z)}{f(z) - w} dz$$

Since  $K = \overline{U} \setminus (\bigcup_{k=1}^{p} B(z_k, \delta[) \text{ is compact, and } w \notin f(K), \text{ we have } \operatorname{dist}(w, f(K)) = \rho > 0; \text{ take } n_1 \in \mathbb{N} \text{ such that } \|f - f_n\|_K < \rho \text{ if } n \ge n_1. \text{ Then } w \notin f_n(K) \text{ if } n \ge n_1, \text{ so that } f_n^{\leftarrow}(w) \cap U \subseteq \bigcup_{k=1}^{p} B(z_k, \delta[ \text{ for } n \ge n_1; \text{ thus, again by the argument principle:}}$ 

$$I_n = \sum \{\nu(f_n, z) : z \in U, f_n(z) = w\} = \sum_{k=1}^p \frac{1}{2\pi i} \int_{\partial B(z_k, \delta]} \frac{f'_n(z)}{f_n(z) - w} \, dz \qquad n \ge n_1$$

Since the sequence  $f'_n(z)/(f_n(z) - w)$  converges to f'(z)/(f(z) - w) uniformly on  $\bigcup_{k=1}^p \partial B(z_k, \delta]$ , the integrals  $I_n$  converge to the integral I, and since  $I_n$  and I are integers, there is  $n_2 \in \mathbb{N}$  such that for  $n \ge n_2$  we have  $I_n = I$ . We conclude, taking  $n(U, w) = \max\{n_1, n_2\}$ .

## 2.7.4. Inward spreading of uniform convergence.

PROPOSITION. Let D be open in  $\mathbb{C}$ , and let  $(f_n)_{n \in \mathbb{N}}$  be a sequence of holomorphic functions on D. Assume that  $f_n$  converges uniformly on some compact subset K of D, and that a bounded component Z of  $\mathbb{C} \setminus K$  is contained in D. Then  $(f_n)_{n \in \mathbb{N}}$  converges uniformly on  $Z \cup K$  to a function continuous on  $Z \cup K$  and holomorphic on Z.

Proof. Recall that  $\partial Z \subseteq \partial K \subseteq K$ , and recall that by the maximum modulus theorem we have  $\|g\|_{\bar{Z}} = \|g\|_{\partial Z} \leq \|g\|_{K}$  for every g continuous on  $Z \cup K$  and holomorphic in Z. If  $f_n$  converges uniformly on K then it is uniformly Cauchy on K, and since  $\|f_n - f_m\|_{K} \geq \|f_n - f_m\|_{\bar{Z}}$  we have  $\|f_n - f_m\|_{K} = \|f_n - f_m\|_{K \cup Z}$ , so that the sequence is uniformly Cauchy on  $K \cup Z$ , hence it converges uniformly on  $K \cup Z$  to a function continuous on  $K \cup Z$  and holomorphic in Z.

COROLLARY. Let D be open in  $\mathbb{C}$  and let S be a locally finite subset of D. Assume that  $(f_n)_{n \in \mathbb{N}}$  is a sequence of functions holomorphic in  $D \setminus S$  which converges compactly on  $D \setminus S$  to  $f \in \mathcal{O}(D \setminus S)$ . If  $c \in S$  is a non-removable singularity for f, then c is a non-removable singularity for all but a finite subfamily of the functions in the sequence.

*Proof.* Pick a disc  $B(c,r] \subseteq D$  such that  $B(c,r] \cap S = \{c\}$ . Assume that for infinitely many  $n \in \mathbb{N}$  the function  $f_n$  has a removable singularity at c. The subsequence so obtained converges uniformly on  $\partial B(c,r]$ , hence, by the above observation on inward spreading of the convergence, also on B(c,r], to a function holomorphic on  $B(c,r[; \text{but on } B(c,r[\smallsetminus \{c\} \text{ this function must coincide with } f, which then has a removable singularity at <math>c$ .

2.7.5. Addendum. We present here a proof of the following fact:

PROPOSITION. The compact-open topology and the topology of uniform convergence on compacta coincide on C(D).

Proof. Assume that  $f \in \bigcap_{j=1}^{m} [K_j, U_j]$ , where  $K_j$  is a compact subset of D and  $U_j$  an open subset of  $\mathbb{C}$ , for each  $j \in \{1, \ldots, m\}$ ; we prove that for some compact  $K \subseteq D$  and some  $\varepsilon > 0$  we have  $B_K(f, \varepsilon] \subseteq \bigcap_{j=1}^{m} [K_j, U_j]$ ; this proves that the the topology of uniform convergence on compacta is finer than the compact–open topology. For every j the set  $f(K_j)$  is a compact subset of the open set  $U_j$ , and hence there exists  $\delta_j > 0$  such that  $f(K_j) + \delta_j B \subseteq U_j$ . Take  $K = \bigcup_{j=1}^{m} K_j$ , and  $\varepsilon < \min\{\delta_j : j = 1, \ldots, m\}$ . Assume that  $||g - f||_K < \varepsilon$ , and let us show that  $g(K_j) \subseteq U_j$ , for every  $j = 1, \ldots, m$ . In fact, if  $z \in K_j$ , we have  $|g(z) - f(z)| < \varepsilon \le \delta_j$ , hence  $g(z) \in f(z_j) + \delta_j B \subseteq U_j$ .

Conversely, let a compact  $K \subseteq D$  and  $\varepsilon > 0$  be given, and let's prove that there exist compact sets  $K_j \subseteq D$  and open sets  $U_j$  such that  $\bigcap_{j=1}^m [K_j, U_j] \subseteq B_K(f, \varepsilon[$ . For every  $z \in K$  pick  $\delta(z) > 0$ such that  $B(z, \delta(z)] \subseteq D$  and  $f(B(z, \delta(z)] \subseteq B(f(z), \varepsilon/2[$ , as allowed by the continuity of f. There exist  $z_1, \ldots, z_m \in K$  such that  $K \subseteq B(z_1, \delta(z_1)] \cup \cdots \cup B(z_m, \delta(z_m)]$ , by compactness of K. Set  $U_j = B(f(z_j), \varepsilon/2[$ , and  $K_j = B(z_j, \delta(z_j)]$ , and suppose  $g \in \bigcap_{j=1}^m [K_j, U_j]$ . If  $z \in K$ , then  $z \in B(z_j, \delta(z_j)]$  for some j, hence:

$$|g(z) - f(z)| \le |g(z) - f(z_j)| + |f(z_j) - f(z)| < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

so that  $g \in B_K(f, \varepsilon[.$ 

2.7.6. Proof of the theorem of Vitali, exercise 2.7.0.5. Since  $\{f_n : n \in \mathbb{N}\}$  is bounded, it has compact closure, and every subsequence of  $f_n$  has a converging subsequence; pick one subsequence converging, let's say, to  $f \in \mathcal{O}(D)$ ; clearly we have  $f(z) = \lim_n f_n(z)$ , for every  $z \in C$ , since every subsequence of  $f_n(z)$  is pointwise convergent, for every  $z \in C$ , to the same limit. Any holomorphic function  $g \in \mathcal{O}(D)$ which is the limit of some subsequence of  $(f_n)_n$  in the compact topology then agrees with f on C; by the identity theorem we have f = g on D. Then  $f_n$  converges to f in the compact–open topology: simply remember the well–known fact that in a topological space a sequence  $f_n$  converges to a point f of the space if every subsequence of this sequence has a subsequence converging to f.

2.7.7. More exercises.

EXERCISE 2.7.7.1. Let D be a region of  $\mathbb{C}$ , and let  $f_n$  be a sequence in  $\mathcal{O}(D)$  such that  $f'_n$  compactly converges in D to a function  $g \in \mathcal{O}(D)$ . Prove that then  $f_n$  compactly converges on D if and only if for at least one  $c \in D$  the sequence  $f_n(c)$  converges in  $\mathbb{C}$ ; and in this case the limit f of  $f_n$  is a primitive of g in D, i.e. f' = g on D.

Solution. Necessity is obvious.

Sufficiency: first of all notice that g indeed has a primitive on D: if  $\gamma$  is any loop in D, we have  $\int_{\gamma} f'_n(z) dz = 0$ , and by uniform convergence on  $[\gamma]$  we get

$$\int_{\gamma} g(z) \, dz = \lim_{n} \int_{\gamma} f'_n(z) \, dz = 0.$$

If  $\lim_n f_n(c) = \ell$  exists in  $\mathbb{C}$ , denote by f that primitive of g on D whose value at c is  $\ell$ . Now the sequence  $f_n$  converges pointwise to f: given  $z \in D$ , we can consider a path  $\alpha$  of origin c end extremity z (by connectedness of D); then

$$f_n(z) = f_n(c) + \int_{\alpha} f'_n(\zeta) \, d\zeta \Rightarrow f(z) = f(c) + \int_{\alpha} g(\zeta) \, d\zeta$$

(we can pass to the limit under the integral, by uniform convergence on the compact set  $[\alpha]$ ). And it is now immediate to prove that we have uniform convergence on every compact disk contained in D: if  $B(a,r] \subseteq D$  we have, for  $z \in B(a,r]$ ,

$$|f(z) - f_n(z)| = |f(z) - f(a) + f(a) - f_n(a) + f_n(a) - f_n(z)| \le |f(z) - f(a) - (f_n(z) - f_n(a))| + |f(a) - f_n(a)|$$
Write now:

Write now:

$$f(z) - f(a) = \int_{[a,z]} g(\zeta) \, d\zeta; \quad f_n(z) - f_n(a) = \int_{[a,z]} f'_n(\zeta) \, d\zeta;$$

so that

$$|f(z) - f(a) - (f_n(z) - f_n(a))| = \left| \int_{[a,z]} (g(\zeta) - f'_n(\zeta)) \, d\zeta \right| \le \int_{[a,z]} |g(\zeta) - f'_n(\zeta)| \, |d\zeta| \le \int_{[a,z]} |g(\zeta) - g'_n(\zeta)| \, |d\zeta| \le \int_{[a,z]} |g(\zeta) - g'_n(\zeta$$

$$||g - f'_n||_{[a,z]}|z - a| \le ||g - f'_n||_{B(a,r]} r;$$

we get

$$|f(z) - f_n(z)| \le ||g - f'_n||_{B(a,r]} r + |f(a) - f_n(a)|$$
 for every  $z \in B(a,r]$  and every  $n \in \mathbb{N}$ 

so that  $f_n$  converges to f uniformly on B(a, r]. Every compact subset of D is covered by a finite set of such discs, so that convergence is uniform on every compact subset of D.

The following exercise is practically contained in the previous one, still it can give another perspective.

EXERCISE 2.7.7.2. Let D be a region of  $\mathbb{C}$ . Denote by  $\mathcal{O}'(D)$  the set of all derivatives of holomorphic functions in D, that is,  $\mathcal{O}'(D)$  is the image of  $\mathcal{O}(D)$  under the differentiation operator  $f \mapsto f'$ .

- (i) Prove that  $\mathcal{O}'(D)$  is a closed linear subspace of  $\mathcal{O}(D)$ .
- (ii) Prove that for every  $c \in D$  there exists a linear continuous map  $I_c : \mathcal{O}'(D) \to \mathcal{O}(D)$  such that  $I_c g(c) = 0$  for every  $g \in \mathcal{O}'(D)$ , and  $I_c$  is a right inverse of the differentiation operator, that is  $(I_c g)' = g$  for every  $g \in \mathcal{O}'(D)$ .

EXERCISE 2.7.7.3. Consider the sequence  $f_n(z) = n/(z+n)$ ; these functions are all holomorphic in  $\mathbb{C} \setminus (-\mathbb{N})$ . Determine the limit on  $\mathbb{C} \setminus (-\mathbb{N})$  and prove that on this open set the sequence converges compactly to this limit.

In the previous example, the limit function is an entire function, even if the functions  $f_n$  are not defined on all of  $\mathbb{C}$ . But for every compact subset K of  $\mathbb{C}$  there exists a natural number  $n_K$  such that for  $n \ge n_K$  all functions  $f_n$  are holomorphic on a neighborhood of K, and converge to the limit uniformly on K. We often say in this case that the sequence  $f_n$  converges to its limit compactly on  $\mathbb{C}$ , even if  $\mathbb{C}$  is not the common domain of all the  $f_n$ 's.

2.7.8. *Compositional factorization of holomorphic functions*. We prove here a "homomorphism theorem" for holomorphic maps.

. Let D, E be open regions of  $\mathbb{C}$ , assume that  $\varphi : D \to E$  is a holomorphic surjective map, and that  $f : D \to \mathbb{C}$  is holomorphic. Then the following are equivalent

(i) f is constant on the fibers of  $\varphi$ , that is,  $\varphi(a) = \varphi(b)$  for  $a, b \in D$  implies f(a) = f(b).

(ii) There exists  $g: E \to \mathbb{C}$ , holomorphic on E, such that  $f = g \circ \varphi$ .

Moreover, g is unique.

Proof. It is well-known and easy to prove that g exists as a map of sets, and is unique, if and only if (i) holds. We only have to prove that if (i) holds then g is holomorphic. Clearly  $\varphi$  is non constant, hence it is an open mapping; then g is continuous (we have  $g^{\leftarrow}(A) = \varphi(f^{\leftarrow}(A))$ , for every  $A \subseteq \mathbb{C}$ ). We first prove that g is holomorphic on the open subset  $W = \varphi(D \setminus Z(\varphi'))$  of E. Given  $w \in W$ , we have  $w = \varphi(z)$  for some  $z \in D \setminus Z(\varphi')$ ; since  $\varphi'(z) \neq 0$ , we have that  $\varphi$  induces a holomorphic isomorphism  $\psi$  of some neighborhood U of z onto some neighborhood V of w; on V we then have  $g_{|V} = f \circ \psi^{-1}$ , composition of holomorphic functions, hence holomorphic, proving that g is holomorphic on W. Consider the complement  $S = E \setminus W$  of W in E, that is the set S of all  $w \in E$  such that  $\varphi^{\leftarrow}(w) \subseteq Z(\varphi')$ . Since  $\varphi$  is non constant in the region D, the zero-set  $Z(\varphi')$  of  $\varphi'$  is (closed and) discrete in D, and this immediately implies that S is also (closed and) discrete in E: pick  $w \in S$ , and a point  $z \in \varphi^{\leftarrow}(w)$ ; there is a neighborhood U of z such that  $U \cap Z(\varphi') = \{z\}$ , hence  $V = \varphi(U)$  is an open set containing w such that  $V \setminus \{w\} \subseteq W$ . All points of S are then isolated singularities for g, and since g is continuous on E, these singularities are all removable.

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**2.8.** Normal convergence of series. Examples. For a series of functions, the handier and simplest criterion is that of *normal convergence*, which we now describe.

DEFINITION. Let D be open in  $\mathbb{C}$  and let  $\sum_{n \ge \nu} f_n$  be series of functions in C(D). We say that this series converges normally in D if for every compact subset K of D the series  $\sum_{n \ge \nu} \|f\|_K$  is convergent.

PROPOSITION. A normally convergent series of functions converges in the compact-open topology to a function f, and every rearrangement has the same sum.

*Proof.* Clearly we have

$$\left\|\sum_{j=\nu}^{k+p} f_j - \sum_{j=\nu}^{k} f_j\right\|_{K} = \left\|\sum_{j=k+1}^{k+p} f_j\right\|_{K} \le \sum_{j=k+1}^{k+p} \|f_j\|_{K},$$

and the latter sum may be made smaller than any previously chosen  $\varepsilon > 0$  for k large, since the series of the seminorms is convergent. Moreover, for every  $z \in D$  the series  $\sum_{n \geq \nu} |f_n(z)|$  is convergent, that is, the numerical series  $\sum_{n \geq \nu} f_n(z)$  is absolutely convergent ( $|f_n(z)| = ||f_n||_{\{z\}}$ ). It is well known that every absolutely convergent series is commutatively convergent, which means that all its rearrangements converge to the same sum.

2.8.1. The zeta function. We define a function  $\zeta(s)$  for  $\sigma = \text{Re } s > 1$  (as customary in number theory the complex variable is denoted by  $s, s = \sigma + it$ , with  $\sigma = \text{Re } s, t = \text{Im } s$ ) by letting

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \dots = \sum_{n=1}^{\infty} \frac{1}{n^s}$$
 (Re  $s > 1$ );

it is of course intended that the power  $n^s$  has the principal value i.e.  $n^s := \exp(s \log n)$ . Since  $|n^s| = |\exp(s \log n)| = \exp((\operatorname{Re} s) \log n) = n^{\sigma}$ , the series is normally convergent in the open half-plane  $\{s \in \mathbb{C} : \operatorname{Re} s = \sigma > 1\}$ ; in fact, if  $\operatorname{Re} s \ge a > 1$  we have

$$\left|\frac{1}{n^s}\right| = \frac{1}{n^{\operatorname{Re} s}} \le \frac{1}{n^a},$$

and the series  $\sum_{n=1}^{\infty} 1/n^a$  is convergent, since a > 1. The sum is the famous zeta function of Riemann; moreover we have

$$\zeta'(s) = \sum_{n=1}^{\infty} \frac{-\log n}{n^s} \qquad (\operatorname{Re} s > 1).$$

2.8.2. A periodic function. Given  $\alpha > 0$  we prove that the formula

$$f_{\alpha}(s) = \sum_{n=-\infty, n \in \mathbb{Z}}^{\infty} e^{-\alpha(s+n)^2}$$

defines an entire function, periodic of period 1. Given any compact subset K of  $\mathbb{C}$ , there obviously exists a > 0 such that  $|\operatorname{Re} s|, |\operatorname{Im} s| \leq a$  for every  $s \in K$ ; we then have

$$|e^{-\alpha(s+n)^2}| = |\exp(-\alpha((\sigma+n)^2 - t^2 + 2it(\sigma+n))| = \exp(-\alpha(\sigma+n)^2 + \alpha t^2) \le e^{\alpha a^2} e^{-\alpha(|n|-a)^2};$$

the (two-sided) series  $\sum_{n=-\infty}^{\infty} e^{\alpha a^2} e^{-\alpha(|n|-a)^2}$  is clearly convergent. Then  $f_{\alpha}$  exists and is holomorphic. Periodicity of 1 is trivial, being a shift of indices in the summation; to be more precise, we could say that  $f_{\alpha}$  is the limit, in the compact-open topology, of the sequence

$$g_m(s) = \sum_{n=-m}^m e^{-\alpha(s+n)^2},$$

so that

$$f_{\alpha}(s+1) - f_{\alpha}(s) = \lim_{m \to \infty} (g_m(s+1) - g_m(s)),$$

and

$$g_m(s+1) - g_m(s) = \sum_{n=-m}^m e^{-\alpha(s+1+n)^2} - \sum_{n=-m}^m e^{-\alpha(s+n)^2} = \sum_{k=-(m-1)}^{m+1} e^{-\alpha(s+k)^2} - \sum_{n=-m}^m e^{-\alpha(s+n)^2} = e^{-\alpha(s+m+1)^2} - e^{-\alpha(s-m)^2},$$

which clearly tends to 0 as m nears infinity.

Some people say that  $f_{\alpha}$  has been obtained by replicating  $g_{\alpha}(s) = e^{-\alpha s^2}$  in period 1; clearly this technique works for any function holomorphic on a strip  $S = \{s \in \mathbb{C} : a < \text{Im } s < b\}$  which falls off fast enough at infinity on the strip, e.g as  $1/|z|^{1+\varepsilon}$ , for some  $\varepsilon > 0$ .

2.8.3. The cotangent series. Let us consider the two-sided series

$$\sum_{n=-\infty}^{\infty} \frac{1}{z-n} \quad \text{on} \quad D = \mathbb{C} \smallsetminus \mathbb{Z}.$$

We consider as partial sums the functions

$$f_m(z) = \sum_{n=-m}^m \frac{1}{z-n}, \qquad m \in \mathbb{N}$$

and we prove that on D this sequence converges compactly to some function  $f \in \mathcal{O}(D)$ . In fact the series

$$\frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2},$$

has the same partial sums as the given two-sided series, and is normally convergent in  $\mathbb{C} \setminus \mathbb{Z}$ , as we now prove. Let K be a compact subset of  $\mathbb{C}$ ; clearly  $\mu = \max\{|z| : z \in K\}$  is finite, and for  $z \in K$  and  $n > \mu$ we then have

$$\left|\frac{2z}{z^2 - n^2}\right| = \frac{2|z|}{|n^2 - z^2|} \le \frac{2\mu}{n^2 - \mu^2},$$

Since the series  $\sum_{n=[\mu]+1}^{\infty} (2\mu)/(n^2 - \mu^2)$  is convergent, we conclude. Clearly the function f has first order poles at every integer, with residue 1, since we can pass to the limit for  $m \to \infty$  in the integral

$$1 = \operatorname{Res}(f_m, n) = \frac{1}{2\pi i} \int_{\partial B(n, 1/2]} f_m(z) \, dz \qquad (m > n).$$

It is also easy to see that f is periodic of period 1:

$$f_m(z+1) - f_m(z) = \sum_{n=-m}^m \frac{1}{(z+1) - n} - \sum_{n=-m}^m \frac{1}{z-n} = \sum_{n=-m}^m \frac{1}{z-(n-1)} - \sum_{n=-m}^m \frac{1}{z-n} = \frac{1}{\frac{1}{z+(m+1)}} - \frac{1}{z-m}$$

and the left-hand side tends to f(z+1) - f(z), the right-hand side to 0. The function  $\pi \cot \pi z$  has exactly the same poles and the same residues, hence  $g(z) = f(z) - \pi \cot \pi z$  has only removable singularities, i.e., it is an entire function. Let us prove that it is identically zero. The pattern of proof is as follows: we prove that it is bounded, and thus constant by Liouville's theorem; next we prove that the constant is 0.

By periodicity, we only need to prove that g is bounded on every vertical strip of width one, let's say  $S = \{z \in \mathbb{C} : -1 \leq \text{Re} z \leq 0\}$ . We observe that  $g(\bar{z}) = \overline{g(z)}$ , hence we need to consider only the half-strip with  $\text{Im} z \geq 0$ ; we take b > 0 and prove that for b large enough g is bounded on the set  $T = \{z \in \mathbb{C} : -1 \leq \text{Re} z \leq 0, \text{Im} z \geq b\}$ ; this concludes the proof, since clearly g is bounded on the compact rectangle  $S \cap \{0 \leq \text{Im} z \leq b\}$ . We use the inequality

$$|f(z)| \le \frac{1}{|z|} + \sum_{n=1}^{\infty} \frac{2|z|}{|z^2 - n^2|}$$

on T, we have  $1/|z| \le 1$ ; if z = x + iy,  $-1 \le x \le 0$ ,  $y \ge b$  we have

$$\frac{2|z|}{z^2 - n^2|} \le \frac{2(|x| + |y|)}{|x^2 - y^2 + 2ixy - n^2|} \le \frac{2(1 + y)}{y^2 + n^2 - x^2} \le \frac{2(1 + y)}{y^2 + n^2 - 1},$$

We set b = 2 and we estimate the sum  $\sum_{n \ge 1} 2(1+y)/(y^2 + n^2 - 1)$  by an integral; since for fixed y > 1 $t \mapsto 2(1+y)/(y^2 + t^2 - 1)$  is decreasing in  $[0, +\infty[$  we have

$$\sum_{n=1}^{\infty} \frac{2(1+y)}{(y^2-1)+n^2} \le \int_0^{\infty} \frac{2(1+y)}{(y^2-1)+t^2} \, dt = \frac{2(1+y)}{\sqrt{y^2-1}} \int_0^{\infty} \frac{dt/\sqrt{y^2-1}}{1+(t/\sqrt{y^2-1})^2} = \pi \frac{1+y}{\sqrt{y^2-1}} \le \sqrt{3}\pi;$$

(the derivative of  $y \mapsto (1+y)/\sqrt{y^2-1}$  is negative on  $[2, +\infty[$ , hence this function decreases to the limit 1 as  $y \to +\infty$ , and assumes its maximum for y = 2). We conclude that for  $\text{Im } z \ge 2$  the function f is dominated by  $\sqrt{3}\pi + 1$ . We recall next that

$$|\cos(x+iy)|^{2} = \cos^{2} x \cosh^{2} y + \sin^{2} x \sinh^{2} y = \cos^{2} x + \sinh^{2} y;$$
$$|\sin(x+iy)|^{2} = \sin^{2} x \cosh^{2} y + \cos^{2} x \sinh^{2} y = \sin^{2} x + \sinh^{2} y;$$

hence (assume  $y \ge 1$ )

$$\pi |\cot(\pi(x+iy))| = \pi \sqrt{\frac{\cos^2(\pi x) + \sinh^2(\pi y)}{\sin^2(\pi x) + \sinh^2(\pi y)}} \le \pi \sqrt{\frac{1 + \sinh^2(\pi y)}{\sinh^2(\pi y)}} = \pi \frac{\cosh(\pi y)}{\sinh(\pi y)} \le \pi \operatorname{cotanh}(\pi).$$

We have proved that g is bounded, and hence constant by Liouville's theorem. To evaluate the constant, let's evaluate g(1/2); here  $\pi \cot(\pi/2) = 0$ ; to compute f(1/2) we use the first version of f:

$$\sum_{k=-m}^{m} \frac{1}{(1/2) - n} = \frac{1}{1/2} + \sum_{n=-1}^{-m} \frac{1}{(1/2) - n} + \sum_{n=1}^{m} \frac{1}{(1/2) - n} = 2 + \sum_{k=1}^{m} \frac{1}{(1/2) + k} - \sum_{n=1}^{m} \frac{1}{n - (1/2)} = 2 + \sum_{k=1}^{m} \frac{1}{(k+1) - 1 + 1/2} - \sum_{n=1}^{m} \frac{1}{n - (1/2)} = 2 + \sum_{k=1}^{m} \frac{1}{(k+1) - 1 + 1/2} - \sum_{n=1}^{m} \frac{1}{n - (1/2)} = 2 + \sum_{k=1}^{m} \frac{1}{(k+1) - 1 + 1/2} - \sum_{n=1}^{m} \frac{1}{n - (1/2)} = 2 + \sum_{k=1}^{m} \frac{1}{(k+1) - 1 + 1/2} - \sum_{n=1}^{m} \frac{1}{n - (1/2)} = 2 + \sum_{k=1}^{m} \frac{1}{(k+1) - 1 + 1/2} - \sum_{n=1}^{m} \frac{1}{(k+1) - 1} - \sum_{n=1}^{m} \frac{1}{(k$$

(put k + 1 = n in the first sum)

$$2 + \sum_{n=2}^{m+1} \frac{1}{n-1/2} - \sum_{n=1}^{m} \frac{1}{n-(1/2)} = 2 + \frac{1}{(m+1)-1/2} - \frac{1}{1-1/2} = \frac{1}{m+1/2};$$

letting m tend to infinity we get 0, hence f(1/2) = 0. It follows that g = 0. We have proved:

$$\pi \cot(\pi z) = \sum_{n=-\infty}^{\infty} \frac{1}{z-n} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} \qquad z \in \mathbb{C} \smallsetminus \mathbb{Z},$$

(where the first sum is intended as  $\lim_{m\to\infty}\sum_{n=-m}^m 1/(z-n)).$ 

We have proved that the two-sided series

$$\sum_{n=-\infty}^{\infty} \frac{1}{z-n}$$

when summed symmetrically compactly converges to a function  $f \in \mathcal{O}(\mathbb{C} \setminus \mathbb{Z})$ . Differentiating term by term the two sided series above we get

$$\sum_{n=-\infty}^{\infty} \frac{-1}{(z-n)^2}$$

where this series is now normally convergent in  $\mathbb{C} \setminus \mathbb{Z}$ , and we obtain

$$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}.$$

With the same ideas we get:

EXAMPLE 2.8.3.1. The function  $\frac{\pi}{\sin(\pi z)}$  has first order poles at every  $n \in \mathbb{Z}$ , with residue  $(-1)^n$ . We conjecture that the sum of the series

$$f(z) := \sum_{n = -\infty}^{\infty} \frac{(-1)^n}{z - n} \left( := \lim_{m \to \infty} \sum_{n = -m}^m \frac{(-1)^n}{z - n} \right) = \frac{1}{z} + \sum_{n = 1}^{\infty} \frac{(-1)^n 2z}{z^2 - n^2},$$

coincides with  $\pi/\sin(\pi z)$ . We can closely follow the argument given for the cotangent, and the same estimates work; we can also observe that if z = x + iy, with  $|y| \ge b > 0$ , b large enough, we have

$$\frac{\pi}{|\sin(\pi z)|} = \frac{\pi}{\sqrt{\sin^2(\pi x) + \sinh^2(\pi y)}} \le \frac{\pi}{\sqrt{\sinh^2(\pi y)}} = \frac{\pi}{|\sinh(\pi y)|} \le \frac{\pi}{\sinh(\pi b)}$$

To conclude we need only to prove that the two functions f and  $\pi/\sin(\pi \cdot)$  agree on a point; to this end we prove that  $g(z) = f(z) - 1/z = \sum_{n=1}^{\infty} \frac{(-1)^n 2z}{z^2 - n^2}$ , agrees with  $\pi/\sin(\pi z) - 1/z = (\pi z - \sin(\pi z))/z$  at some point; it is immediate to see that at 0 both functions are 0.

2.8.4. Another proof of the preceding formulae. There is another way of proving the preceding formulae for the cotangent and sine functions, perhaps easier, but less direct.

For every integer  $n \ge 1$  we consider a rectangular loop  $\gamma_n$ , the polygonal path of vertices (n+1/2) + n i, -(n+1/2) + n i, -(n+1/2) - n i, (n+1/2) - n i, (n+1/2) + n i. Let us estimate  $\cot(\pi z)$  on this path; as observed we have, if z = x + iy

$$|\cot an(\pi z)|^{2} = \frac{\cos^{2} \pi x + \sinh^{2}(\pi y)}{\sin^{2}(\pi x) + \sinh^{2}(\pi y)}$$

so that on the vertical sides of the rectangle we get

$$|\cot an(\pi(\pm(n+1/2)+iy))|^2 = \frac{\sinh^2(\pi y)}{1+\sinh^2(\pi y)} \le 1,$$

while on the horizontal sides:

$$\left| \cot(\pi(x \pm n \, i)) \right|^2 = \frac{\cos^2 \pi x + \sinh^2(\pi n)}{\sin^2(\pi x) + \sinh^2(\pi n)} \le \frac{1 + \sinh^2(\pi n)}{\sinh^2(\pi n)} \le 2.$$

Thus  $\|\cot a(\pi \cdot)\|_{\gamma_n} \leq \sqrt{2}$  for  $n \geq 1$ . Consider the integral

$$\int_{\gamma_n} \frac{\cot(\pi\zeta)}{\zeta^2 - z^2} \, d\zeta;$$

with  $z \notin \mathbb{Z}$ , and n > |z|. By the residue theorem this integral is (we write for simplicity  $f(\zeta)$  in place of  $\cot(\pi\zeta)/(\zeta^2 - z^2)$ ):

$$\int_{\gamma_n} \frac{\cot(\pi\zeta)}{\zeta^2 - z^2} d\zeta = 2\pi i \left( \operatorname{Res}(f, z) + \operatorname{Res}(f, -z) + \sum_{k=-n}^n \operatorname{Res}(f, k) \right);$$

the residues are easily computed:

$$\operatorname{Res}(f,z) = \frac{\operatorname{cotan}(\pi z)}{2z}; \quad \operatorname{Res}(f,-z) = \frac{\operatorname{cotan}(\pi(-z))}{-2z} = \frac{\operatorname{cotan}(\pi z)}{2z};$$

moreover

$$\operatorname{Res}(f,k) = \frac{\cos(\pi k)/(k^2 - z^2)}{\pi \cos(\pi k)} = \frac{1}{\pi (k^2 - z^2)}$$

If we estimate the integral:

$$\left| \int_{\gamma_n} \frac{\cot(\pi\zeta)}{\zeta^2 - z^2} \, d\zeta \right| \le \int_{\gamma_n} \frac{|\cot(\pi\zeta)|}{|\zeta^2 - z^2|} \, |d\zeta| \le \int_{\gamma_n} \frac{\sqrt{2}}{n^2 - |z|^2} \, |d\zeta| = \sqrt{2} \frac{8n + 2}{n^2 - |z|^2},$$

we see that the integral tends to 0 as n tends to infinity. We thus get

$$0 = \frac{\cot(\pi z)}{z} + \frac{1}{\pi} \sum_{k=-\infty}^{\infty} \frac{1}{k^2 - z^2} \qquad (z \in \mathbb{C} \setminus \mathbb{Z});$$

but of course

$$\sum_{k=-\infty}^{\infty} \frac{1}{k^2 - z^2} = -\frac{1}{z^2} + \sum_{n=1}^{\infty} \frac{2}{n^2 - z^2},$$

so that:

$$\pi \operatorname{cotan}(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} \quad (z \notin \mathbb{Z}).$$

A similar computation yields the result for  $\pi/\sin(\pi z)$ .

2.8.5. Fourier series. Functions which are holomorphic and periodic can be expanded into Fourier series. We want to prove it by using only complex-variable methods. Assume that  $f: S \to \mathbb{C}$  is holomorphic and 1-periodic, where S is an open strip which for simplicity will be assumed to contain the real axis; i.e. there exist a, b, with  $-\infty \leq a < 0 < b \leq \infty$ , such that  $S = \{s \in \mathbb{C} : a < \text{Im } s < b\}$ . The function ex:  $s \mapsto e^{2\pi i s}$  maps the strip S onto the annulus  $A = \{z \in \mathbb{C} : e^{-2\pi b} < |z| < e^{-2\pi a}\}$ . Moreover, for every 1-periodic holomorphic function  $f: S \to \mathbb{C}$  there exists a holomorphic  $g: A \to \mathbb{C}$  such that  $g(e^{2\pi i s}) = f(s)$ , for every  $s \in S$ . In fact, if  $e^{2\pi i s} = e^{2\pi i t}$  then  $e^{2\pi i (t-s)} = 1$ , which happens iff t - s = n is an integer; but then f(t) = f(s+n) = f(s). In other words, f is constant on the fibers  $F(z) = \exp^{-1}(z) = \{s \in S : e^{2\pi i s} = z\}$ , of the map ex, for  $z \in A$ : we then get a well defined map  $g: A \to \mathbb{C}$  if we set g(z) = f(s), whenever  $z = e^{2\pi i s}$ , and g is holomorphic (see 2.7.8). As every function

holomorphic in an annulus, g can be developed in a Laurent series; that is, with normal convergence on compact subsets of A we have:

$$g(z) = \sum_{n=-\infty}^{\infty} c_n z^n$$
 where  $c_n = \frac{1}{2\pi i} \int_{\gamma} \frac{g(\zeta)}{\zeta^{n+1}} d\zeta$ ,

with  $\gamma(t) = e^{2\pi i t} = ex(t), t \in [0, 1]$  the unit circle. Substitution into the formula for  $c_n$  gives

$$c_n = \frac{1}{2\pi i} \int_0^1 \frac{g(e^{2\pi it})}{e^{2\pi i(n+1)t}} (2\pi i) e^{2\pi it} dt = \int_0^1 f(t) e^{-2\pi int} dt,$$

the *n*-th Fourier coefficient of f.

We then have

. FOURIER SERIES Assume that  $f: S \to \mathbb{C}$  is holomorphic and 1-periodic, where  $S = \{s \in \mathbb{C} : a < Im \ s < b\}, a < 0 < b$ . Then, with normal convergence on closed substripts of S:

$$f(s) = \sum_{n=-\infty}^{+\infty} c_n e^{2\pi i n s} \qquad \text{where } c_n = \int_0^1 f(t) e^{-2\pi i n t} dt, \text{ for every } n \in \mathbb{Z}.$$

2.8.6. Poisson formula. Assume now that the periodic  $f : S \to \mathbb{C}$  has been obtained by replication in period 1 of a holomorphic function  $h : S \to \mathbb{C}$  falling to zero quickly enough at infinity (see 2.8.2). Then

$$c_n = \int_0^1 \left( \sum_{m=-\infty}^\infty h(t+m) \right) \, e^{-2\pi i n t} \, dt;$$

clearly normal convergence of the series allows the exchange of series and integral; hence

$$c_n = \sum_{m=-\infty}^{\infty} \int_0^1 h(t+m) e^{-2\pi i n t} dt = \sum_{m=-\infty}^{\infty} \int_m^{m+1} h(\theta) e^{-2\pi i n(\theta-m)} d\theta = \sum_{m=-\infty}^{\infty} \int_m^{m+1} h(\theta) e^{-2\pi i n \theta} d\theta = \int_{-\infty}^{\infty} h(\theta) e^{-2\pi i n \theta} d\theta.$$

We have obtained

. POISSON'S FORMULA Let  $h: S \to \mathbb{C}$  be holomorphic on the open strip  $S = \{s \in \mathbb{C} : a < \text{Im } s < b\}$ , with a < 0 < b, and such that if  $s \in S$  we have  $|h(s)| \leq K/(1+|s|)^{1+\varepsilon}$  for some  $K, \varepsilon > 0$ . Then for every  $s \in S$  we have, with normal convergence on (compact subsets of) the strip:

$$\sum_{n=-\infty}^{\infty} h(s+n) = \sum_{n=-\infty}^{\infty} \widehat{h}(n) e^{2\pi i n s} \quad in \ particular, \ if \ s = 0: \quad \sum_{n=-\infty}^{\infty} h(n) = \sum_{n=-\infty}^{\infty} \widehat{h}(n),$$

where  $\hat{h}(n) = \int_{-\infty}^{+\infty} h(x) e^{-2\pi i n x} dx$  is the value at n of the Fourier transform  $\hat{h}$  of h, the function  $\hat{h} : \mathbb{R} \to \mathbb{C}$  defined by  $\hat{h}(\xi) = \int_{-\infty}^{+\infty} h(x) e^{-2\pi i \xi x} dx$ .

2.8.7. Example. With 
$$h(s) = e^{-\alpha s^2}$$
 we get  
 $c_n = \int_{-\infty}^{\infty} e^{-\alpha s^2} e^{-2\pi i n s} ds = \int_{-\infty}^{\infty} e^{-\alpha (s^2 + 2(\pi i n/\alpha)s)} ds = \int_{-\infty}^{\infty} e^{-\alpha (s^2 + 2(\pi i n/\alpha)s - \pi^2 n^2/\alpha^2 + \pi^2 n^2/\alpha^2)} ds = e^{-\pi^2 n^2/\alpha^2} \int_{-\infty}^{\infty} e^{-\alpha (s + \pi i n/\alpha)^2} ds$ 

It is now easy to see that, for every  $w \in \mathbb{C}$  we have

$$\int_{-\infty}^{\infty} e^{-\alpha(x+w)^2} dx = \int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\pi/\alpha};$$

in fact, if w = p + iq then clearly, by translation invariance of the integral:

$$\int_{-\infty}^{\infty} e^{-\alpha(x+w)^2} \, dx = \int_{-\infty}^{\infty} e^{-\alpha((x+p)+iq)^2} \, dx = \int_{-\infty}^{\infty} e^{-\alpha(t+iq)^2} \, dt,$$

and if we integrate  $e^{-\alpha z^2}$  on polygonal paths [-r, r, r + iq, -r + iq, -r] and let r tend to infinity we see that

$$\int_{-\infty}^{\infty} e^{-\alpha t^2} dt = \int_{-\infty}^{\infty} e^{-\alpha (t+iq)^2} dt$$

We have obtained  $c_n = \sqrt{\pi/\alpha} e^{-(\pi^2/\alpha)n^2}$ , so that

$$g(z) = \sqrt{\pi/\alpha} \sum_{n = -\infty}^{\infty} e^{-(\pi^2/\alpha)n^2} z^n; \qquad z \in \mathbb{C} \setminus \{0\};$$

substituting  $z = e^{2\pi i s}$  we get:

$$f_{\alpha}(s) := \sum_{n=-\infty}^{\infty} e^{-\alpha(s+n)^2} = \sqrt{\pi/\alpha} \sum_{n=-\infty}^{\infty} e^{-(\pi^2/\alpha)n^2} e^{2\pi i n s} \qquad s \in \mathbb{C}, \, \alpha > 0.$$

In particular, for s = 0:

$$\sum_{n=-\infty}^{\infty} e^{-\alpha n^2} = \sqrt{\pi/\alpha} \sum_{n=-\infty}^{\infty} e^{-(\pi^2/\alpha)n^2} \qquad \alpha > 0.$$

2.8.8. A theta function formula. In the derivation of the functional equation for the Riemann's zeta function we shall use the function  $\theta(x) = \sum_{n=-\infty}^{\infty} e^{-n^2 \pi x}$ , i.e. with  $\alpha = \pi x$ ; then, putting  $\pi/x$  in place of  $\alpha$  in the preceding formula we get:

$$\theta(1/x) := \sum_{n = -\infty}^{\infty} e^{-n^2 \pi/x} = \sqrt{x} \sum_{n = -\infty}^{\infty} e^{-n^2 \pi x} = \sqrt{x} \ \theta(x) \qquad x > 0,$$

a formula which we shall later put to use.

2.8.9. The cotangent development by Poisson's formula. If a > 0 the function  $h_a(s) = 1/(a^2 + s^2)$  is holomorphic on the strip  $\{s \in \mathbb{C} : -a < \text{Im } s < a\}$ ; we have for its Fourier transform, for every  $\xi \in \mathbb{R}$ :

$$\widehat{h}_a(\xi) = \int_{-\infty}^{\infty} \frac{e^{-2\pi i\xi x}}{a+x^2} \, dx = \frac{\pi}{a} e^{-2\pi a|\xi|}$$

(this is a classical exercise in the computation of integrals via residue theorem), and Poisson's formula 2.8.6 then gives

$$\sum_{n=-\infty}^{+\infty} \frac{1}{a^2 + n^2} = \frac{\pi}{a} \sum_{n=-\infty}^{+\infty} e^{-2\pi a|n|};$$

a little work on the right-hand side yields

$$\sum_{n=-\infty}^{+\infty} e^{-2\pi a|n|} = 1 + 2\sum_{n=1}^{\infty} e^{2\pi an} = 1 + 2\frac{e^{-2\pi a}}{1 - e^{-2\pi a}} = \frac{1 + e^{-2\pi a}}{1 - e^{-2\pi a}} = \frac{e^{\pi a} + e^{-\pi a}}{e^{\pi a} - e^{-\pi a}} = \operatorname{cotanh}(\pi a),$$

while the left-hand side is

$$\sum_{n=-\infty}^{+\infty} \frac{1}{a^2 + n^2} = \frac{1}{a^2} + 2\sum_{n=1}^{\infty} \frac{1}{a^2 + n^2},$$

so that we get

$$\pi \operatorname{cotanh}(\pi a) = \frac{1}{a} + \sum_{n=1}^{\infty} \frac{2a}{a^2 + n^2} \qquad (a > 0).$$

Interpreted as a series of functions of the complex variable a, it is easy to see that the right-hand side converges normally on  $\mathbb{C} \setminus i\mathbb{Z}$  to a holomorphic function, which coincides with  $\pi \operatorname{cotanh}(\pi a)$  on the positive real line. By the identity theorem we then have on all of  $\mathbb{C} \setminus i\mathbb{Z}$ , with normal convergence:

$$\pi \operatorname{cotanh}(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 + n^2};$$

and setting iz in place of z we get, noting that  $\cosh(iw) = \cosh w$  and  $\sinh(iw) = i \sin w$ :

$$\frac{\pi}{i} \operatorname{cotan}(\pi z) = \frac{1}{iz} + i \sum_{n=1}^{\infty} \frac{2z}{n^2 - z^2} \quad \text{hence} \quad \pi \operatorname{cotan}(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}.$$

#### 2.9. Holomorphic functions defined by integrals.

THEOREM. Let D be open in  $\mathbb{C}$ , let I be an interval of  $\mathbb{R}$ , and let  $f : D \times I \to \mathbb{C}$  be a function such that for every given  $x \in I$  the function  $f(\cdot, x) : D \to \mathbb{C}$  given by  $s \mapsto f(s, x)$  is holomorphic on D. Assume that

- for every compact subset K of D there exists a function  $\rho_K : I \to [0, +\infty[$  such that  $\rho_K \in L^1(I)$ and for every  $(s, x) \in K \times I$  we have  $|f(s, x)| \leq \rho_K(x)$ .
- for every  $s \in D$  the function  $x \mapsto f(s, x)$  is measurable;

Then the formula

$$F(s) = \int_{I} f(s, x) \, dx$$

defines a holomorphic function  $F: D \to \mathbb{C}$ , whose derivative is obtained by differentiating with respect to s under the integral sign:

$$F'(s) = \int_I \partial_s f(s, x) \, dx.$$

*Proof.* Given  $s \in D$ , and a sequence  $s_i$  in  $D \setminus \{s\}$  converging to s, we prove that

$$\lim_{j \to \infty} \frac{F(s_j) - F(s)}{s_j - s} = \int_I \partial_s f(s, x) \, dx$$

(it will be proved also that the integral on the right side exists). Pick r > 0 such that  $B(s, r] \subseteq D$ ; then  $s_j \in B(s, r]$  for j large enough and

$$\frac{F(s_j) - F(s)}{s_j - s} = \int_I \frac{f(s_j, x) - f(s, x)}{s_j - s} \, dx$$

if we let j tend to infinity, inside the integral the limit is exactly  $\partial_s f(s, x)$ ; if we find a function  $u \in L^1(I)$  such that

$$\left|\frac{f(s_j, x) - f(s, x)}{s_j - s}\right| \le u(x) \quad \text{for } j \in \mathbb{N} \text{ large enough, and (almost) every } x \in I,$$

then Lebesgue dominated convergence theorem allows us to conclude.

By the hypothesis there is  $\rho \in L^1(I)$  such that  $|f(s, x)| \leq \rho(x)$  for every  $(s, x) \in B(s, r] \times I$ . Pick t, with 0 < t < r; we then have, if  $s_j \in B(s, t]$ :

$$\left|\frac{f(s_j, x) - f(s, x)}{s_j - s}\right| \le \max\{|\partial_s f(z, x)| : z \in [s, s_j]\} \le \max\{|\partial_s f(z, x)| : z \in B(s, t]\}:$$

the first inequality is simply the mean value theorem, the second is due to the fact that the segment  $[s, s_j]$  is contained in the disc B(s, t] if  $s_j \in B(s, t]$ . Remember now that for every g holomorphic on B(s, r] we have (recall 2.7)

$$||g'||_{B(s,t]} \le \frac{||g||_{B(s,r]}}{r-t};$$

thus

$$\max\{|\partial_s f(z,x)|: z \in B(s,t]\} \le \frac{1}{r-t} \max\{|f(z,x)|: z \in B(s,r]\} \le \frac{\rho(x)}{r-t} = u(x).$$

REMARK. In exactly the same way the following more general theorem could be proved:

. Let  $(X, \mathcal{S}, \mu)$  be a measure space. Let D be open in  $\mathbb{C}$  and let  $f : D \times X \to \mathbb{C}$  be a function such that for every  $x \in X$  the function  $s \mapsto f(s, x)$  is holomorphic on D. Assume that for every compact subset Kof D there exists a function  $\rho_K \in L^1_\mu(X)$  such that for every  $(s, x) \in K \times X$  we have  $|f(s, x)| \leq \rho_K(x)$ , and that for every  $s \in D$  the function  $x \mapsto f(s, x)$  is measurable. Then the formula

$$F(s) = \int_X f(s, x) \, d\mu(x),$$

defines a holomorphic function  $F: D \to \mathbb{C}$ , whose derivative is obtained by differentiating with respect to s under the integral sign:

$$F'(s) = \int_X \partial_s f(s, x) \, d\mu(x).$$

Holomorphic functions defined by normally convergent series fall in the scope of this theorem (with  $\mu$  the counting measure on  $\mathbb{N}$ ).

EXAMPLE 2.9.0.1. Let  $I = ]1, +\infty[$  and let  $\omega : I \to \mathbb{C}$  be bounded, measurable and such that  $\omega \in O(e^{-\varepsilon\xi})$  for a given  $\varepsilon > 0$  as  $\xi \to \infty$  (that is, there exist L, a > 0 such that  $|\omega(\xi)| \leq Le^{-\varepsilon\xi}$  for  $\xi \geq a$ ). Then, for every  $a, b \in \mathbb{C}$  the formula

$$F(s) = \int_{1}^{\infty} \xi^{as+b} \omega(\xi) \, d\xi \qquad (s \in \mathbb{C})$$

defines an entire function  $F \in \mathcal{O}(\mathbb{C})$ . In fact, let a compact  $K \subseteq \mathbb{C}$  be given; since  $\xi > 0$  we have

$$|\xi^{as+b}\omega(\xi)| = |\xi^{as+b}| |\omega(\xi)| = \xi^{\operatorname{Re}(as+b)} |\omega(\xi)| = \xi^{\operatorname{RE$$

now since  $\xi > 1$  we have:

$$\xi^{\operatorname{Re}(as+b)} \le \xi^{|\operatorname{Re}(as+b)|} \le \xi^{|as+b|} \le \xi^{|a||s|+|b|} \le \xi^{|a|\mu+|b|},$$

where  $\mu = \max\{|s|: s \in K\}$ ; then, if  $s \in K$  and  $\xi \in [1, +\infty[$  we have

$$|\xi^{as+b}\omega(\xi)| \le L\,\xi^{|a|\mu+|b|}e^{-\varepsilon\xi},$$

(since  $\omega$  is bounded it is clear that by enlarging L we can make  $|\omega(\xi)| \leq L e^{-\varepsilon\xi}$  on all of  $[1, +\infty[)$ . This last function  $\xi \mapsto L \xi^{|a|\mu+|b|} e^{-\varepsilon\xi}$  is clearly in  $L^1([1, +\infty[), \text{ since } \varepsilon > 0.$ 

EXAMPLE 2.9.0.2. Let I = [0,1]; assume that  $\omega : I \to \mathbb{C}$  is bounded and measurable. Then the formula

$$F(s) = \int_0^1 \xi^{s-1} \omega(\xi) \, d\xi,$$

defines a function F holomorphic at least in the right half-plane  $S = \{s \in \mathbb{C} : \operatorname{Re}(s) > 0\}$ . In fact, if  $K \subseteq S$  is compact, then  $a = \min\{\operatorname{Re} s : s \in K\} > 0$  and we have:

$$\xi^{s-1}| = \xi^{\operatorname{Re} s-1} \le \xi^{a-1}$$
 for every  $s \in K$  and  $0 < \xi \le 1$ .

Then, in  $K \times [0, 1]$ , if  $M = \|\omega\|_{\infty} = \sup\{|\omega(\xi)| : \xi \in [0, 1]\}$ :

$$|\xi^{s-1}\omega(\xi)| \le \xi^{a-1}M \qquad \text{and since } a-1 > -1 \text{ the function } \xi \mapsto M\xi^{a-1} \text{ is in } L^1([0,1]).$$

If  $\omega$  is also bounded away from 0 in a right neighborhood of 0, then in general the function F will have a singularity at s = 0; for instance, it is easy to see that  $s \mapsto \int_0^1 \xi^{s-1} e^{-\xi} d\xi$  tends to  $\infty$  if  $s \to 0^+$ (with  $s \in \mathbb{R}$ ). If on the contrary  $\lim_{\xi \to 0^+} \omega(\xi) = 0$ , then F may be holomorphic in a larger halfplane. For instance, if for some  $\alpha > 0$  we have that  $\omega$  is  $O(\xi^{\alpha})$  as  $\xi \to 0^+$ , then F is holomorphic in  $S_{\alpha} = \{s \in \mathbb{C} : \operatorname{Re} s > -\alpha\}$ . In fact we have  $|\omega(\xi)| \leq \lambda \xi^{\alpha}$  for  $\xi \in ]0, \delta]$ , for some  $\delta, 0 < \delta < 1$ , and some  $\lambda > 0$  and if  $\operatorname{Re} s \geq a > -\alpha$  we have

$$|\xi^{s-1}\,\omega(\xi)| = \xi^{\operatorname{Re} s-1}|\omega(\xi)| \le \xi^{a-1}\lambda\xi^{\alpha} = \lambda\xi^{(a+\alpha)-1} \quad \text{for } 0 < \xi < \delta.$$

As integrable dominating function we can take  $\rho(\xi) = \lambda \xi^{(a+\alpha)-1}$  for  $\xi \in ]0, \delta]$ ,  $\rho(\xi) = M \xi^{a-1}$  in  $[\delta, 1]$ . For instance the function

$$F(s) = \int_0^1 \xi^{s-1} (e^{\xi} - 1) \, d\xi,$$

is holomorphic in the half plane  $\{\operatorname{Re} s > -1\}$ 

EXAMPLE 2.9.0.3. Prove that the formula (B(p,q)) is to be read "beta of p,q")

$$B(p,q) = \int_0^1 \xi^{p-1} (1-\xi)^{q-1} d\xi \qquad \text{Re}\,p, \text{Re}\,q > 0,$$

defines a function on  $S \times S$  holomorphic in both variables (keeping one fixed, it is holomorphic in the other; S is the right half-plane); verify that B(p,q) = B(q,p); this is Euler's beta function.

2.9.1. The Gamma function as an integral. The gamma function will be defined as an infinite product (3.4.3); we describe here its integral representation. The integral

$$\Gamma(s) = \int_0^\infty \xi^{s-1} e^{-\xi} d\xi,$$

is absolutely convergent if  $\operatorname{Re} s > 0$ , and so defines a function  $\Gamma : S \to \mathbb{C}$ , where  $S = {\operatorname{Re} s > 0}$  is the right open half-plane. And  $\Gamma$  is holomorphic on this half-plane: to take advantage of the previous examples write

$$\Gamma(s) = \int_0^1 \xi^{s-1} e^{-\xi} d\xi + \int_1^\infty \xi^{s-1} e^{-\xi} d\xi = F_1(s) + F_2(s),$$

with obvious meaning of the symbols; by 2.9.0.1  $F_2$  is an entire function, and by 2.9.0.2  $F_1$  is holomorphic on Re s > 0, so that the sum is holomorphic on the right half-plane. Integrating by parts we get the functional equation for  $\Gamma$ :

$$\Gamma(s+1) = \int_0^\infty \xi^s \, e^{-\xi} \, d\xi = \left[-\xi^s \, e^{-\xi}\right]_0^\infty + \int_0^\infty s \, \xi^{s-1} \, e^{-\xi} \, d\xi = s \, \Gamma(s)$$
  
that is  $\Gamma(s+1) = s \, \Gamma(s)$  (Re  $s > 0$ ).

By induction we immediately get

$$\Gamma(s+m+1) = \Gamma(s) \, s(s+1) \cdots (s+m) = \Gamma(s) \, \prod_{k=0}^{m} (s+k),$$

which, coupled with  $\Gamma(1) = 1$  says that  $\Gamma(m+1) = m!$  for every  $m \in \mathbb{N}$ , but more importantly allows us to extend  $\Gamma$  to a holomorphic function on  $\mathbb{C} \setminus (-\mathbb{N}) = \mathbb{C} \setminus \{0, -1, -2, ...\}$  simply by defining

$$\Gamma(s) = \frac{\Gamma(s+m+1)}{s(s+1)\cdots(s+m)} \quad \text{if} - (m+1) < \operatorname{Re} s.$$

We easily get that  $\Gamma$  has a simple pole at -m, and  $\operatorname{Res}(\Gamma, -m) = (-1)^m/m!$ . Next, an easy and nice application of the Fubini's theorem (see. e.g. [Remmert]) gives the addition formula for  $\Gamma$  in terms of the beta function defined in 2.9.0.3:

$$\frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)} = B(p,q)\left(:=\int_0^1 \xi^{p-1}(1-\xi)^{q-1}\,d\xi\right) \qquad \operatorname{Re} p, \operatorname{Re} q > 0;$$

from which it follows that, if  $0 < \sigma < 1$  we have

$$\Gamma(\sigma)\Gamma(1-\sigma) = B(\sigma, 1-\sigma) = \int_0^1 \xi^{\sigma-1} (1-\xi)^{-\sigma} d\xi = \int_0^\infty \frac{t^{\sigma-1}}{1+t} dt$$

(the last equality is obtained by the change of variable  $\xi = t/(1+t)$ ). This integral may be computed via residue theorem, and we get:

$$\Gamma(\sigma) \Gamma(1 - \sigma) = \frac{\pi}{\sin(\pi\sigma)} \qquad 0 < \sigma < 1$$

Since  $s \mapsto \Gamma(s) \Gamma(1-s)$  is holomorphic in the region  $\mathbb{C} \setminus \mathbb{Z}$ , by the identity theorem we get the remarkable formula:

(EULER'S SUPPLEMENT) 
$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \qquad s \in \mathbb{C} \smallsetminus \mathbb{Z}.$$

This formula shows, among other things, that  $\Gamma$  is zero-free; moreover we get  $\Gamma(1/2)^2 = \pi$ , and since  $\Gamma(\sigma) > 0$  for  $\sigma > 0$ , we get  $\Gamma(1/2) = \sqrt{\pi}$ .

2.9.2. The functional equation for  $\zeta$ . Following the original procedure by Riemann, we now obtain the functional equation for the  $\zeta$  function. Start with  $\Gamma(s/2) = \int_0^\infty \xi^{(s/2)-1} e^{-\xi} d\xi$ , and substitute  $\xi = n^2 \pi x$  in the integral, obtaining

$$\Gamma(s/2) = \int_0^\infty n^{s-2} \pi^{(s/2)-1} x^{(s/2)-1} e^{-n^2 \pi x} (n^2 \pi) \, dx = n^s \pi^{s/2} \int_0^\infty x^{(s/2)-1} e^{-n^2 \pi x} \, dx \qquad (n = 1, 2, 3, \dots),$$

equivalently

$$\pi^{-s/2} \frac{\Gamma(s/2)}{n^s} = \int_0^\infty x^{(s/2)-1} e^{-n^2 \pi x} \, dx \qquad (n = 1, 2, 3, \dots).$$

Summing over all integers from 1 to  $\infty$  we get, assuming that  $\sigma > 1$ :

$$\pi^{-s/2} \Gamma(s/2) \zeta(s) = \sum_{n=1}^{\infty} \int_0^\infty x^{(s/2)-1} e^{-n^2 \pi x} \, dx.$$

Moreover for  $\sigma > 1$  the series of  $L^1$ -norms

$$\sum_{n=1}^{\infty} \int_{0}^{\infty} |x^{(s/2)-1}| e^{-n^2 \pi x} \, dx = \sum_{n=1}^{\infty} \int_{0}^{\infty} x^{(\sigma/2)-1} \, e^{-n^2 \pi x} \, dx = \sum_{n=1}^{\infty} \pi^{-\sigma/2} \frac{\Gamma(\sigma/2)}{n^{\sigma}}$$

is convergent; then we can exchange the series and the integral, obtaining:

$$\pi^{-s/2} \Gamma(s/2) \zeta(s) = \int_0^\infty x^{(s/2)-1} \,\omega(x) \, dx \quad \text{where} \quad \omega(x) = \sum_{n=1}^\infty e^{-n^2 \pi x}.$$

We can write:

$$\int_0^\infty x^{(s/2)-1}\,\omega(x)\,dx = \int_0^1 x^{(s/2)-1}\,\omega(x)\,dx + \int_1^\infty x^{(s/2)-1}\,\omega(x)\,dx = F_1(s) + F_2(s);$$

we observe that  $\omega(x)$  is  $O(e^{-\pi x})$  as  $x \to \infty$ ; in fact, if  $x \ge 1$ :

$$\omega(x) = e^{-\pi x} \left( 1 + \sum_{n=2}^{\infty} e^{-(n^2 - 1)\pi x} \right) \le L e^{-\pi x};$$

where  $L = 1 + \sum_{n=2}^{\infty} e^{-(n^2 - 1)\pi}$ . Hence the second term  $F_2(s)$  is an entire holomorphic function. We now put x = 1/t in the first of these integrals, obtaining:

$$F_1(s) = \int_0^1 x^{(s/2)-1} \,\omega(x) \, dx = \int_1^\infty t^{1-s/2} \,\omega(1/t) \, \frac{dt}{t^2}$$

If we consider  $\theta(x) = \sum_{n=-\infty}^{\infty} e^{-n^2 \pi x}$ , for x > 0 (one of Jacobi's theta functions), we have  $\theta(x) = 1 + 2\omega(x)$ , equivalently  $\omega(x) = (\theta(x) - 1)/2$ . It was proved in 2.8.8, with the help of the Poisson summation formula, that:

$$\theta(1/x) = \sqrt{x} \,\theta(x) \quad \text{for } x > 0;$$

thus:

$$\omega(1/x) = \frac{1}{2}\theta(1/x) - \frac{1}{2} = \frac{1}{2}\sqrt{x}\,\theta(x) - \frac{1}{2} = \frac{\sqrt{x}}{2}(1+2\omega(x)) - \frac{1}{2} = -\frac{1}{2} + \frac{\sqrt{x}}{2} + \sqrt{x}\,\omega(x)$$

Hence:

$$\begin{split} F_1(s) &= \int_1^\infty t^{1-s/2} \,\omega(1/t) \,\frac{dt}{t^2} = \\ &\int_1^\infty t^{-1-(s/2)} \left( -\frac{1}{2} + \frac{t^{1/2}}{2} + t^{1/2} \,\omega(t) \right) \,dt = \\ &- \frac{1}{2} \int_1^\infty t^{-1-(s/2)} \,dt + \frac{1}{2} \int_1^\infty t^{-1/2-(s/2)} \,dt + \int_1^\infty t^{-1/2-(s/2)} \omega(t) \,dt = \\ &\frac{1}{2(-s/2)} - \frac{1}{2(1/2 - (s/2))} + \int_1^\infty t^{-s/2-1/2} \,\omega(t) \,dt = \\ &- \frac{1}{s} + \frac{1}{s-1} + \int_1^\infty t^{-s/2-1/2} \,\omega(t) \,dt. \end{split}$$

Writing again x in place of t in the second integral, we have obtained, if  $\sigma = \operatorname{Re} s > 1$ :

$$\pi^{-s/2} \Gamma(s/2) \zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty (x^{s/2-1} + x^{-s/2-1/2}) \,\omega(x) \, dx = G(s).$$

But, as observed,  $\omega$  is  $O(e^{-\pi x})$  as  $x \to +\infty$ , so that the integral on the right hand side defines an entire function of s, and the right hand side is a meromorphic function G with two simple poles at s = 0 and s = 1. This means that we can define  $\zeta(s)$  as

$$\zeta(s) := \frac{\pi^{s/2}}{\Gamma(s/2)} G(s) = \frac{\pi^{s/2}}{2(s-1)\Gamma(s/2+1)} (1 + s(s-1)\varphi(s)),$$

where  $\varphi(s) = \int_1^\infty (x^{s/2-1} + x^{-s/2-1}) \,\omega(x) \, dx$  is an entire function. Since  $\Gamma$  is zero-free, the only possible pole of  $\zeta$  is 1. Now 1 is indeed a pole for  $\zeta$ , with residue  $\pi^{1/2}/\Gamma(1/2) = 1$ . Notice that  $\zeta(0) = -1/2$ . Moreover the simple poles of  $\Gamma(s/2+1)$ , that is s = -2n,  $n = 1, 2, 3, \ldots$  are zeros for  $\zeta$  (the so called *trivial zeros* of  $\zeta$ ). Notice that the meromorphic function G does not change if we exchange s with 1-s, i. e. G is symmetric with respect to the point s = 1/2. We then get the

Functional equation for 
$$\zeta$$
  $\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s)$ 

which may also be written as

$$\zeta(1-s) = \gamma(s)\,\zeta(s), \qquad \text{where} \quad \gamma(s) = \pi^{1/2-s}\,\frac{\Gamma(s/2)}{\Gamma((1-s)/2)};$$

notice that  $\gamma$  is meromorphic on  $\mathbb{C}$ , with one simple pole at s = 0 (Res $(\gamma, 0) = 2$ ), and other simple poles at the negative even integers, and simple zeros at the positive odd integers 2n + 1, with n = 0, 1, 2, ...

It will be proved, with the representation of  $\zeta$  as Euler's infinite product, to be discussed later in 3.2.3, that  $\zeta$  has no zeros in the half-plane  $\sigma > 1$ . From this and the functional equation it follows that the only zeros of  $\zeta$  for Re s < 0 are the trivial ones, and that they are all simple; with a little more work one shows that there are no zeros on the line Re s = 1, so that the only non-trivial zeros of  $\zeta$  must reside on the open strip 0 < Re s < 1, the *critical strip*.

Moreover we have that  $\zeta(\bar{s}) = \zeta(s)$  for every  $s \in \mathbb{C} \setminus \{1\}$ . Recall in fact that if D is an open subset of  $\mathbb{C}$ ,  $f \in \mathcal{O}(D)$ , and  $D^* = \{\bar{s} : s \in D\}$  is the symmetrical of D with respect to the real axis, then the formula  $f^*(s) = \overline{f(\bar{s})}$  defines a holomorphic function  $f^* : D^* \to \mathbb{C}$ . If  $D = \mathbb{C} \setminus \{1\}$  then  $D^* = D$ , and since  $\zeta^*(\sigma) = \zeta(\sigma)$  for  $\sigma > 1$  (this can be shown in many ways, directly from the definition as series) we have  $\overline{\zeta(\bar{s})} = \zeta(s)$  for every  $s \in \mathbb{C} \setminus \{1\}$ , which we can write also  $\zeta(\bar{s}) = \overline{\zeta(s)}$ , for every  $s \in \mathbb{C} \setminus \{1\}$ .

Of course, if s is a zero of  $\zeta$  in the critical strip, then also  $\zeta(1-s) = 0$ , from the functional equation, and as above proved we also have  $\zeta(\bar{s}) = 0$ . So: zeros on the critical strip must be symmetrical with respect to the point s = 1/2, and also with respect to the line Re s = 1/2. We close with a proof of the fact that  $\zeta(\sigma) < 0$  for  $0 < \sigma < 1$ : we estimate  $\omega(x)$  for x > 1:

$$\omega(x) = \sum_{n=1}^{\infty} e^{-n^2 \pi x}$$

with an integral: observe that  $t \mapsto e^{-t^2 \pi x}$  is strictly decreasing and positive on  $[0, +\infty)$  so that

$$\sum_{n=1}^{\infty} e^{-n^2 \pi x} < \int_0^{\infty} e^{-t^2 \pi x} \, dt = \frac{1}{2\sqrt{x}};$$

and we can estimate the integral in the expression of G by

$$\int_{1}^{\infty} (x^{\sigma/2-1} + x^{-\sigma/2-1/2}) \,\omega(x) \, dx < \frac{1}{2} \int_{1}^{\infty} (x^{\sigma/2-3/2} + x^{-\sigma/2-1}) \, dx = \frac{1}{2} \left( \frac{-1}{\sigma/2 - 1/2} + \frac{-1}{-\sigma/2} \right) = \frac{1}{\sigma(1 - \sigma)}.$$

whence

$$G(\sigma) < \frac{1}{\sigma(\sigma-1)} + \frac{1}{\sigma(1-\sigma)} = 0$$

EXERCISE 2.9.2.1. Write again  $\Gamma(s) = \int_0^1 \xi^{s-1} e^{-\xi} d\xi + \int_1^\infty \xi^{s-1} e^{-\xi} d\xi$ ; prove that

$$\int_0^1 \xi^{s-1} e^{-\xi} d\xi = \sum_{n=0}^\infty \frac{(-1)^n}{(s+n) n!} \quad (\operatorname{Re} s > 0).$$

Prove that the series on the right hand-side is normally convergent in  $\mathbb{C} \setminus (-\mathbb{N})$ . Deduce another representation on  $\mathbb{C} \setminus (-\mathbb{N})$  of the  $\Gamma$  function (partial fraction development of  $\Gamma$ ):

$$\Gamma(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(s+n)\,n!} + \varphi(s),$$

where  $\varphi(s) = \int_1^\infty \xi^{s-1} e^{-\xi} d\xi$  is an entire function; check the residues at poles of  $\Gamma$  with this formula, and verify the functional relation  $\Gamma(s+1) = s \Gamma(s)$ .

2.9.3. The function  $\zeta_a$ . Let us prove that the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$$

compactly converges in the open half-plane  $S = \{s \in \mathbb{C} : \text{Re } s > 0\}$  to a holomorphic function  $\zeta_a(s)$  (zeta function with alternating signs). We use Abel's summation formula: let  $\alpha_n$  be a sequence such that  $\alpha_n - \alpha_{n-1} = (-1)^{n-1}$ , e.g. take  $\alpha_0 = 0, \alpha_1 = 1, \alpha_2 = 0$ , and in general  $\alpha_n = 0$  for n even,  $\alpha_n = 1$  for n odd. Consider the m-th partial sum of the above series:

$$\sum_{n=1}^{m} \frac{(-1)^{n-1}}{n^s} = \sum_{n=1}^{m} \frac{\alpha_n - \alpha_{n-1}}{n^s} = \sum_{n=1}^{m} \frac{\alpha_n}{n^s} - \sum_{n=1}^{m} \frac{\alpha_{n-1}}{n^s} = \sum_{n=1}^{m} \frac{\alpha_n}{n^s} - \sum_{k=0}^{m-1} \frac{\alpha_k}{(k+1)^s} = \frac{\alpha_m}{m^s} + \sum_{n=1}^{m-1} \frac{\alpha_n}{n^s} - \sum_{n=1}^{m-1} \frac{\alpha_n}{(n+1)^s} = \frac{\alpha_m}{m^s} - \sum_{n=1}^{m-1} \alpha_n \left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right).$$

We now prove that this last sequence compactly converges in S to a holomorphic function. Let K be a compact subset of S; then  $\mu = \min\{\operatorname{Re} s : s \in K\} > 0$ , hence  $|\alpha_m/m^s| \leq 1/m^{\mu}$  for  $s \in K$ , so that the sequence  $\alpha_m/m^s$  compactly converges to the zero function; the second term is the partial sum of the series

$$\sum_{n=1}^{\infty} \alpha_n \left( \frac{1}{(n+1)^s} - \frac{1}{n^s} \right);$$

the conclusion is reached if we prove that this series normally converges in S. We have:

$$\left|\frac{1}{(n+1)^s} - \frac{1}{n^s}\right| = \left|\int_n^{n+1} \frac{-s}{x^{s+1}} \, dx\right| \le |s| \int_n^{n+1} \frac{dx}{x^{\sigma+1}},$$

so that

$$\left\|\alpha_n \left(\frac{1}{(n+1)^s} - \frac{1}{n^s}\right)\right\|_K \le M \int_n^{n+1} \frac{dx}{x^{\mu+1}}$$

where  $M = \max\{|s| : s \in K\}$ . We then have:

$$\sum_{n=1}^{\infty} \left\| \alpha_n \left( \frac{1}{(n+1)^s} - \frac{1}{n^s} \right) \right\|_K \le \sum_{n=1}^{\infty} M \int_n^{n+1} \frac{dx}{x^{\mu+1}} = M \int_1^{\infty} \frac{dx}{x^{\mu+1}} = \frac{M}{\mu},$$

and normal convergence in  $S = \{s \in \mathbb{C} : \text{Re } s > 0\}$  is proved. Observe that if  $\text{Re } s \leq 1$  the series  $\sum_{n=1}^{\infty} (-1)^{n-1}/n^s$  does not converge absolutely, so that whenever a compact subset K of S contains a point with  $\text{Re } s \leq 1$  the series of the K-norms will not converge. Notice that for Re s > 1 we have

$$\zeta(s) - \zeta_a(s) = \sum_{k=1}^{\infty} \frac{2}{(2k)^s} = 2^{1-s} \sum_{k=1}^{\infty} \frac{1}{k^s} = 2^{1-s} \zeta(s) \quad \text{whence} \quad (1 - 2^{1-s}) \zeta(s) = \zeta_a(s),$$

so that

$$\zeta(s) = \frac{\zeta_a(s)}{1 - 2^{1-s}} \qquad \text{Re}\,s > 1;$$

if  $1 - 2^{1-s} \neq 0$ , i.e. if  $s \neq 1 + (2\pi/\log 2)k i$  but  $\operatorname{Re} s > 0$  the above formula makes sense, and allows us to extend the zeta function to the half-plane  $\operatorname{Re} s > 0$ , perhaps with poles at  $1 + (2\pi/\log 2)k i$ ,  $k \in \mathbb{Z}$ . Now s = 1 is indeed a simple pole: we have  $\zeta_a(1) = \log 2 \neq 0$ , and the denominator  $1 - 2^{1-s}$  has a simple zero at 1; the derivative of the denominator is  $2^{1-s} \log 2$ , so that the residue is 1. Since we have already seen that  $\zeta$  has a holomorphic extension to  $\mathbb{C} \setminus \{1\}$ , we can use this fact to show that  $\zeta_a$  can be extended to an entire function by the formula

$$\zeta_a(s) = (1 - 2^{1-s})\,\zeta(s);$$

hence  $\zeta_a$  has zeros at  $1 + (2\pi/\log 2)ki$  if  $k \neq 0$ .

2.9.4. Continuity of the composition. For every pair D, E of open subsets of  $\mathbb{C}$  we denote by C(D, E) the subset of  $C(D, \mathbb{C})$  consisting of functions such that  $f(D) \subseteq E$ ; C(D, E) is topologized with the topology induced by the compact-open topology. We want to prove the following result:

. COMPOSITION THEOREM Let D, E, G be open subsets of  $\mathbb{C}$ ; then the composition map  $(g, f) \mapsto g \circ f$  is continuous as a map of  $C(E, G) \times C(D, E)$ , with the product topology, into C(D, G).

In other words, if  $g_n \in C(E, G)$  converges compactly to  $g \in C(E, G)$  and  $f_n \in C(D, E)$  converges compactly to  $f \in C(D, E)$ , then  $g_n \circ f_n$  converges compactly to  $g \circ f \in C(D, G)$ . To prove this we use the auxiliary notion of *continuous convergence*.

DEFINITION. Let X be a metrizable space; denote by  $C(X, \mathbb{C})$  the set of continuous functions from X to  $\mathbb{C}$ . If  $(f_n)_n$  is a sequence of  $C(X, \mathbb{C})$  and  $f \in C(X, \mathbb{C})$ , we say that the sequence  $(f_n)_n$  converges continuously to f on X if for every  $x \in X$  and every sequence  $(x_n)_n$  in X converging to x we have that  $\lim_n f_n(x_n) = f(x)$ .

We have the

LEMMA. If f and  $f_n$  are as above then  $(f_n)_n$  converges continuously to f if and only if  $(f_n)_n$  converges to f uniformly on every compact subset of X.

Proof. Necessity Assume that  $(f_n)_n$  converges continuously to f; arguing by contradiction, assume that it does not converge to f uniformly on every compact subset of X; this means that there exists a compact subset K of X such that  $||f - f_n||_K$  does not tend to 0 as n tends to  $\infty$ ; in turn, this means that there is  $\alpha > 0$  and a sequence  $\nu(0) < \nu(1) < \ldots$  of natural numbers such that  $||f - f_{\nu(k)}||_K > \alpha$  for every  $k \in \mathbb{N}$ ; and then there exists a sequence  $(x_{\nu(k)})_k$  of points of K such that  $||f(x_{\nu(k)}) - f_{\nu(k)}(x_{\nu(k)})| > \alpha$ for every  $k \in \mathbb{N}$ . By compactness of K, the sequence  $x_{\nu(k)}$  has a subsequence  $(x_{\nu(\mu(j))})_j$  converging to  $x \in K$ . By continuous convergence we have  $\lim_{j\to\infty} |f(x) - f_{\nu(\mu(j))}(x_{\nu(\mu(j))})| = 0$ , but:

$$\begin{aligned} |f(x) - f_{\nu(\mu(j))}(x_{\nu(\mu(j))})| &= |f(x) - f(x_{\nu(\mu(j))}) + f(x_{\nu(\mu(j))}) - f_{\nu(\mu(j))}(x_{\nu(\mu(j))})| \ge \\ &\geq |f(x_{\nu(\mu(j))}) - f_{\nu(\mu(j))}(x_{\nu(\mu(j))})| - |f(x) - f(x_{\nu(\mu(j))})| > \alpha - |f(x) - f(x_{\nu(\mu(j))})|, \end{aligned}$$

and by continuity of f the right hand side tends to  $\alpha - 0 = \alpha > 0$ , a contradiction.

Sufficiency Assume uniform convergence on compact of  $(f_n)_n$ , and let  $(x_n)_n$  be a sequence of X converging to x in X. Then  $K = \{x\} \cup \{x_n : n \in \mathbb{N}\}$  is a compact subset of X, on which we have uniform convergence of  $(f_n)_n$  to f. Then:

$$|f(x) - f_n(x_n)| = |f(x) - f(x_n) + f(x_n) - f_n(x_n)| \le |f(x) - f(x_n)| + |f(x_n) - f_n(x_n)| \le |f(x) - f(x_n)| + ||f - f_n||_K,$$

and the lemma is proved.

Proof. (of the continuity of composition) Assume  $(z_n)_n$  is a sequence in D converging to  $z \in D$ . Then  $f_n(z_n)$  converges to f(z), since uniform convergence on compacta implies continuous convergence; for the same reason  $g_n(f_n(z_n))$  converges to g(f(z)). This implies that  $g_n \circ f_n$  converges continuously to  $g \circ f$ ; and since continuous convergence implies uniform convergence on compacta, the proof is concluded.

2.9.5. *Composition lemma*. A particular case is obtained by keeping fixed the function which operates second (continuity of the post–composition map):

# . COMPOSITION LEMMA Let E, D be open sets of $\mathbb{C}$ . Given $\varphi \in C(E, \mathbb{C})$ the mapping $\varphi_* : C(D, E) \to C(D, \mathbb{C})$ given by $\varphi_*(f) = \varphi \circ f$ ,

is continuous in the compact-open topologies.

2.9.6. Composition in the other direction. We also have continuity of the pre-composition map  $g \mapsto g \circ \varphi$ , which is an algebra homomorphism, in particular a linear map. Given open subsets D and E of  $\mathbb{C}$  and a continuous map  $\varphi : D \to E$  we have the map  $\varphi^* : C(E) \to C(D)$  given by  $\varphi^*(g) = g \circ \varphi$ , which clearly is a homomorphism of complex algebras. Continuity (with respect to the compact-open topologies) is in this case immediate from the fact that  $\|\varphi^*(g)\|_K = \|g\|_{\varphi(K)}$ .

#### 3. Infinite products

When D is a region, the identity theorem shows that the ring  $\mathcal{O}(D)$  is an integral domain. However the ring does not satisfy any finiteness condition; e.g., in the ring  $\mathcal{O}(\mathbb{C})$  of entire functions the function  $\sin(\pi z)$  has infinitely many non-trivial divisors; the functions  $f_n(z) = \sin(\pi z)/p_n(z)$ , where  $p_n(z) =$  $z \prod_{k=1}^{n} (z^2 - k^2)$  give a non stationary ascending chain of principal ideals:

$$(f_0) \subsetneq (f_1) \subsetneq (f_2) \subsetneq \ldots,$$

showing that the ring is non noetherian, and that is not a UFD. However, functions may be written as infinite products; e.g we shall show that we have:

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{n^2} \right), \qquad z \in \mathbb{C},$$

where the right-hand side has to be carefully explained.

## 3.1. Convergence of infinite products.

3.1.1. Heuristic foreword. The concept we want is of course the multiplicative analog of a convergent infinite sum, i.e. of a convergent series. Tentatively, we might give the following definition: given a sequence  $(a_n)_{n\in\mathbb{N}}$  of complex numbers, we form the sequence  $p_0 = a_0, p_1 = a_0 a_1, \ldots, p_m = a_0 \cdots a_m$ of partial products, and say that the product converges (to p) if this sequence converges to p. Some complication is however introduced by the presence of 0. We want to be able to say that a converging infinite product has value 0 if and only if some  $a_n$  is 0. Now if some  $a_n$  is 0 then clearly  $p_m = 0$  for  $n \geq m$ , regardless of the other members of the sequence  $a_n$ , so that two sequences might be eventually equal, but have sequences of partial products with different asymptotic behaviour, or conversely have the same sequence of partial products, being however wildly different in asymptotic behaviour. Moreover the sequence 1/2, 1/2, 1/2, ... has the sequence  $p_m = (1/2)^{m+1}$  as sequence of partial products, so that very easily a sequence of non-zero numbers would have, with the above definition, a zero product. Clearly the role of 0 has to be peculiar when speaking of products; 0 is not in the multiplicative group  $\mathbb{C}^{\times}$  of  $\mathbb{C}$ , where products of non-zero numbers are. Since spontaneously we also get sequences indexed not only by N, but by sets like  $\{n \in \mathbb{Z} : n \geq \nu\}$ , for some  $\nu \in \mathbb{Z}$ , we consider index sets like these.

3.1.2. Definition of convergence.

DEFINITION. Let  $(a_n)_{n\geq\nu}$  be a sequence of complex numbers. We say that the infinite product  $\prod_{n=\nu}^{\infty} a_n$  is convergent if there exists m such that the sequence of partial products

$$p_{m,n} = a_m \cdots a_n = \prod_{j=m}^n a_j \qquad n \ge m$$

converges to a non-zero number  $a \in \mathbb{C}^{\times}$  as n tends to infinity.

The value of the infinite product is then, by definition,  $p = \prod_{n=\nu}^{\infty} a_n = \prod_{i=\nu}^{m-1} a_i \cdot a_i$ .

This definition implies of course immediately that a convergent infinite product has value 0 if and only if at least one of its members is 0, that  $a_n \neq 0$  but for a finite set of indices, and that eventually equal sequences have the same convergence behaviour also as infinite products, as we have the right to expect. Notice also

**PROPOSITION.** If the infinite product  $\prod_{n=\nu}^{\infty} a_n$  is convergent, then  $\lim_{n\to\infty} a_n = 1$ .

*Proof.* Simply note that  $a_n = p_{m,n}/p_{m,n-1}$  for n > m, and that both  $p_{m,n}$  and  $p_{m,n-1}$  converge to the same non-zero complex number a as n tends to infinity. 

3.1.3. Logarithms of infinite products. Let us prove

**PROPOSITION.** Let  $(a_n)_{n>\nu}$  be a sequence of complex numbers. Then the following are equivalent:

- (i) The infinite product ∏<sup>∞</sup><sub>n=ν</sub> a<sub>n</sub> is convergent.
  (ii) (CAUCHY'S CONDITION) For every neighborhood V of 1 in C there exists an index n<sub>V</sub> such that for  $n_V \leq m \leq n$  then  $p_{m,n} \in V$  (all the partial products are in V, for an index large enough).
- (iii) There is an index  $\bar{m}$  such that for  $n \geq \bar{m}$  the logarithm  $\log a_n$  is defined and moreover the series

$$\log a_{\bar{m}} + \log a_{\bar{m}+1} + \dots = \sum_{n=\bar{m}}^{\infty} \log a_n,$$

is convergent.

Moreover, if (iii) holds, then the value of the product is  $\left(\prod_{j=\nu}^{\bar{m}-1} a_j\right) e^s$ , where s is the sum of the series in (iii).

*Proof.* (i) implies (ii) There is an index  $\overline{m}$  such that the sequence  $p_{\overline{m},n}$  converges to a non zero  $a \in \mathbb{C}$ . By continuity of the division (the map  $(w, z) \mapsto w/z = w z^{-1}$ , from  $\mathbb{C}^{\times} \times \mathbb{C}^{\times}$  to  $\mathbb{C}^{\times}$ ) at the point (a, a), given a neighborhood V of 1 we find a neighborhood U of a in  $\mathbb{C}^{\times}$  such that  $UU^{-1} \subseteq V$ ; and since  $\lim_{n\to\infty} p_{\overline{m},n} = a$ , given U we find  $n_U = n_V$  such that if  $n \ge n_V$  then  $p_{\overline{m},n} \in U$ . If  $n_V < m \le n$  we then have

 $p_{\bar{m},n} p_{\bar{m},m-1}^{-1} \in U U^{-1} \subseteq V$ , that is,  $p_{\bar{m},n} p_{\bar{m},m-1}^{-1} = p_{m,n} \in V$ .

(ii) implies (iii). First of all, let  $T = \{z \in \mathbb{C} : \text{Re } z > 0\}$  be the right-half plane; take an index  $\overline{m}$  such that  $p_{\overline{m},n} \in T$  for  $n \ge \overline{m}$ . Given  $\varepsilon > 0$ , we can find a neighborhood V of 1 such that  $\log V \subseteq B(0, \varepsilon]$ ; we can also assume that V is contained in T. By (ii) we have  $p_{m,n} \in V$  for  $m \le n$  and m large enough; since  $V \subseteq T$ ,  $\log p_{m,n}$  is then defined, and since  $a_n = p_{n-1,n} \in T$  also  $\log a_n$  is defined; moreover

$$\log p_{m,n} = \log a_m + \log a_{m+1} + \dots + \log a_n \in B(0,\varepsilon[$$

(if all the  $a_j$ s, and their product belong to T, then the logarithm of the product is the sum of the logarithms of the  $a_j$ s). This proves that the series

$$\sum_{n=\bar{m}}^{\infty} \log a_n$$

is convergent, having a sequence of partial sums which satisfies the Cauchy condition.

(iii) implies (i). Simply apply the function exp to the partial sums of the series  $\sum_{n=m}^{\infty} \log a_n$ ; the sequence so obtained will converge to  $\exp s \neq 0$ , if  $s = \sum_{n=m}^{\infty} \log a_n$  is the sum of the series, by continuity of the exponential function.

EXAMPLE 3.1.3.1. Consider the sequence  $a_n = 1 + 1/n (= (n+1)/n)$  for  $n \ge 2$ . Then

$$p_{2,n} = \frac{2+1}{2} \cdot \frac{3+1}{3} \cdots \frac{n+1}{n} = \frac{n+1}{2}$$

since  $\lim_{n\to\infty} p_{2,n} = \infty$ , the infinite product  $\prod_{n=2}^{\infty} (1+1/n)$  diverges (to infinity).

Consider now  $a_n = 1 - 1/n (= (n-1)/n)$  for  $n \ge 2$ . Then

$$p_{2,n} = \frac{2-1}{2} \cdot \frac{3-1}{3} \cdots \frac{n-1}{n} = \frac{1}{n};$$

thus  $\lim_{n\to\infty} p_{2,n} = 0$ ; the infinite product  $\prod_{n=2}^{\infty} (1-1/n)$  diverges to 0. Notice that these products diverge, although in both cases we have  $\lim_{n\to\infty} a_n = 1$ .

Finally consider  $a_n = 1 - 1/n^2$ ,  $n \ge 2$ . By the preceding examples we immediately have

$$p_{2,n} = \frac{n+1}{2} \cdot \frac{1}{n} = \frac{1+1/n}{2};$$

then the infinite product converges and  $\prod_{n=2}^{\infty}(1-1/n^2) = 1/2$ .

3.1.4. Absolute and commutative convergence. Of course the infinite product of the absolute values of the elements of a convergent infinite product will always converge to the absolute value of the product; and convergence of  $\prod_{n\geq\nu} |a_n|$  does not imply convergence of  $\prod_{n\geq\nu} a_n$ . The notion of absolute convergence for a product has to be given through the corresponding infinite series of logarithms.

REMARK. What we are exploring is convergence in the multiplicative group  $\mathbb{C}^{\times}$  of nonzero complex numbers. The metric |w - z| that this groups inherits from  $\mathbb{C}$  is not suitable for it; e.g. with this metric  $\mathbb{C}^{\times}$  is non complete, and the metric is not adapted to the group structure, that is, it is not invariant under the "translations" of the group, the mappings  $z \mapsto az$  (with  $a \in \mathbb{C}^{\times}$  fixed). A good metric for  $\mathbb{C}^{\times}$  is defined as follows: let  $\rho(z) = |\log z|$  for  $z \in \mathbb{C}^{\times}$ ; prove that  $\rho$  is continuous on  $\mathbb{C}^{\times}$ , and that  $\rho(wz) \leq \rho(w) + \rho(z)$  for every  $w, z \in \mathbb{C}^{\times}$ ; observe that  $\rho(z^{-1}) = \rho(z)$  and that  $\rho(z) = 0$  if and only if z = 1. The metric  $d_{\rho}(w, z) = \rho(w/z) = |\log(w/z)|$  is translation invariant, topologically compatible, and makes  $\mathbb{C}^{\times}$  complete. This remark should motivate the subsequent definition.

DEFINITION. Let  $(a_n)_{n \ge \nu}$  be a sequence of complex numbers. We say that the infinite product  $\prod_{n=\nu}^{\infty} a_n$  is absolutely convergent if there exists  $m \in \mathbb{N}$  such that  $\operatorname{Re} a_n > 0$  for  $n \ge m$ , and the series  $\sum_{n=m}^{\infty} \log a_n$  is absolutely convergent.

It is customary to write the infinite product  $\prod_{n \ge \nu} a_n$  in the form  $\prod_{n \ge \nu} (1+b_n)$ , with  $b_n = (a_n - 1)$ . For the next proposition we need the following

LEMMA. If  $|z| \leq 1/2$ , then

$$\frac{|z|}{2} \le |\log(1+z)| \le \frac{3}{2} |z|.$$

*Proof.* Put  $u(z) = z - \log(1 + z)$ . Then, if |z| < 1:

$$u(z) = \int_{[0,z]} u'(\zeta) \, d\zeta = \int_{[0,z]} \frac{\zeta}{1+\zeta} \, d\zeta;$$

so that if  $|z| \leq 1/2$  we have (use the parametrization  $\zeta = t z, t \in [0, 1]$  for the last equality):

$$|u(z)| = \left| \int_{[0,z]} \frac{\zeta}{1+\zeta} \, d\zeta \right| \le \int_{[0,z]} \frac{|\zeta|}{|1+\zeta|} \, |d\zeta| \le \int_{[0,z]} \frac{|\zeta|}{1/2} \, |d\zeta| = 2|z|^2 \int_0^1 t \, dt = |z|^2.$$

Thus  $|u(z)| \le |z|^2 \le (1/2)|z|$  if  $|z| \le 1/2$ ; then  $||z| - |\log(1+z)|| \le |z - \log(1+z)| \le |z|/2$ , which gives easily the statement.  $\square$ 

We recall also that a *rearrangement* of a sequence  $(b_n)_{n \geq \nu}$  is a sequence  $(b_{\sigma(n)})_n \geq \nu$ , where  $\sigma$  is a *permutation*, that is, a self-bijection, of the set  $\{\nu, \nu + 1, \nu + 2, ...\}$  of indices.

**PROPOSITION.** Any absolutely convergent product  $\prod_{n>\nu} a_n = \prod_{n>\nu} (1+b_n)$  is convergent. Moreover, the following are equivalent:

- (i)  $\prod_{n>\nu} (1+b_n)$  is absolutely convergent.
- (i)  $\prod_{n \ge \nu} (1 + \sigma_n)^{\infty}$  is absolutely convergent. (ii) The series  $\sum_{n \ge \nu}^{\infty} b_n$  is absolutely convergent. (iii) Every rearrangement  $\prod_{n \ge \nu} (1 + b_{\sigma(n)})$  is convergent.

And all the rearrangements converge to the same product.

*Proof.* The first assertion is an obvious consequence of the proposition in the preceding section, since absolute convergence of a series implies convergence.

(i) is equivalent to (ii): use the lemma. (iii) is equivalent to (i): recall that a series is commutatively convergent iff it is absolutely convergent; and that all rearrangements of an absolutely convergent series converge to the same sum.

EXERCISE 3.1.4.1. Let  $(b_n)_{n\geq\nu}$  be a sequence of positive real numbers, converging to 0. Prove that the infinite products

$$\prod_{n=\nu}^{\infty} (1+b_n) \qquad \prod_{n=\nu}^{\infty} (1-b_n)$$

converge if and only if the series  $\sum_{n=\nu}^{\infty} b_n$  is convergent.

Solution. By the hypothesis  $\lim_{n\to\infty} b_n = 0$ , we can pick m so that for  $n \ge m$  we have  $0 \le b_n \le 1/2$ . We know that the infinite products are convergent if and only if the series

$$\sum_{n \ge m} \log(1 + b_n), \quad \text{respectively} \quad \sum_{n \ge m} \log(1 - b_n)$$

are convergent (see proposition 3.1.3). These series have terms of constant sign, all positive the first, all negative the second, so that their convergence is equivalent to absolute convergence, which is equivalent, by the preceding theorem, to convergence of the series  $\sum_{n>m} b_n$ .  $\square$ 

**3.2.** Convergence of infinite products of holomorphic functions. Since we shall use almost only the notion of normally convergent product we define it first; the general notion of compact convergence for an infinite product will be relegated to an exercise (3.2.5).
3.2.1. Normally convergent infinite products. The infinite product  $\prod_{n\geq\nu} f_n$  will be frequently written  $\prod_{n>\nu} (1+g_n)$ , with  $g_n = f_n - 1$ .

DEFINITION. Let D be an open subset of  $\mathbb{C}$ , and let  $(f_n)_{n \geq \nu}$  be a sequence of functions holomorphic on D. We say that the infinite product  $\prod_{n \geq \nu} f_n = \prod_{n \geq \nu} (1 + g_n)$  converges normally on D if the series  $\sum_{n \geq \nu} g_n = \sum_{n \geq \nu} (f_n - 1)$  is normally convergent on D.

Then we have:

THEOREM. Let D be a region of  $\mathbb{C}$ , and let  $(f_n)_{n\geq\nu}$  be a sequence of functions holomorphic on D, none of which is identically zero. Assume that the infinite product  $\prod_{n\geq\nu} f_n = \prod_{n\geq\nu} (1+g_n)$  is normally convergent on D. Then there exists a holomorphic function  $f: D \to \mathbb{C}$ , not identically zero, such that the sequence  $p_m = \prod_{n=\nu}^m f_n$  converges to f, uniformly on compact subsets of D. Moreover:

- (i) For every compact subset K of D there exists  $m_K$  such that if  $n \ge m_K$  then  $f_n$  has no zero on K, and the sequence  $p_{m_K,m} = \prod_{n=m_K}^m f_n$  converges uniformly on K to a function nowhere zero on K.
- (ii)  $c \in D$  is a zero of f if and only if c is a zero of some function  $f_n$ , and we have  $\operatorname{ord}(f, c) = \sum_{n \geq \nu} \operatorname{ord}(f_n, c);$
- (iii) The product is commutatively convergent: every rearrangement compactly converges to the same function.
- (iv) LOGARITHMIC DIFFERENTIATION The series  $\sum_{n \ge \nu} \frac{f'_n}{f_n}$  of the logarithmic derivatives of the functions  $f_n$  converges normally on  $D \smallsetminus Z_D(f)$  to the logarithmic derivative f'/f of the infinite product:

$$\frac{f'}{f} = \sum_{n \ge \nu} \frac{f'_n}{f_n} \quad \text{with normal convergence in } D \smallsetminus Z_D(f).$$

*Proof.* Given a compact subset K of D there exists an index  $m_K$  such that for  $n \ge m_K$  the functions  $f_n$  have no zero on K. In fact, since the series  $\sum_{n\ge\nu} \|g_n\|_K$  is convergent, there exists  $m_K$  such that for  $n\ge m_K$  we have  $\|g_n\|_K \le 1/2$ ; then  $\operatorname{Re} f_n(z) \ge 1/2$  for  $z \in K$ , in particular  $f_n$  is non-zero; and the series

$$\sum_{n \ge m_K} \log(1 + g_n)$$

is normally convergent on K, to a function  $s_K$ , since by the lemma 3.1.4 we have

$$\|\log(1+g_n)\|_K \le (3/2)\|g_n\|_K.$$

By the composition lemma 2.9.5 we have that the sequence  $\exp\left(\sum_{n=m_K}^m \log(1+g_n)\right) = \prod_{n=m_K}^m (1+g_n) = \prod_{n=m_K}^m f_n$  converges uniformly on K to the function  $\exp s_K$ , never zero on K. So (i) has been proved; and (ii) is now easy: on K we have

$$f = \left(\prod_{n=\nu}^{m_K-1} f_n\right) \exp(s_K);$$

if f(c) = 0 we take K = B(c, r], with  $B(c, r] \subseteq D$ . Then  $\exp(s_K)$  is holomorphic on B(c, r] and then

$$\operatorname{ord}(f,c) = \operatorname{ord}\left(\prod_{n=\nu}^{m_K-1} f_n, c\right) + \operatorname{ord}(\exp(s_K), c) = \sum_{n=\nu}^{m_K-1} \operatorname{ord}(f_n, c) + 0.$$

Statement (iii) is immediate: pointwise, the product converges absolutely, hence commutatively.

Statement (iv) is simply obtained differentiating termwise the series of the logarithms, as we now explain in detail. Given a compact subset K of D let r > 0 be such that  $L = K + rB \subseteq D$ ; there exists then  $m_L$  such that for  $n \ge m_L$  we have  $||g_n||_L \le 1/2$ , thus, by the Cauchy estimate for the derivative:

$$\left\|\frac{f'_n}{f_n}\right\|_K = \left\|\frac{g'_n}{1+g_n}\right\|_K \le \frac{1}{r} \|\log(1+g_n)\|_L \le \frac{3}{2r} \|g_n\|_L,$$

As above the series  $\sum_{n \ge m_L} \log(1 + g_n)$  converges normally on K to a function  $s_K$  continuous and holomorphic on the interior of K; in this interior we have

$$s'_K = \sum_{n \ge m_L} \frac{g'_n}{1+g_n} = \sum_{n \ge m_L} \frac{f'_n}{f_n},$$

with normal convergence of the series. Since we have

$$f = \left(\prod_{n=\nu}^{m_L-1} f_n\right) \exp(s_K) \quad \text{on} \quad K,$$

logarithmic differentiation gives, in the interior of K

$$\frac{f'}{f} = \sum_{n=\nu}^{m_L-1} \frac{f'_n}{f_n} + s'_K = \sum_{n \ge \nu} \frac{f'_n}{f_n}.$$

Statement (ii) is one of the most important facts to be remembered: the zero set of the product is the union of the zero sets of the factors, and the multiplicities are obtained by adding multiplicities: exactly as in the case of finite products!

EXAMPLE 3.2.1.1. The infinite product  $z \prod_{n=1}^{\infty} (1-z^2/n^2)$  is plainly normally convergent on all of  $\mathbb{C}$ . In fact if K is compact and  $\mu = \max\{|z| : z \in K\}$  we have  $|-z^2/n^2| \le \mu^2/n^2$ , and the series  $\sum_{n\ge 1} \mu^2/n^2$  is convergent. The product is an entire function with simple zeros at every integer, and no other zero. Euler proved that this product is  $\sin(\pi z)/\pi$ . In fact, calling the product f, by logarithmic differentiation we get

$$\frac{f'(z)}{f(z)} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{-2z/n^2}{1 - z^2/n^2} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2};$$

but in 2.8.3 we saw that the right hand–side is  $\pi \cot(\pi z) = \pi \cos(\pi z)/\sin(\pi z)$ , which is the logarithmic derivative of  $\sin(\pi z)$ . Two functions with the same logarithmic derivative differ by a multiplicative constant, so  $f(z) = k\pi \sin(\pi z)$  for every  $z \in \mathbb{C}$ . We have that  $\lim_{z\to 0} f(z)/z = \prod_{n=1}^{\infty} 1 = 1$ , while  $\lim_{z\to 0} \sin(\pi z)/z = \pi$ , so that  $k = 1/\pi^2$ , hence  $f(z) = \sin(\pi z)/\pi$ . We have the formula

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty} (1 - z^2/n^2).$$

EXAMPLE 3.2.1.2. The infinite product  $z \prod_{n=1}^{\infty} (1+z/n) e^{-z/n}$  normally converges on all of  $\mathbb{C}$ . Let us estimate  $(1+z/n)e^{-z/n} - 1$ ; put  $u(z) = (1+z)e^{-z}$ ; we have  $u'(z) = e^{-z} - (1+z)e^{-z} = -ze^{-z}$ ; since we have

$$|u'(z)| \le |z| e^{-\operatorname{Re} z} \le |z|e^{|z|}, \text{ we get } |u(z) - u(0)| \le |z|^2 e^{|z|};$$

(mean value theorem); thus on the disc rB we have, since  $|z|/n \le r/n$ :

$$\left| \left( 1 + \frac{z}{n} \right) e^{-z/n} - 1 \right| \le \frac{r^2}{n^2} e^{r/n}.$$

Notice that the factors  $e^{-z/n}$  do not introduce zeros, but force the product to converge: without them, the product  $\prod_{n\geq 2}^{\infty}(1+x/n)$  does not converge, for no x > 0 (3.1.4.1). The value H(z) of the infinite product is an entire function with simple zeros at  $0, -1, -2, -3, \ldots$ . It has half the factors of the sine function, and in fact we easily have

$$H(z) H(-z) = -z^2 \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) = -\frac{z}{\pi} \sin(\pi z).$$

The value H(1) is  $\prod_{n=1}^{\infty} (1 + 1/n) e^{-1/n}$ ; thus

$$\log H(1) = \sum_{n=1}^{\infty} \left( \log \left( 1 + \frac{1}{n} \right) - \frac{1}{n} \right) = \lim_{m \to \infty} \left( \sum_{n=1}^{m} \left( \log(n+1) - \log n - \frac{1}{n} \right) \right) = \lim_{m \to \infty} \left( \log(m+1) - \sum_{n=1}^{m} \frac{1}{n} \right) = -\gamma,$$

where  $\gamma$  is called Euler's constant; it is known that  $0 < \gamma < 1$ . Since  $H(1) = e^{-\gamma}$ , we have that

$$\Delta(z) = e^{\gamma z} H(z) = z e^{\gamma z} \prod_{n=1}^{\infty} (1 + z/n) e^{-z/n},$$

is an entire function such that  $\Delta(1) = 1$ , which has simple zeros at  $0, -1, -2, \ldots$ . This is the Weierstrass  $\Delta$ -function, whose reciprocal is the Euler's  $\Gamma$  function.

3.2.2. Unordered normally convergent products. Since a normally convergent product is commutatively convergent, its definition may be given for an unordered index set (necessarily countable). If we have a family  $(f_d)_{d\in S}$  of functions holomorphic on a region D, indexed by some countable set S, we shall say that the infinite product  $\prod_{d\in S} f_d = \prod_{d\in S} (1+g_d)$  is normally convergent on D if for every compact subset K of D the sum

$$\sum_{d \in S} \|g_d\|_K \left( := \sup\left\{ \sum_{d \in F} \|g_d\|_K : F \text{ finite subset of } S \right\} \right) \text{ is finite.}$$

The same considerations may of course be applied to normally convergent series.

. Let D be a region of  $\mathbb{C}$ , and let  $(f_d)_{d\in S}$  be a family on holomorphic functions on D such that the infinite product  $\prod_{d\in S} f_d$  is normally convergent on D to  $f \in \mathcal{O}(D)$ . Assume that no  $f_d$  is identically zero on D. Then  $\prod_{d\in S} (1/f_d)$  converges normally on  $D \setminus Z_D(f)$  to 1/f.

Proof. Exercise.

3.2.3. The Euler's product. We now consider the infinite product

$$\prod_{p \in \text{primes}} (1 - p^{-s})^{-1},$$

and prove that the product normally converges in the open half-plane  $S = \{\text{Re } s > 1\}$ . Moreover, the convergence is to the Riemann's zeta function, thus proving that  $\zeta$  has no zeros in this half-plane. First, take a compact subset  $K \subseteq S$ , and let  $a = \min\{\text{Re } s : s \in K\}$ ; we have a > 1. Let us estimate the sup-norm on K of  $(1 - p^{-s})^{-1} - 1$ :

$$\left|\frac{1}{1-p^{-s}}-1\right| = \left|\frac{p^{-s}}{1-p^{-s}}\right| \le \frac{|p^{-s}|}{1-|p^{-s}|} = \frac{p^{-\sigma}}{1-p^{-\sigma}} \le \frac{(1-2^{-\sigma})^{-1}}{p^{\sigma}} \le \frac{L}{p^a} \quad (L = (1-2^{-a})^{-1}).$$

Clearly the series  $\sum_{p \in \text{primes}} 1/p^a$  is convergent for a > 1, since  $\sum_{p \in \text{primes}} 1/p^a < \sum_{n \ge 1} 1/n^a$ . Fix now a finite set F of primes; we prove that

$$\zeta(s) = \left(\sum_{n \in F'} \frac{1}{n^s}\right) \prod_{p \in F} (1 - p^{-s})^{-1}, \qquad (\operatorname{Re} s > 1)$$

where F' is the set of positive integers whose factorization does not contain primes belonging to F. We prove the formula by induction over the cardinality of F; it is trivially true by definition of  $\zeta(s)$  when  $F = \emptyset$ . Assuming it true for a finite set F of primes, we consider  $G = F \cup \{q\}$  where q is a prime not in F; we notice that any number in F' has a unique representation as a product  $m q^k$ , where  $m \in G'$  and  $k \ge 0$  is an integer: this is a consequence of the existence of a unique factorization of a positive integer into (positive) primes. Thus we have

$$\sum_{n \in F'} \frac{1}{n^s} = \sum_{m \in G'} \left( \sum_{k \ge 0} \frac{1}{(m q^k)^s} \right) = \sum_{m \in G'} \frac{1}{m^s} \left( \sum_{k \ge 0} \frac{1}{(q^s)^k} \right) = \frac{1}{1 - q^{-s}} \sum_{m \in G'} \frac{1}{m^s},$$

which immediately yields the above formula for G. It is clear that as the set F gets larger, e.g. by taking an increasing sequence  $F_m$  of finite sets of primes whose union is the set of all primes, then the right hand side of the above formula tends to the infinite product  $\prod_{p \in \text{primes}} (1 - p^{-s})^{-1}$ ; in fact  $\sum_{n \in F'_m} 1/n^s$ tends to 1, because  $F'_m$  is a decreasing sequence and  $\bigcap_m F'_m = \{1\}$  (this last because every integer has a finite factorization into primes). Thus we get

EULER'S PRODUCT FORMULA 
$$\zeta(s) = \prod_{p \in \text{primes}} (1 - p^{-s})^{-1}$$
 with normal convergence in  $\{\text{Re } s > 1\}$ .

This formula prompted Euler to give another proof of Euclid's theorem about the existence of infinitely many primes: if the set of primes were finite then the right-hand side would have a finite limit for  $s \to 1$ , but the representation of  $\zeta$  as a series easily implies that  $\lim_{\sigma \to 1^+, \sigma \in \mathbb{R}} \zeta(\sigma) = +\infty$ . As a consequence the infinite product  $\prod_{p \in \text{primes}} (1-p^{-1})^{-1}$  is divergent, and hence the series  $\sum_{p \in \text{primes}} 1/p$  diverges to  $+\infty$ .

3.2.4. Logarithms of an infinite product. If we have on a region a finite product of zero-free functions, and each function has a logarithm, then the product also has a logarithm, namely the sum of the logarithms of the factors. Some caution must be used for an infinite product, even normally convergent: not necessarily a sum of arbitrary logarithms of the factors will converge. However, it is always possible to choose the arbitrary additive constants in  $2\pi i\mathbb{Z}$  to make the series of the logarithms converge normally to a given logarithm of the product: simply fix a point  $c \in D$ ; the series  $\sum_{n\geq\nu} \log f_n(c)$ , where log is the principal logarithm, is convergent, hence the series of the primitives of  $f'_n/f_n$  with value  $\log f_n(c)$  at c is normally convergent in D (see exercise 2.7.7.1); it clearly converges to a primitive of f'/f, by the above theorem on logarithmic differentiation and its value at c is  $\sum_{n\geq\nu} \log f_n(c)$ , which clearly is a logarithm of f(c); by adding to the logarithm of the first function a suitable multiple of  $2\pi i$  by an integer we can of course make the sum of this series equal to any prescribed logarithm of f(c). Summing up:

PROPOSITION. Let D be a region of  $\mathbb{C}$ , and let  $f = \prod_{n \geq \nu} f_n$  be a normally convergent product of zero-free functions, each admitting a logarithm on D. Then f has a logarithm on D, and for any fixed logarithm  $\operatorname{Log} f$  of f we can choose logarithms  $\operatorname{Log} f_n$  of the factors such that

$$\operatorname{Log} f = \sum_{n \ge \nu} \operatorname{Log} f_n$$
 with normal convergence on  $D$ .

3.2.5. Compactly convergent infinite products. As stated in 3.2, the notion of normally convergent product is the most important one. For completeness we say that one can prove the following:

PROPOSITION. Let D be a region of  $\mathbb{C}$ , and let  $(f_n)_{n\geq\nu}$  be a sequence of functions holomorphic on D. Assume that the sequence  $p_m = \prod_{n=\nu}^m f_n$  converges compactly on D to a function f not identically zero. Then:

- (i)  $c \in D$  is a zero of f if and only if c is a zero of some function  $f_n$ , and we have  $\operatorname{ord}(f,c) = \sum_{n \geq \nu} \operatorname{ord}(f_n,c)$ ;
- (ii) For every compact subset K of D there exists  $m_K$  such that if  $n \ge m_K$  then  $f_n$  has no zero on K, and the sequence  $p_{m_K,m} = \prod_{n=m_K}^m f_n$  converges uniformly on K to a function nowhere zero on K.
- (iii) LOGARITHMIC DIFFERENTIATION The series  $\sum_{n \ge \nu} \frac{f'_n}{f_n}$  of the logarithmic derivatives of the functions  $f_n$  converges compactly on  $D \smallsetminus Z_D(f)$  to the logarithmic derivative f'/f of the infinite product:

$$\frac{f'}{f} = \sum_{n \ge \nu} \frac{f'_n}{f_n} \quad \text{with compact convergence in } D \smallsetminus Z_D(f).$$

Proof. It is a (quite strenuous, if not overly difficult) exercise.

3.2.6. Zeroes of  $\zeta$  on Re s = 1. To complete our collection of results on the zeta function we prove that it has no zero of real part 1. We use Euler's product, which has been proved to be normally convergent to  $\zeta(s)$  for  $\sigma = \text{Re } s > 1$  (see 3.2.3). We can take a logarithm of this product (see 3.2.4), where we assume the logarithms in the series to be real for  $s = \sigma > 1$  real:

$$\log \zeta(s) = \sum_{p \in \text{primes}} \log(1 - p^{-s})^{-1} = \sum_{p \in \text{primes}} \sum_{m=1}^{\infty} \frac{1}{m p^{ms}};$$

We get

$$3\log \zeta(s) = \sum_{p,m} \frac{3}{m \, p^{m \, s}}; \quad 4\log \zeta(s) = \sum_{p,m} \frac{4}{m \, p^{m s}} \qquad \text{Re} \, s > 1;$$

we recall that, whatever the determination of the logarithm  $\log w$ , we have  $\operatorname{Re} \log w = \log |w|$ , so that, given  $s = \sigma + it$  with  $\sigma > 1$ :

$$\log |\zeta^{3}(\sigma) \zeta^{4}(\sigma + it) \zeta(\sigma + 2it)| = \sum_{p,m} \frac{1}{m p^{m\sigma}} \operatorname{Re}(3 + 4p^{-imt} + p^{-2imt});$$

now we have

$$\operatorname{Re}(3+4p^{-imt}+p^{-2imt}) = 3+4\cos(mt\log p) + \cos(2mt\log p) = 3+4\cos\vartheta + \cos(2\vartheta) = 3+4\cos\vartheta + 2\cos^2\vartheta - 1 = 2(\cos\vartheta + 1)^2 \ge 0, \quad \text{for every } \vartheta \in \mathbb{R}.$$

Exponentiating we have

$$|\zeta^3(\sigma)\,\zeta^4(\sigma+i\,t)\,\zeta(\sigma+2i\,t)| \ge 1; \qquad \sigma > 1, \,t \in \mathbb{R}$$

from this we get

$$|(\sigma-1)\zeta(\sigma)|^3 \left| \frac{\zeta(\sigma+i\,t)}{\sigma-1} \right|^4 |\zeta(\sigma+2i\,t)| \ge \frac{1}{\sigma-1} \quad \sigma > 1;$$

this gives the required contradiction: if  $\zeta(1+it) = 0$  for some t (necessarily nonzero) then the left hand side has a finite limit for  $\sigma \to 1^+$ , namely  $|\zeta'(1+it)|^4 |\zeta(1+2it)|$ , whereas the right hand side tends to  $+\infty$ . If we recall that

$$\zeta(1-s) = \gamma(s)\,\zeta(s), \qquad \text{where} \quad \gamma(s) = \pi^{1/2-s}\,\frac{\Gamma(s/2)}{\Gamma((1-s)/2)};$$

and that  $\gamma$  only zeros are at the positive odd integers, we obtain that  $\zeta$  has no purely imaginary zero: the non trivial zeros of  $\zeta$  are in the interior of the critical strip.

**3.3. The Weierstrass product theorem.** The identity theorem makes holomorphic functions extremely rigid: in general we cannot manufacture holomorphic functions to specification. Quite surprisingly we can however construct holomorphic functions with prescribed set of zeros, with prescribed multiplicities, if that set is of course locally finite in the region D.

3.3.1. Factorization in the ring of holomorphic functions. The ring of complex polynomials  $\mathbb{C}[z]$  is a factorial domain (even a principal ideal domain), and every nonzero polynomial can be written as  $p(z) = a (z - d_1) \cdots (z - d_m)$ , where  $d_1, \ldots, d_m$  are the (not necessarily distinct) zeros of the polynomial, and a is a unit of  $\mathbb{C}[z]$ , i.e., a nonzero constant. The elements z - d are exactly the irreducible elements of the ring  $\mathbb{C}[z]$ . If we consider the integral domain  $\mathcal{O}(D)$  of holomorphic functions over a region D, if a function  $f \in \mathcal{O}(D)$  has a finite set of zeros  $d_1, \ldots, d_m$ , each repeated according to multiplicity, we can easily prove that we have

$$f(z) = u(z) \prod_{n=1}^{m} (z - d_k)$$
 where u is a unit of  $\mathcal{O}(D)$ , i.e.  $u \in \mathcal{O}(D)$  has no zeros in D.

(notice that in every region D functions like  $e^h$ , with  $h \in \mathcal{O}(D)$ , are units of  $\mathcal{O}(D)$ ; and when D is simply connected, then every unit in  $\mathcal{O}(D)$  is of this form). A function with infinitely many zeros will have infinitely many irreducible factors, e.g.  $\sin(\pi z)$  has all z - n,  $n \in \mathbb{Z}$  as factors. Can we hope for a factorization in infinitely many factors? In general, of course, a product such as  $z \prod_{n=1}^{\infty} (z^2 - n^2)$  will not converge; however in this case we can write it as  $k z \prod_{n=1}^{\infty} (1 - z^2/n^2)$ , with k a suitable constant, and the product will converge to  $\sin(\pi z)$ , as we have seen. For other functions with infinitely many zeros the situation is not so simple. But: assume that the function  $f \in \mathcal{O}(D)$  has infinitely many zeros, and arrange them in a sequence  $d_1, \ldots, d_m, \ldots$ , where each zero is repeated according to multiplicity. We shall prove that there exists a sequence  $u, u_j$  of units of  $\mathcal{O}(D)$  such that

$$f(z) = u(z) \prod_{n=1}^{\infty} (z - d_n) u_n(z),$$

where the convergence of the infinite product is normal in D. And even more, given a sequence  $d_1, \ldots, d_n, \ldots$  in D, which has no cluster point in D, we can find a sequence  $u_n$  of units of  $\mathcal{O}(D)$  such that the product

$$\prod_{n=1}^{\infty} (z - d_n) u_n(z)$$

converges normally on D, necessarily to a function  $f \in \mathcal{O}(D)$  with zero-set  $Z(f) = \{d_1, d_2, ...\}$ , and each  $d \in Z(f)$  has multiplicity equal to the number of times it appear in the sequence,  $\operatorname{ord}(f, d) = \operatorname{Card}(\{n \in \mathbb{N} : d_n = d\})$ . We first have to observe the following elementary fact: divisibility in the ring  $\mathcal{O}(D)$  is completely determined by the order function, in this sense:

PROPOSITION. Let D be a region, and let f, g be holomorphic on D and not identically zero. Then f divides g in the ring  $\mathcal{O}(D)$  if and only if  $\operatorname{ord}(f, z) \leq \operatorname{ord}(g, z)$  for every  $z \in D$ ; and f, g are associate in  $\mathcal{O}(D)$  if and only if  $\operatorname{ord}(f, z) = \operatorname{ord}(g, z)$ , for every  $z \in D$ . Proof. If g = f h with  $h \in \mathcal{O}(D)$  then  $\operatorname{ord}(g, z) = \operatorname{ord}(f, z) + \operatorname{ord}(h, z)$  so that  $\operatorname{ord}(g, z) \ge \operatorname{ord}(f, z)$ . Conversely, consider h = g/f; it is a function holomorphic on  $D \smallsetminus Z_D(f)$ ; every point of  $Z_D(f)$  is an isolated singularity for h, and if  $c \in Z_D(f)$  then  $\operatorname{ord}(g, c) \ge \operatorname{ord}(f, c)$  shows that the singularity is removable. Then  $h \in \mathcal{O}(D)$ . The remaining part of the proof is left to the reader: we simply recall that f and g are said to be associate in the ring  $\mathcal{O}(D)$  if there exists a unit  $u \in \mathcal{O}(D)$  such that g = f u.  $\Box$ 

3.3.2. Weierstrass factors. For n = 1, 2, 3, ... define  $E_n(z) = (1-z) \exp(z+z^2/2+z^3/3+\cdots+z^n/n)$ , while  $E_0(z) = 1-z$ . These factors are interpolated in infinite products to ensure convergence.

LEMMA. For  $|z| \leq 1$  and every  $n \in \mathbb{N}$  we have  $|E_n(z) - 1| \leq |z|^{n+1}$ .

*Proof.* Differentiating we get

$$E'_n(z) = (-1 + (1 - z)(1 + z + \dots + z^n)) \exp(z + z^2/2 + z^3/3 + \dots + z^n/n) = -z^n \exp(z + z^2/2 + z^3/3 + \dots + z^n/n).$$

Thus

$$|E'_n(z)| = |z|^n |\exp(z + z^2/2 + z^3/3 + \dots + z^n/n)| \le |z|^n \exp(|z| + |z|^2/2 + \dots + |z|^n/n).$$

Since  $E_n(0) = 1$  we have

$$E_n(z) - 1 = E_n(z) - E_n(0) = \int_{[0,z]} E'_n(\zeta) \, d\zeta = \int_0^1 z \, E'_n(tz) \, dt,$$

so that

$$E_n(z) - 1 \le |z| \int_0^1 |E'_n(tz)| dt;$$

now, the above estimate for  $E'_n(z)$  implies:

$$|E'_n(tz)| \le t^n |z|^n \exp(t|z| + t^2 |z|^2 / 2 + \dots + t^n |z|^n / n);$$

if  $|z| \leq 1$  then we have

$$\exp(t|z| + t^2|z|^2/2 + \dots + t^n|z|^n/n) \le \exp(t + t^2/2 + \dots + t^n/n),$$

so that

$$|E'_n(tz)| \le |z|^n (t^n \exp(t + t^2/2 + \dots + t^n/n)) = |z|^n (-E'_n(t)).$$

Thus, if  $|z| \leq 1$ 

$$|E_n(z) - 1| = |E_n(z) - E_n(0)| \le |z| \int_0^1 |z|^n (-E'_n(t)) \, dt = |z|^{n+1} (E_n(0) - E_n(1)) = |z|^{n+1}.$$

#### 3.3.3. The construction for a diverging sequence.

LEMMA. Let  $d_1, d_2, \ldots$  be a sequence of non zero complex numbers, such that  $\lim_{n\to\infty} d_n = \infty$ . Let  $(k_n)_{n\geq 1}$  be a sequence of natural numbers such that  $\sum_{n=1}^{\infty} (r/|d_n|)^{k_n+1} < \infty$ , for every r > 0. Then the infinite product

$$f(z) = \prod_{n=1}^{\infty} E_{k_n}(z/d_n)$$

is normally convergent in  $\mathbb{C}$ , and defines an entire function whose zero set is exactly  $\{d_n : n \ge 1\}$ , each zero with a multiplicity equal to the number of repetitions in the sequence,  $\operatorname{ord}(f,d) = \operatorname{Card}(\{n \ge 1 : d_n = d\})$ .

*Proof.* By the lemma, the infinite product is normally convergent: since  $d_n$  diverges, given r > 0 there exists  $n_r \in \mathbb{N}$  such that  $r/|d_n| \leq 1$  for  $n \geq n_r$ , hence  $|z/d_n| \leq 1$  for  $n \geq n_r$  and  $|z| \leq r$ , and the lemma implies

$$E_{k_n}(z/d_n) - 1 \le (|z|/|d_n|)^{k_n+1} \le (r/|d_n|)^{k_n+1} \quad n \ge n_r, |z| \le r.$$

Notice also that the factor  $E_{k_n}(z/d_n)$  has a first order zero at  $d_n$ , and no zeros in  $\mathbb{C} \setminus \{d_n\}$ , so the zeros of the product are the  $d_n$ , counted as many times as they appear in the sequence.

REMARK. Notice that since the sequence diverges no term can be repeated infinitely many times. Observe also that given an arbitrary divergent sequence we can find a sequence  $k_n$  of non negative integers such that  $\sum_{n=1}^{\infty} (r/|d_n|)^{k_n+1} < \infty$ , for every r > 0: it suffices to take  $k_n = n-1$ : if  $|d_n| \ge 2r$  for  $n \ge \bar{n}$  we have  $(r/|d_n|)^{k_n+1} \le (1/2)^n$  for  $n \ge \bar{n}$ . In general of course it is better to choose  $k_n$  as small as possible, to make the Weierstrass factors as simple as possible.

. WEIERSTRASS PRODUCT THEOREM FOR  $\mathbb{C}$ . For every entire function f there exists a sequence of entire functions  $v, v_1, v_2, \ldots$  such that

$$f(z) = e^{v(z)} z^m \prod_{n=1}^{\infty} (z - d_n) e^{v_n(z)} \qquad m = \operatorname{ord}(f, 0),$$

where  $d_n$  is the sequence of all non zero-zeros of f, each repeated according to multiplicity, and the product is normally convergent in  $\mathbb{C}$ .

*Proof.* We may take, for instance,  $v_n$  in such a way that  $E_{k_n}(z/d_n) = (z - d_n) e^{v_n(z)}$ .

3.3.4. Orders. Given an open region D of  $\mathbb{C}$  an order on D is a function  $\mathfrak{d} : D \to \mathbb{Z}$  whose support  $S = \{z \in D : \mathfrak{d}(z) \neq 0\}$  is a locally finite subset of D. Every non identically zero meromorphic function  $f \in \mathcal{M}(D)$  gives an order, its order function,  $z \mapsto \operatorname{ord}(f, z)$ . Such orders are called principal orders. The Weierstrass product theorem will imply that on  $\mathbb{C}$  every order is a principal order; and we shall prove that this is true for every region in 3.3.5. Positive principal orders on D correspond then to functions holomorphic on D, without poles. We have that  $\operatorname{ord}(f, z) = 0$  identically if and only if f is a unit in  $\mathcal{O}(D)$ ; on simply connected regions we know that this is equivalent to the existence of a logarithm, i.e. units are exactly elements of the form  $\exp g(z)$ , for some  $g \in \mathcal{O}(D)$ . Every order  $\mathfrak{d}$  may be written as the difference of two positive orders,  $\mathfrak{d} = \mathfrak{d}^+ - \mathfrak{d}^-$ , in the usual way. Given a positive order  $\mathfrak{d} : D \to \mathbb{Z}$ , let  $d_1, d_2, \ldots$ , be a sequence (which terminates if the support of  $\mathfrak{d}$  is finite) in which all nonzero elements d in the support of  $\mathfrak{d}$  appear exactly  $\mathfrak{d}(d)$  times. A Weierstrass product for the order  $\mathfrak{d}$  is a product (finite or infinite)

$$f(z) = z^{\mathfrak{d}(0)} \prod_{n \ge 1} (z - d_n) u_n(z),$$

which is normally convergent on D, and where each  $u_n$  is a unit of  $\mathcal{O}(D)$ . It is immediate to check that if f is a Weierstrass product for  $\mathfrak{d}$ , then  $\operatorname{ord}(f, \cdot) = \mathfrak{d}$ .

. FACTORIZATION THEOREM FOR A REGION For every positive order on a region D there exists a Weierstrass product. For every order  $\mathfrak{d}$  on D there exist functions  $f, g \in \mathcal{O}(D)$  such that  $\operatorname{ord}(f/g, z) = \mathfrak{d}(z)$ , for every  $z \in D$ . In particular, for every  $f \in \mathcal{O}(D)$  there exists a sequence  $u, u_n$  of units of  $\mathcal{O}(D)$  such that

$$f(z) = u(z) z^m \prod_{n=1}^{\infty} (z - d_n) u_n(z) \qquad m = \operatorname{ord}(f, 0).$$

where  $d_n$  is the sequence of all non-zero zeros of f, each repeated according to multiplicity, and the product is normally convergent in D.

Proof. In 3.3.6.

Here is a reap of corollaries, whose proofs are left as exercises; D is a region:

- Every meromorphic function on D is the quotient of two functions holomorphic on D, hence  $\mathcal{M}(D)$  is the field of fractions of the integral domain  $\mathcal{O}(D)$ .
- If D is simply connected, then a function f has an m-th root in  $\mathcal{O}(D)$  if and only if  $\operatorname{ord}(f, z)$  is divisible by m, for every  $z \in D$  (here  $m \geq 2$  is an integer).
- As in every integral domain we can speak of divisibility in  $\mathcal{O}(D)$ : f divides g means that there exists  $h \in \mathcal{O}(D)$  such that g = h f. We have observed that f divides g if and only if  $\operatorname{ord}(f, z) \leq \operatorname{ord}(g, z)$  for all  $z \in D$ , and that two nonzero elements are associate if and only if they have the same order function. Recalling that a greatest common divisor (gcd) for a subset  $S \subseteq \mathcal{O}(D)$  is a divisor of every element of S, which is divided by every other common divisor of the elements of S, it is immediate to see that any function f such that

$$\operatorname{ord}(f, z) = \min\{\operatorname{ord}(g, z) : g \in S\},\$$

is a gcd for S: notice that to prove existence of f we need the product theorem, even if S has only two elements!

3.3.5. Product theorem in bounded regions. Observe that if  $d,c\in\mathbb{C}$  ,  $c\neq d,$  and  $m\geq 1$  then the function

$$E_m\left(\frac{d-c}{z-c}\right) \qquad \text{where} \quad E_m(s) = (1-s)\exp(s/1+s^2/2+\dots+s^m/m),$$

has a simple zero for z = d, and is holomorphic in  $\mathbb{C} \setminus \{c\}$ . These functions will serve as Weierstrass factors for the product theorem in arbitrary regions. Suppose that we have an infinite discrete subset S of  $\mathbb{C}$  (discrete means discrete in the induced topology, i.e., no point of S is an accumulation point of S). If S', set of accumulation point of S, is empty, then every surjective sequence  $n \mapsto d_n$  on  $\mathbb{N}$  onto S, in which no value is assumed infinitely many times (i.e., for no  $d \in S$  the set  $\{n \in \mathbb{N} : d_n = d\}$  is infinite) will have  $\infty$  as limit. Assume on the contrary that S' is non empty, and moreover that we can arrange the elements of S in a sequence  $d_n$  such that  $\lim_{n\to\infty} \operatorname{dist}(d_n, S') = 0$ . For every  $n \in \mathbb{N}$  pick a point  $c_n \in S'$  such that  $|d_n - c_n| = \operatorname{dist}(d_n, S')$  (this is possible because S' is closed in  $\mathbb{C}$ ). Let  $k_n$  be a sequence of natural numbers such that the series  $\sum_{n=1}^{\infty} (r|d_n - c_n|)^{k_n+1}$  converges for every r > 0. In these hypotheses

. The infinite product

$$\prod_{n=1}^{\infty} E_{k_n} \left( \frac{d_n - c_n}{z - c_n} \right),$$

converges normally on  $\mathbb{C} \setminus S'$  to a holomorphic function, which has zeros exactly at the points of S, with a multiplicity equal to the number of times they appear in the sequence  $d_n$ .

*Proof.* Given a compact subset K of  $\mathbb{C} \setminus S'$  we have  $\min\{\operatorname{dist}(z, S') : z \in K\} = \rho > 0$ , so that, if  $z \in K$  we have  $|z - c_n| \ge \rho$  for every n, hence

$$\left|\frac{d_n - c_n}{z - c_n}\right| \le \frac{|d_n - c_n|}{\rho};$$

since  $|d_n - c_n| = \text{dist}(d_n, S')$  tends to 0 as  $n \to \infty$ , there is  $\bar{n} \in \mathbb{N}$  such that  $\rho^{-1}|d_n - c_n| < 1$  for  $n \ge \bar{n}$ ; thus we get

$$\left| E_{k_n} \left( \frac{d_n - c_n}{z - c_n} \right) - 1 \right| \le (\rho^{-1} |d_n - c_n|)^{k_n + 1} \quad \text{for} \quad n \ge \bar{n},$$

and since the series  $\sum_{n \ge \bar{n}} (\rho^{-1} | d_n - c_n |)^{k_n + 1}$  is convergent, we conclude.

EXERCISE 3.3.5.1. Let S be discrete in  $\mathbb{C}$ , and let S' be the set of accumulation points of S in  $\mathbb{C}$ ; assume that S' is non empty. Call proper enumeration of S a map  $n \mapsto z_n$  of  $\mathbb{N}$  onto S, such that no element of S is repeated an infinite number of times. Prove that the following are equivalent:

- (i) There is a proper enumeration  $n \mapsto z_n$  of S such that  $\lim_{n\to\infty} \operatorname{dist}(z_n, S') = 0$ .
- (ii) For every  $\varepsilon > 0$  the set  $\{z \in S : \operatorname{dist}(z, S') \ge \varepsilon\}$  is finite.
- (iii) For every proper enumeration  $n \mapsto z_n$  of S we have  $\lim_{n\to\infty} \operatorname{dist}(z_n, S') = 0$ .

3.3.6. Product theorem in arbitrary regions. We prove here the factorization theorem for a region, stated in 3.3.4. Given a region D of  $\mathbb{C}$ , and an infinite subset S of D discrete in D, its derived set S' (=set of accumulation points of S), if non empty, is contained in the boundary  $\partial D$  of D. We want to be able to apply the preceding arguments for  $\mathbb C$  and for bounded domains (Weierstrass product theorem and 3.3.5). To do this we split S into two disjoint subsets A and B, where A has no accumulation point in  $\mathbb{C}$ , and B is such that B' = S', and if B' is non empty, then we can arrange B in a sequence  $d_n$  such that  $\lim_{n\to\infty} \operatorname{dist}(d_n, B') = 0$ ; either one of these sets may be empty, in which case we could apply directly one of the preceding theorems. Simply take  $A = \{z \in S : |z| \operatorname{dist}(z, S') \geq 1\}$  and  $B = \{z \in S : |z| \operatorname{dist}(z, S') < 1\}$ . Then A has no accumulation point in  $\mathbb{C}$ ; in fact, if c is such a point, and  $z_i$  a sequence in A converging to c, then  $|z_i| \operatorname{dist}(z_i, S')$  tends to  $|c| \operatorname{dist}(c, S') = |c| = 0$ , and cannot be larger than 1. Next, for every  $\varepsilon$  with  $0 < \varepsilon < 1$  the set  $B_{\varepsilon} = \{z \in B : \operatorname{dist}(z, S') \ge \varepsilon\}$  is finite: in fact it is bounded, since if  $z \in B_{\varepsilon}$  we have  $|z| \leq 1/\varepsilon$ , and if it were infinite it would have an accumulation point c, for which we have  $\operatorname{dist}(c, S') = 0$ . It is now clear, given an order function  $\mathfrak{d}: D \to \mathbb{Z}$ , how to construct a meromorphic function with that order; we can of course assume that the order is positive and split the support S of the order into two disjoint subsets A and B as above; arrange the terms of A, if A is infinite, in a sequence repeated according to multiplicity, and use 3.3.3 to get an entire function  $f_A$  with A as a zero set, with the prescribed multiplicities. For B do the same, but use 3.3.5, to get a function  $f_B$  holomorphic on  $\mathbb{C} \setminus B' \supseteq D$  with zero-set B, with the right multiplicities; finally take  $f = f_A f_B$ .

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## **3.4.** The $\Gamma$ function.

3.4.1. Weierstrass  $\Delta$  function. We consider again the infinite product:

$$\Delta(z) = e^{\gamma z} z \prod_{n=1}^{\infty} \left( 1 + \frac{z}{n} \right) e^{-z/n}, \quad \gamma := \lim_{m \to \infty} \left( \sum_{n=1}^{m} \frac{1}{n} - \log m \right);$$

which in 3.2.1.2 has been shown to be normally convergent in  $\mathbb{C}$ ; we can confirm it by observing that  $\Delta(z) = ze^{\gamma z} \prod_{n=1}^{\infty} E_1(-z/n)$ , and that the series  $\sum_{n=1}^{\infty} (r/n)^2$  is convergent for every r > 0. We note Gauss limit formula:

$$\Delta(z) = \lim_{m \to \infty} \frac{z(z+1)\cdots(z+m)}{m! \, m^z} \qquad z \in \mathbb{C};$$

In fact we can write

$$e^{\gamma z} z \prod_{n=1}^{m} \frac{n+z}{n} \exp\left(-z \sum_{n=1}^{m} \frac{1}{n}\right) = \frac{z(z+1)\cdots(z+m)}{m!} \exp\left(-z \left(\sum_{n=1}^{m} \frac{1}{n} - \gamma\right)\right);$$

we multiply and divide by  $m^z = \exp(z \log m)$  and we obtain for the *m*-th partial product:

$$\frac{z(z+1)\cdots(z+m)}{m!\,m^z}\exp\left(-z\,\left(\sum_{n=1}^m\frac{1}{n}-\log m-\gamma\right)\right);$$

for any given z the last term tends to 1, so the limit formula is proved. Notice that  $\Delta(1) = 1$ . This immediately implies the functional equation

$$z \Delta(z+1) = \Delta(z)$$
 for every  $z \in \mathbb{C}$ .

in fact, assuming  $z \neq 0$ :

$$\Delta(z+1) = \lim_{m \to \infty} \frac{(z+1)((z+1)+1)\cdots((z+1)+m)}{m!\,m^{z+1}} = \frac{1}{z} \lim_{m \to \infty} \frac{z\,(z+1)(z+2)\cdots(z+(m+1))}{(m+1)!\,(m+1)^z} \frac{(m+1)^{z+1}}{m^{z+1}} = \frac{\Delta(z)}{z}.$$

We can also observe that the factors of  $\Delta$  are roughly half the sine factors; considering the product  $\Delta(z)\Delta(-z)$  we get

$$\Delta(z)\Delta(-z) = e^{\gamma z} z \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n} e^{-\gamma z} (-z) \prod_{n=1}^{\infty} \left(1 - \frac{z}{n}\right) e^{z/n} = -z^2 \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) = -\frac{z}{\pi} \left(\pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)\right) = -\frac{z}{\pi} \sin(\pi z).$$

Since  $\Delta(-z)/(-z) = \Delta(1-z)$  we get

$$\Delta(z)\Delta(1-z) = \frac{\sin(\pi z)}{\pi}.$$

3.4.2. The multiplication formula. For future use we prove that:

. If  $m \ge 2$  is an integer then:

$$\prod_{k=1}^{m-1} \Delta(k/m) = \frac{\sqrt{m}}{(\sqrt{2\pi})^{m-1}}.$$

*Proof.* Observe that

$$\prod_{k=1}^{m-1} \Delta(k/m) = \prod_{k=1}^{m-1} \Delta(1 - k/m),$$

so that

$$\left(\prod_{k=1}^{m-1} \Delta(k/m)\right)^2 = \prod_{k=1}^{m-1} \Delta(k/m) \Delta(1-k/m) = \prod_{k=1}^{m-1} \frac{\sin(\pi k/m)}{\pi} = \frac{1}{\pi^{m-1}} \prod_{k=1}^{m-1} \sin(\pi k/m);$$

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now we have

$$\begin{split} \prod_{k=1}^{m-1} \sin(\pi k/m) &= \prod_{k=1}^{m-1} \frac{e^{i\pi k/m} - e^{-i\pi k/m}}{2i} = \frac{1}{(2i)^{m-1}} \prod_{k=1}^{m-1} e^{i\pi k/m} (1 - e^{-2i\pi k/m}) = \\ \frac{1}{(2i)^{m-1}} \prod_{k=1}^{m-1} e^{i\pi k/m} \prod_{k=1}^{m-1} (1 - e^{-2i\pi k/m}) = \frac{e^{i\pi(m-1)/2}}{(2i)^{m-1}} \prod_{k=1}^{m-1} (1 - e^{-2i\pi k/m}) = \\ \frac{1}{2^{m-1}} \prod_{k=1}^{m-1} (1 - e^{-2i\pi k/m}). \end{split}$$

Since  $e^{-2\pi i k/m}$ , for k = 1, ..., m-1 are exactly the non trivial m-th roots of 1, we have, for every indeterminate X,  $\prod_{k=1}^{m-1} (X - e^{-2i\pi k/m}) = \sum_{j=0}^{m-1} X^j$ ; thus the last product has value m. We have obtained

$$\prod_{k=1}^{m-1} \sin(\pi k/m) = \frac{m}{2^{m-1}},$$

hence

$$\left(\prod_{k=1}^{m-1} \Delta(k/m)\right)^2 = \frac{m}{(2\pi)^{m-1}},$$

and observing that  $\Delta(x) > 0$  for x > 0 we conclude.

3.4.3. Euler's gamma function. For  $s \in \mathbb{C} \setminus (-\mathbb{N})$  we define:

$$\Gamma(s) := \frac{1}{\Delta(s)} = \frac{e^{-\gamma s}}{s} \prod_{n=1}^{\infty} \frac{e^{s/n}}{1+s/n}.$$

The infinite product on the right is normally convergent in  $\mathbb{C} \setminus (-\mathbb{N})$  (see 3.2.2). The functional equation  $z\Delta(z+1) = \Delta(z)$  becomes

Functional equation for  $\Gamma$ 

which by induction gives

$$\Gamma(s+m+1) = s(s+1)\cdots(s+m)\,\Gamma(s),$$

and immediately implies

$$\operatorname{Res}(\Gamma, -m) = \frac{(-1)^m}{m!} \qquad m \in \mathbb{N}$$

Next we have:

EULER'S SUPPLEMENT

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$
  $s \in \mathbb{C} \setminus \mathbb{Z}.$ 

 $\Gamma(s+1) = s \,\Gamma(s).$ 

Moreover we have

GAUSS PRODUCT 
$$\Gamma(s) = \lim_{m \to \infty} \frac{m! \, m^{\circ}}{s(s+1) \cdots (s+m)}$$

Notice, from the very definition as a product, that  $\Gamma(\bar{s}) = \overline{\Gamma(s)}$ , hence  $\Gamma(\sigma)$  is real if  $\sigma$  is real, and from Gauss product representation we get, observing that  $|n^s| = n^{\operatorname{Re} s}$  and that  $|n+s| \ge n + \operatorname{Re} s$  for  $\operatorname{Re} s > 0$ :

$$|\Gamma(s)| \le \Gamma(\sigma)$$
 if  $\sigma = \operatorname{Re} s > 0;$ 

so that  $\Gamma$  is bounded on every vertical strip  $a \leq \operatorname{Re} s \leq b$ , if  $0 < a < b < \infty$ ; this is important in the characterization of  $\Gamma$  (Wielandt's uniqueness theorem).

3.4.4. *Integral representation*. The Gauss product representation yields easily (if a little misteriously) the integral representation. Consider in fact the sequence of functions

$$u_m(\xi) = \xi^{s-1} \left(1 - \frac{\xi}{m}\right)^m \chi_m(\xi)$$
, where  $\chi_m$  is the characteristic function of the interval  $[0, m]$ .

A classical result of analysis says that  $\left(1 - \frac{\xi}{m}\right)^m \chi_m(\xi)$  is increasing and converges on  $[0, \infty]$  to  $e^{-\xi}$ , so that if  $\sigma = \operatorname{Re} s > 0$  we can apply the dominated convergence theorem and deduce that

$$\lim_{m \to \infty} \int_{[0,\infty[} \xi^{s-1} \left(1 - \frac{\xi}{m}\right)^m \chi_m(\xi) \, d\xi = \int_0^\infty \xi^{s-1} \, e^{-\xi} \, d\xi$$

First substitute  $\xi = mt$  to get

$$\int_0^m \xi^{s-1} \left(1 - \frac{\xi}{m}\right)^m d\xi = m^s \int_0^1 t^{s-1} (1-t)^m dt,$$

and then integrate by parts m times:

$$\int_0^1 t^{s-1} (1-t)^m dt = \frac{1}{s} [t^s (1-t)^m]_0^1 + \frac{m}{s} \int_0^1 t^s (1-t)^{m-1} dt = \frac{m}{s(s+1)} [t^{s+1} (1-t)^{m-1}]_0^1 + \frac{m(m-1)}{s(s+1)} \int_0^1 t^{s+1} (1-t)^{m-2} dt.$$

This is the second iteration; after m iterations we then get

$$\int_0^1 t^{s-1} (1-t)^m \, dt = \frac{m!}{s(s+1)\cdots(s+m-1)} \int_0^1 t^{s+m-1} \, dt = \frac{m!}{s(s+1)\cdots(s+m-1)(s+m)};$$

we have obtained

$$\int_0^m \xi^{s-1} \left(1 - \frac{\xi}{m}\right)^m d\xi = \frac{m! \, m^s}{s(s+1)\cdots(s+m)}$$

exactly Gauss representation. Passing to the limit we obtain:

$$\Gamma(s) = \int_0^\infty \xi^{s-1} e^{-\xi} d\xi \qquad \operatorname{Re} s > 0.$$

3.4.5. Wielandt's uniqueness theorem. To what extent does the functional equation determine the  $\Gamma$  function? clearly any constant multiple of  $\Gamma$  verifies the same functional equation, so we have to fix a value at some point, let us say at 1. Still this does not determine the function; it is easy to see that multiplying by any function  $\varphi$  periodic of period 1 the functional equation is still verified, e.g.  $\Gamma$  and  $\cos(2\pi s)\Gamma(s)$  both verify the functional equation, and have value 1 at 1. Assuming that we have a function f holomorphic in the right half plane  $T = \{\text{Re } s > 0\}$ , which satisfies the functional equation f(s+1) = s f(s) for  $s \in T$ , it is easy to see that it can be uniquely extended to a function holomorphic in  $\mathbb{C} \setminus (-\mathbb{N})$ , using the functional equation, by the formula

$$f(s) = \frac{f(s+m)}{s(s+1)\cdots(s+m-1)} \qquad m \ge 1, \operatorname{Re} s > -m, \quad s \notin -\mathbb{N},$$

and that if f(1) = 0 then the formula actually defines f as an entire function (if  $\operatorname{Re} s > -1$ , then we define f(s) = f(s+1)/s, and if f(1) = 0 we have that 0 is a removable singularity for f, with f(0) = f'(1); by induction we get a similar extension to every negative integer). Having extended f to  $\mathbb{C} \setminus (-\mathbb{N})$  we can consider  $g : \mathbb{C} \setminus \mathbb{Z} \to \mathbb{C}$  defined by g(s) = f(s) f(1-s). We prove that g(s+1) = -g(s); hence g(s+2) = -g(s+1) = g(s), so that g is periodic of period 2, and in general  $g(s+m) = (-1)^m g(s)$  for every integer m. In fact we have

$$g(s+1) = f(s+1)f(1-(1+s)) = sf(s)f(-s) = -f(s)(-sf(-s)) = -f(s)f(1-s) = -g(s)$$

Notice also that, by linearity of the functional equation with respect to the unknown function, every linear combination of solutions of the functional equation is still a solution. Now we can prove:

. WIELANDT'S UNIQUENESS THEOREM The  $\Gamma$  function is the unique solution of the functional equation  $\Gamma(s+1) = s \Gamma(s)$  which is bounded on the strip  $S = \{1 \le \text{Re} \ s \le 2\}$ , and is such that  $\Gamma(1) = 1$ . *Proof.* If  $\Phi$  is another solution bounded in the same strip, with  $\Phi(1) = 1$ , the difference u is a solution of the functional equation with u(1) = 0, bounded in S, say  $|u(s)| \leq L$  for  $s \in S$ ; by the observation above, u extends to an entire function. Now we have that u is also bounded in the strip  $S_0 = \{0 \leq \operatorname{Re} s \leq 1\}$ : it is clearly bounded on the compact intersection of this strip with  $\{-1 \leq \operatorname{Im} s \leq 1\}$ , and if  $0 \leq \operatorname{Re} s \leq 1$ , and  $|\operatorname{Im} s| > 1$  we have

$$|u(s)| = \frac{|u(s+1)|}{|s|} \le \frac{L}{|\operatorname{Im} s|} < L$$

Consider now the function v(s) = u(s) u(1 - s); this entire function is symmetrical with respect to the point s = 1/2, and is then also bounded in  $S_0$ . But then v is bounded by the same constant on every strip  $\{m \le \text{Re} \ s \le m+1\}$ , since  $v(s+m) = (-1)^m v(s)$ , hence v is bounded in  $\mathbb{C}$ . By Liouville's theorem v is constant; since v(1) = u(1)u(0) = 0, v is identically zero, and hence u is identically zero.  $\Box$ 

EXERCISE 3.4.5.1. Given  $a \in \mathbb{C}$ , with  $a \neq 0, 1$ , consider the functional equation

$$f(s+1) = a f(s);$$

it has an obvious solution: the entire function  $s \mapsto a^s$ ; by definition,  $a^s := \exp(s \log a)$ , with  $\log a = \log |a| + i \arg a$  the principal logarithm. Prove that all holomorphic solutions of the functional equation are of the form  $a^s \varphi(s)$ , where  $\varphi$  is a holomorphic function periodic with period 1. Assuming a > 0 (hence  $a \in \mathbb{R}$ ), prove that  $k a^s$  are the only solutions bounded on vertical strips with compact base;  $k \in \mathbb{C}$  is a constant.

3.4.6. Multiplication formula. Taking reciprocals in the multiplication formula for  $\Delta$  we get

$$\prod_{k=1}^{m-1} \Gamma(k/m) = \frac{(\sqrt{2\pi})^{m-1}}{\sqrt{m}},$$

but, as was first proved by Gauss, more is true.

We seek a function  $\varphi_m$  such that:

$$\Gamma(z) = F(z) := \prod_{k=0}^{m-1} \Gamma((z+k)/m) \varphi_m(z).$$

Imposing the functional equation:

$$\begin{split} F(z+1) &= \prod_{k=0}^{m-1} \Gamma((z+1+k)/m) \,\varphi_m(z+1) = \varphi_m(z+1) \prod_{j=1}^m \Gamma((z+j)/m) = \\ &\varphi_m(z+1) \,\prod_{j=1}^{m-1} \Gamma((z+j)/m) \,\Gamma(z/m+1) = \varphi_m(z+1) \,\frac{z}{m} \prod_{j=0}^{m-1} \Gamma((z+j)/m) = \\ &z \,\frac{\varphi_m(z+1)}{m\varphi_m(z)} \,F(z), \end{split}$$

so that the functional equation is verified by F if  $\varphi_m(z) = c m^z$ , with c a suitable constant. The function

$$F(z) = c m^{z} \prod_{k=0}^{m-1} \Gamma((z+k)/m)$$

is clearly bounded in the strip  $\{1 \leq \text{Re} z \leq 2\}$  (observe that  $|m^z| = m^{\text{Re} z}$ , and that  $\Gamma$  is bounded in strips like  $\{1/m \leq \text{Re} z \leq 2\}$ . Next we have

$$F(1) = c m \prod_{k=0}^{m-1} \Gamma((1+k)/m) = \frac{c m}{\prod_{j=1}^{m-1} \Delta(j/m)} = \frac{c m}{\sqrt{m}/(\sqrt{2\pi})^{m-1}}$$

so that to have F(1) = 1 we need  $c = 1/(\sqrt{m} (\sqrt{2\pi})^{m-1})$ . Setting z = ms we conclude with:

$$\Gamma(ms) = \frac{m^{ms}}{(\sqrt{2\pi})^{m-1}\sqrt{m}} \prod_{k=0}^{m-1} \Gamma(s+k/m) \quad m \ge 2,$$

Multiplication Formula

Remark. For m = 2 we get

#### LEGENDRE'S DUPLICATION FORMULA

$$\Gamma(2s) = \frac{2^{2s-1}}{\sqrt{\pi}} \Gamma(s+1/2) \Gamma(s).$$

discovered by Legendre in 1811, a year before Gauss gave the more general multiplication formula.

3.4.7. Logarithmic derivative of  $\Gamma$ . As recalled in 3.4.3, we have  $\Gamma(s) = (e^{-\gamma s}/s) \prod_{n=1}^{\infty} e^{s/n}/(1+s/n)$  with normal convergence in  $\mathbb{C} \setminus (-\mathbb{N})$ . Hence, with normal convergence in the same region:

$$\psi(s) := \frac{\Gamma'(s)}{\Gamma(s)} = -\gamma - \frac{1}{s} + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{s+n}\right).$$

Substituting s = 1 in this formula we get  $\Gamma'(1) = -\gamma$ . Differentiating we get

$$\psi'(s) = \frac{\Gamma''(s)\Gamma(s) - (\Gamma'(s))^2}{(\Gamma(s))^2} = \frac{1}{s^2} + \sum_{n=1}^{\infty} \frac{1}{(s+n)^2},$$

which is strictly positive for  $s = \sigma > 0$ , thus showing that  $\Gamma$  is *logarithmically convex* in  $]0, +\infty[$ , i.e., the function  $\log \Gamma$  is a (strictly) convex function on  $]0, +\infty[$ . There is the following characterization of the real gamma function, due to the danish mathematicians Bohr and Möllerup:

. If  $f: [0, +\infty[\rightarrow]0, +\infty[$  is logarithmically convex (that is,  $\log f$  is a convex function), f(x+1) = x f(x) for every x > 0, and f(1) = 1, then  $f(x) = \Gamma(x)$  for every x > 0.

For a proof see [Conway], 7.13.

3.4.8. de Moivre-Stirling formula. We want an asymptotic estimate of  $\Gamma$  for values of the variable with large absolute value. The character of the function, which has poles on the negative integers, makes the estimate good only far from the negative real axis. Heuristically, we start with the integral representation and real positive values  $s = \sigma$  of the variable:

$$\Gamma(\sigma+1) = \int_0^\infty \xi^\sigma \, e^{-\xi} \, d\xi,$$

and we presume that for large values of  $\sigma$  the integral concentrates around the maximum of the integrand, which occurs for  $\xi = \sigma$ ; to better study this guess, we set this maximum at 1 by means of  $\xi = \sigma x$ , obtaining

$$\begin{split} \Gamma(\sigma+1) = & \sigma^{\sigma+1} \int_0^\infty x^\sigma e^{-\sigma x} \, dx = (x = 1 + u) = \sigma^{\sigma+1} e^{-\sigma} \int_{-1}^\infty (1 + u)^\sigma \, e^{-\sigma u} \, du \approx \\ & \sigma^{\sigma+1} e^{-\sigma} \int_{-1}^1 (1 + u)^\sigma \, e^{-\sigma u} \, du = \\ & \sigma^{\sigma+1} e^{-\sigma} \int_{-1}^1 \exp(-\sigma (u - \log(1 + u))) \, du; \end{split}$$

using Taylor approximation of  $u - \log(1 + u)$  near 0 we write  $u^2/2$  in place of  $u - \log(1 + u)$ :

$$\sigma^{\sigma+1} e^{-\sigma} \int_{-1}^{1} e^{-\sigma u^2/2} \, du \approx \sigma^{\sigma+1} e^{-\sigma} \int_{-\infty}^{\infty} e^{-\sigma u^2/2} \, du = \sigma^{\sigma+1} e^{-\sigma} \sqrt{2\pi/\sigma}.$$

We thus guess that the asymptotic formula

$$\Gamma(\sigma+1) \sim \sqrt{2\pi} \, \sigma^{\sigma+1/2} e^{-\sigma} \quad \text{equivalently} \quad \Gamma(\sigma) \sim \sqrt{2\pi} \, \sigma^{\sigma-1/2} e^{-\sigma} \quad (\sigma \to +\infty)$$

holds. Therefore we try to write

$$\Gamma(s) = \sqrt{2\pi} \, s^{s-1/2} e^{-s} \, e^{\mu(s)},$$

where  $\mu$  is holomorphic in the slit plane  $\mathbb{C}_{-}$ , and  $\mu(s)$  tends to 0 as s tends to infinity (in a sense to be made precise, remaining far from the negative real axis). We put  $F(s) = \sqrt{2\pi} s^{s-1/2} e^{-s} e^{\mu(s)}$  and we impose to it the functional equation. To determine  $\mu$ , we observe that we must have, by the functional equation:

$$\begin{split} s &= \frac{(s+1)^{s+1-1/2} \, e^{-(s+1)} \, e^{\mu(s+1)}}{s^{s-1/2} \, e^{-s} \, e^{\mu(s)}} = \\ &\exp\left((s+1/2) \log(s+1) - (s-1/2) \log s - 1 + \mu(s+1) - \mu(s)\right), \end{split}$$

which, writing s as  $e^{\log s}$  certainly holds if

$$\mu(s+1) - \mu(s) = 1 - (s+1/2)(\log(s+1) - \log s).$$

To solve this difference equation we use the following representation of the principal logarithm:

$$\log s = \int_0^\infty \left(\frac{1}{\xi+1} - \frac{1}{\xi+s}\right) d\xi, \qquad s \in \mathbb{C}_-,$$

easily obtained from  $\log s = \int_{[1,s]} d\zeta/\zeta = (s-1) \int_0^1 dt/(1+t(s-1))$  by the change of variable  $t = 1/(1+\xi)$ . We have from this formula:

$$\log(s+1) - \log s = \int_0^\infty \left(\frac{1}{\xi+s} - \frac{1}{\xi+(s+1)}\right) d\xi = \lim_{r \to +\infty} \int_0^r \left(\frac{1}{\xi+s} - \frac{1}{\xi+(s+1)}\right) d\xi;$$

but

$$\int_0^r \frac{d\xi}{\xi + (s+1)} = \int_1^{r+1} \frac{d\eta}{\eta + s},$$

so that

$$\int_0^r \left(\frac{1}{\xi+s} - \frac{1}{\xi+(s+1)}\right) d\xi = \int_0^1 \frac{d\xi}{\xi+s} - \int_r^{r+1} \frac{d\xi}{\xi-s},$$

and we finally obtain

$$\log(s+1) - \log s = \int_0^1 \frac{d\xi}{\xi+s}.$$

Substitution into the above formula for the increment of  $\mu$  gives

$$\mu(s+1) - \mu(s) = 1 - (s+1/2) \int_0^1 \frac{d\xi}{\xi+s} = \int_0^1 \left(1 - \frac{s+1/2}{\xi+s}\right) d\xi = \int_0^1 \frac{\xi - 1/2}{\xi+s} d\xi,$$

so that

$$\mu(s+1) - \mu(s) = \int_0^1 \frac{\xi - 1/2}{\xi + s} \, d\xi$$

inspired by the above formula for the increment of the logarithm we observe that the right hand side is exactly the increment of the function defined by

$$\mu(s) = \int_0^\infty \frac{p(x)}{x+s} \, dx,$$

where  $p : \mathbb{R} \to \mathbb{R}$  is the sawtooth function defined as p(x) = 1/2 - x for  $x \in ]0, 1[, p(0) = 0$ , and then extended by 1-periodicity. It is easy to see that the preceding formula actually defines a function holomorphic in the slit plane; observe in fact that p has a bounded primitive on  $\mathbb{R}$ , the function q(x) = x(x-1)/2 for  $x \in [0,1]$ , and then extended by periodicity; then an integration by parts gives

$$\mu(s) = \left[\frac{q(x)}{x+s}\right]_0^\infty + \int_0^\infty \frac{q(x)}{(x+s)^2} \, dx = \int_0^\infty \frac{q(x)}{(x+s)^2} \, dx,$$

and it is now easy to see that  $\mu$  is holomorphic in the slit plane. If  $s = \sigma + it$ , we have, assuming first t > 0:

$$\begin{aligned} |\mu(s)| &\leq \frac{1}{8} \int_0^\infty \frac{dx}{|x+s|^2} = \frac{1}{8} \int_0^\infty \frac{dx}{(x+\sigma)^2 + t^2} = \\ &\frac{1}{8t} \left[ \arctan((x+\sigma)/t) \right]_0^\infty = \frac{1}{8t} \left( \frac{\pi}{2} - \arctan(\sigma/t) \right) = \frac{1}{8t} \operatorname{arccot}(\sigma/t). \end{aligned}$$

If  $s = |s|e^{i\varphi}$ , with  $-\pi < \varphi < \pi$  we get  $\sigma = |s|\cos\varphi$  and  $t = |s|\sin\varphi$  so that  $\operatorname{arccot}(\sigma/t) = \varphi$  if  $0 < \varphi < \pi$ ; hence

$$|\mu(s)| \le \frac{1}{8|s|} \frac{\arg s}{\sin \arg s} \qquad s \notin \mathbb{R}.$$

(since  $\mu(\bar{s}) = \overline{\mu(s)}$  we get the formula also for  $\arg s < 0$ ; and if  $\arg s = 0$  it is understood that  $\arg s / \sin \arg s = 1$ ). Since  $\theta \mapsto \theta / \sin \theta$  is increasing in  $[0, \pi[$ , in the angular sector  $W_{\delta} = \{s \in \mathbb{C} : -\pi + \delta \leq \arg s \leq \pi - \delta\}$ , for  $0 < \delta < \pi$ , we get the estimate

$$|\mu(s)| \leq \frac{1}{8|s|} \frac{\pi - \delta}{\sin \delta} \quad \text{in particular in the right half-plane} \quad |\mu(s)| \leq \frac{\pi}{16|s|}.$$

We have to complete the proof: we have seen that  $F(s) = \sqrt{2\pi} s^{s-1/2} e^{-s} e^{\mu(s)}$  verifies the functional equation of the  $\Gamma$  function. Let us verify that F is bounded on the strip  $\{1 \leq \text{Re } s \leq 2\}$ . For future use we prove more, studying also the asymptoticity of  $|F(\sigma + it)|$  as  $t \to \pm\infty$ .

. For every  $\sigma \in \mathbb{R}$  we have

$$|F(\sigma + it)| \sim \sqrt{2\pi} |t|^{\sigma - 1/2} e^{-\pi |t|/2} \qquad |t| \to \infty,$$

uniformly when  $\sigma$  varies in a compact subset of  $\mathbb{R}$ .

*Proof.* Since  $F(\sigma - it) = \overline{F(\sigma + it)}$  we can consider only  $t \to +\infty$ ; now

$$|F(\sigma + it)| = \sqrt{2\pi} |e^{(s-1/2)\log s}| |e^{-s}| |e^{\mu(s)}|;$$

and

$$|e^{(s-1/2)\log s}| = \exp(\operatorname{Re}((\sigma - 1/2 + it)(\log |s| + i \arg s))) = \exp((\sigma - 1/2)\log |s| - t \arg s) = |s|^{\sigma - 1/2} e^{-t \arg s}$$
  
so that, recalling that  $\arg s = \operatorname{cotan}(\sigma/t) = \pi/2 - \arctan(\sigma/t)$ , since  $t > 0$ :

$$\begin{aligned} |F(\sigma+it)| = &\sqrt{2\pi} |s|^{\sigma-1/2} \, e^{-\sigma - t\pi/2 + t \arctan(\sigma/t)} \, e^{\operatorname{Re}(\mu(s))} = \\ &\sqrt{2\pi} \, t^{\sigma-1/2} \, e^{-t \, \pi/2} \, |s/t|^{\sigma-1/2} \exp(-t(\sigma/t - \arctan(\sigma/t)) + \operatorname{Re}(\mu(s)). \end{aligned}$$

Assuming that  $\sigma \in [-a, a]$  for some given large a > 0, we have to prove that given  $\varepsilon > 0$  there is  $t_{\varepsilon} > 0$  such that

$$\left||s/t|^{\sigma-1/2}\exp(-t(\sigma/t - \arctan(\sigma/t)) + \operatorname{Re}(\mu(s)) - 1\right| \le \varepsilon \quad \text{for} \quad t \ge t_{\varepsilon}.$$

In fact

$$|s/t|^{\sigma-1/2} = \exp((2\sigma - 1)\log(1 + (\sigma/t)^2)/4),$$

and, if a > 1:

$$|(2\sigma - 1)\log(1 + (\sigma/t)^2)/4| \le ((2a + 1)/4)(a/t)^2;$$

and since  $|x - \arctan x| \le |x|^3/3$  for  $0 \le |x| \le 1$  we also have

$$|-t(\sigma/t - \arctan(\sigma/t))| \le t |(\sigma/t)|^3/3 \le a^3/(3t^2)$$

moreover  $\mu(s)$  tends to 0 as s tends to infinity, on every vertical strip. Asymptoticity is proved.

That F is bounded in the strip  $\{1 \le \operatorname{Re} s \le 2\}$  is now clear, since by continuity F is bounded on compact rectangles  $\{1 \le \operatorname{Re} s \le 2\}, \{-b \le \operatorname{Im} s \le b\}$ , for every b > 0. It remains to prove that F(1) = 1. Assuming F(1) = a, Wielandt's theorem implies that  $F(s)/a = \Gamma(s)$ ; by Legendre's duplication formula we get, for x > 0:

$$\frac{1}{a}\sqrt{2\pi} (2x)^{2x-1/2} e^{-2x} e^{\mu(2x)} = 2\pi \frac{1}{a^2} \frac{2^{2x-1}}{\sqrt{\pi}} (x+1/2)^x e^{-(x+1/2)} e^{\mu(x+1/2)} x^{x-1/2} e^{-x} e^{\mu(x)};$$

thus

$$a \, 2^{2x} \, x^{2x-1/2} e^{-2x} \, e^{\mu(2x)} = 2^{2x} \, (x+1/2)^x \, x^{x-1/2} \, e^{-2x} \, e^{-1/2} \, e^{\mu(x+1/2)+\mu(x)};$$

simplifying

$$a e^{\mu(2x) - \mu(x+1/2) - \mu(x)} = (1/\sqrt{e}) \left(1 + \frac{1}{2x}\right)^x$$

Taking the limit of both sides as x tends to  $\infty$  we get

$$a = (1/\sqrt{e})\sqrt{e}$$
 i.e.  $a = 1$ .

Summing up:

. DE MOIVRE-STIRLING FORMULA. In the slit plane  $\mathbb{C}_{-}$  there is a holomorphic function  $\mu$  such that

$$\Gamma(s) = \sqrt{2\pi} \, s^{s-1/2} e^{-s} \, e^{\mu(s)}$$

where  $\mu$  tends to 0 as s tends to infinity in every sector which excludes the negative real axis; more precisely we have, given  $0 < \delta < \pi$ 

$$|\mu(s)| \le \left(\frac{\pi-\delta}{8\sin\delta}\right) \frac{1}{|s|}$$
 if  $-\pi+\delta \le \arg s \le \pi-\delta.$ 

In the course of the proof we have also obtained:.

COROLLARY. For every  $\sigma \in \mathbb{R}$  we have

$$|\Gamma(\sigma + it)| \sim \sqrt{2\pi} \, |t|^{\sigma - 1/2} \, e^{-\pi |t|/2} \qquad |t| \to \infty,$$

uniformly when  $\sigma$  varies in a compact subset of  $\mathbb{R}$ .

3.4.9. Hankel contour integral for the Gamma function. The integral representation of the Gamma function is valid only in the right half-plane. There is another representation with a contour integral which is valid in all of  $\mathbb{C} \setminus (-\mathbb{N})$ , found by the german mathematician Hankel (*circa* 1860). Given c = a + i b, with  $a, b \in \mathbb{R}$ , and b > 0, we call  $\gamma = \gamma_c$  the contour obtained by the juxtaposition of the following lines:

- The half line  $s_{-}(t) = \bar{c} + t = (a + t) ib, t \in ]-\infty, 0];$
- The arc of circle ρ(θ) = ε e<sup>iθ</sup>, θ ∈ [- arg c, arg c] of radius ε = |c|, counterclockwise from c̄ to c;
  The half line s<sub>+</sub>(t) = c − t = (a − t) + ib, t ∈ [0, +∞].

We can assume that  $\gamma$  is parametrized as  $t \mapsto \gamma(t)$ , with  $t \in \mathbb{R}$  and  $\lim_{t \to \pm\infty} \operatorname{Re} \gamma(t) = -\infty$ , whereas  $\lim_{t \to \pm\infty} \operatorname{Im} \gamma(t) = \pm b$ . We now consider the function  $u : \mathbb{C} \times \mathbb{C}_{-} \to \mathbb{C}$ , where  $\mathbb{C}_{-} = \mathbb{C} \setminus \mathbb{R}_{-}$  is the slit plane, given by  $u(s, z) = z^{-s}e^{z}$  (of course  $z^{-s} := \exp(-s\log z)$ , where  $\log z = \log |z| + i \arg z$  is the principal branch of the logarithm). If  $s = \sigma + it$  and z = x + iy, with  $z \notin \mathbb{R}_{-}$ , we have  $|z^{-s}| = |\exp(-s\log z)| = \exp(-\sigma\log |z| + t\arg z) = |z|^{-\sigma}e^{t\arg z} \leq |z|^{-\sigma}e^{\pi|t|}$ ; Then we have

$$|z^{-s}e^{z}| \le |z|^{-\sigma}e^{\pi|t|} e^{x},$$

and if  $z \in [\gamma]$  we have:

$$|u(s,z)| < |z|^{-\sigma} e^{\pi|t|} e^x.$$

Assume now that s is contained in a vertical strip with compact base,  $S_k = \{-k \le \text{Re} \ s \le k\}$ . If  $z \in [\gamma]$ 

$$|z|^{-\sigma} = (x^2 + y^2)^{-\sigma/2} \le (1 + x^2 + |c|^2)^{k/2};$$

since on  $[\gamma]$  we have  $x = \operatorname{Re} z \leq |c|$  we obtain that there is a constant  $L_k > 0$  such that

(\*) 
$$|u(s,z)| \le L_k e^{\pi |t|} e^{x/2}$$
 for every  $s \in S_k, z \in [\gamma]$ 

(simply observe that  $\lim_{x\to-\infty} (1+x^2+|c|^2)^{k/2} e^{x/2} = 0$ , and take  $L_k = \max\{(1+x^2+|c|^2)^{k/2} e^{x/2} : x \le |c|\}$ ). If s varies in a given compact subset K of  $\mathbb{C}$ , there is k > 0 such that  $K \subseteq kB$ , hence

$$|u(s,z)| \le L_k e^{\pi k} e^{x/2}$$
 for every  $s \in K, z = x + i y \in [\gamma]$ 

so that the function

$$h(s) := \frac{1}{2\pi i} \int_{\gamma} u(s, z) \, dz = \frac{1}{2\pi i} \int_{\gamma} z^{-s} \, e^z \, dz$$

is an entire function. Observe also that if  $s \in S_k$  we have, from inequality (\*):

(\*\*) 
$$|h(s)| \le \frac{L_k}{2\pi} e^{\pi|t|} \int_{\gamma} e^{-x/2} |dz| \le M_k e^{\pi|t|}.$$

Notice that the integrand  $z \mapsto z^{-m} e^z$  is holomorphic on the punctured plane  $\mathbb{C} \setminus \{0\}$  when s = m is an integer. We now prove that h(1) = 1, and that h(-n) = 0, for every  $n \in \mathbb{N}$ . Given r > |a| we consider the loop  $\alpha_r$  given by the segment from -r - ib to  $\bar{c} = a - ib$ , then the arc of circle  $\sigma$  from  $\bar{c}$  to c, then the segment from c to -r + ib and then the segment from -r - ib to -r - ib. If  $\beta_r$  is the contour given by the horizontal half–line from  $-\infty$  to -r - ib, followed by the segment from -r - ib to -r + ib, next by the half–line from -r + ib to  $-\infty$  we have

$$h(m) = \frac{1}{2\pi i} \int_{\alpha_r} z^{-m} e^z \, dz + \frac{1}{2\pi i} \int_{\beta_r} z^{-m} e^z \, dz \quad \text{for every} \quad r > |a|.$$

It is easy to see that for every given integer m we have

$$\lim_{r \to +\infty} \int_{\beta_r} z^{-m} e^z \, dz = 0$$

(the integral is dominated by  $2L_k e^{\pi k} \int_{-\infty}^r e^{x/2} dx + 2bL_k e^{\pi k} e^{-r/2}$ , if  $|m| \leq k$ ), so that

$$h(m) = \frac{1}{2\pi i} \lim_{r \to +\infty} \int_{\alpha_r} z^{-m} e^z dz.$$

If  $s = m \in \mathbb{Z}$ ,  $z \mapsto z^{-m} e^z$  is an entire function if  $-m \ge 0$ , so that the integral on  $\alpha_r$  is always 0; hence h(m) = 0 if  $m \le 0$  is an integer; otherwise 0 is a pole of order m, with residue 1/(m-1)!; thus h(m) = 1/(m-1)! if  $m \ge 1$  is an integer, in particular, h(1) = 1.

We now get the functional equation h(s) = s h(s+1). Integrate by parts:

$$\int_{\gamma} z^{-s} e^{z} dz = \left[ z^{-s} e^{z} \right]_{z=\gamma(-\infty)}^{z=\gamma(+\infty)} + s \int_{\gamma} z^{-s-1} e^{z} dz = 0 + s h(s+1)$$

(notice in fact that  $|\gamma(t)^{-s} e^{\gamma(t)}| \leq L_k e^{\pi b} e^{\operatorname{Re} \gamma(t)/2}$  tends to 0 as  $\operatorname{Re} \gamma(t)$  tends to  $-\infty$ ). Next we consider:

$$F(s) = \frac{1}{2i\,\sin(\pi s)} \int_{\gamma} z^{s-1} \, e^z \, dz = \pi \, \frac{h(1-s)}{\sin(\pi s)}.$$

The functional equation for h gives:

$$F(1+s) = \pi \frac{h(-s)}{\sin(\pi(s+1))} = \pi \frac{(-s)h(-s+1)}{-\sin(\pi s)} = s \pi \frac{h(1-s)}{\sin(\pi s)} = s F(s).$$

Notice that h(1-s) has zeros at every strictly positive integers; these are then removable singularities for F; and F has first order poles exactly at the points of  $-\mathbb{N}$ , where h(1-s) is non zero, as observed (h(n) = 1/(n-1)! for  $n \ge 1$  an integer). In particular, F is holomorphic on the right half-plane {Re s > 0}. Observe that F is bounded on the strip  $\{1 \le \text{Re } s \le 2\}$ ; in fact from (\*\*) we have  $|h(1-s)| \le M_2 e^{\pi|t|}$ is  $s = \sigma + it$  is in this strip; moreover

$$|\sin(\pi s)| = \frac{|e^{i\pi s} - e^{-i\pi s}|}{2} \ge \frac{||e^{i\pi s}| - |e^{-i\pi s}||}{2} = \frac{|e^{-\pi t} - e^{\pi t}|}{2} = \sinh(\pi |t|).$$

Hence, assuming  $t \neq 0$ :

$$|F(s)| \le \pi \, \frac{M_2 \, e^{\pi |t|}}{\sinh(\pi |t|)} = 2\pi \frac{M_2}{1 - e^{-2\pi |t|}} \qquad 1 \le \operatorname{Re} s \le 2, \, t = \operatorname{Im} s;$$

continuity of F on the strip shows boundedness also near t = 0. The uniqueness theorem of Wielandt then shows that for every  $s \in \mathbb{C} \setminus (-\mathbb{N})$  we have  $F(s) = d\Gamma(s)$ , with d = F(1). Write Euler's complement formula:

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$
 whence  $\Gamma(s) d\Gamma(1-s) = \frac{\pi d}{\sin(\pi s)};$ 

then

$$\Gamma(s) F(1-s) = \frac{\pi d}{\sin(\pi s)},$$
 that is  $\Gamma(s) \pi \frac{h(s)}{\sin(\pi s)} = \frac{\pi d}{\sin(\pi s)}$ 

which simplifies to  $\Gamma(s) h(s) = d$ ; for s = 1 we get h(1) = d; since h(1) = 1, we conclude that  $h(s) = 1/\Gamma(s) = \Delta(s)$ , and that  $F(s) = \Gamma(s)$ .

We observe that there is considerable latitude for the choice of the contour: since the integrand is holomorphic in the slit plane, we only have to take a contour  $\gamma : \mathbb{R} \to \mathbb{C}_-$  which turns around the origin and such that  $\lim_{t\to\pm\infty} \operatorname{Re}(\gamma(t)) = -\infty$ , while  $\operatorname{Im}(\gamma(t))$  remains bounded. In particular, we can take both half-lines on the negative real axis, taking care of considering the two values of  $z^{-s}$  which come from taking limits on the upper or on the lower half-plane. To be more precise: assume that  $\operatorname{Re} c = a < 0$ , and consider the contour given by the half-line  $\lambda_-(t) = a + t$  with  $t \in ]-\infty, 0]$ , the circle  $\rho(t) = |a| e^{i\theta}$ ,  $\theta \in [-\pi, \pi]$ , and the half-line  $\lambda_+(t) = a - t$ ,  $t \in [0, +\infty[$ . It is not difficult to see that one has

$$2\pi ih(s) = \int_{-\infty}^{0} \exp(-s(\log(-(a+t)) - i\pi))e^{a+t} dt + \int_{0}^{\infty} \exp(-s(\log(t-a) + i\pi))e^{a-t} (-dt) + \int_{\rho} z^{-s}e^{z} dz$$

Put  $a + t = \xi$  in the first of these integrals, and  $a - t = \xi$  in the second, obtaining

$$2\pi ih(s) = e^{i\pi s} \int_{-\infty}^{a} |\xi|^{-s} e^{\xi} d\xi - e^{-i\pi s} \int_{-\infty}^{a} |\xi|^{-s} e^{\xi} d\xi + \int_{\rho} z^{-s} e^{z} dz$$

that is, putting also  $a = -\varepsilon$ , and  $\rho_{\varepsilon}$  in place of  $\rho$ :

$$h(s) = \frac{\sin(\pi s)}{\pi} \int_{-\infty}^{-\varepsilon} |\xi|^{-s} e^{\xi} d\xi + \frac{1}{2\pi i} \int_{\rho_{\varepsilon}} z^{-s} e^{z} dz \quad \text{for every} \quad s \in \mathbb{C}, \ \varepsilon > 0.$$

Recalling that  $\Gamma(s) = \pi h(1-s)/\sin(\pi s)$  we get also

$$\Gamma(s) = \int_{-\infty}^{-\varepsilon} |\xi|^{s-1} e^{\xi} d\xi + \frac{1}{2i\sin(\pi s)} \int_{\rho_{\varepsilon}} z^{s-1} e^{z} dz \quad \text{for every} \quad s \in \mathbb{C} \smallsetminus (-\mathbb{N}), \, \varepsilon > 0.$$

Now the first term will not have a finite limit for  $\varepsilon \to 0^+$  unless Re s > 0. If this happens also the integral over the circle tends to 0, and we obtain

$$\Gamma(s) = \int_{-\infty}^{0} |\xi|^{s-1} e^{\xi} d\xi = \int_{0}^{\infty} t^{s-1} e^{-t} dt \qquad \text{Re}\, s > 0.$$

**3.5. Jensen's formula.** The Weierstrass product theorems say that we can give arbitrarily the zeros for a holomorphic functions on a region D, provided that the obvious necessary condition, that they accumulate only outside of D, holds. If we impose some other condition to the holomorphic function, e.g. boundedness, or some growth condition, the situation is quite different. A quantitative condition which gives some information is Jensen's formula, which we now derive. Recall that we proved that there is a Möbius involution of the unit disk  $\Delta$  which exchanges 0 and d, with 0 < |d| < 1, given by  $g_d(z) = (z - d)/(\overline{d}z - 1)$  (see 1.1.6). By conjugation with  $z \to r z$  we get:

. For every r > 0 and every  $d \in r\Delta \setminus \{0\}$  there exists a unique Möbius involution of the disk  $r\Delta$ , which preserves the boundary of  $r\Delta$  and exchanges 0 and d. This involution is given by

$$g_{d,r} = r^2 \, \frac{z-d}{\bar{d} \, z - r^2}$$

What we are asserting is then:  $g_{d,r}(r\Delta) = r\Delta$ ;  $g_{d,r}^{-1} = g_{d,r}$ , and  $|g_{d,r}(z)| = r$  if |z| = r. We need the simple:

LEMMA. (i) If g is holomorphic and never zero on an open set containing the closed disk rB, we have

(*ii*)  
$$\log |g(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |g(r e^{i\vartheta})| \, d\vartheta.$$
$$\int_0^{2\pi} \log |1 - e^{i\vartheta}| \, d\vartheta = 0.$$

*Proof.* (i) g is nonzero on an open disk  $(r + \varepsilon)\Delta$ , for some  $\varepsilon > 0$ . Then g has a logarithm  $\log g$  on this convex set. Cauchy formula says that  $\log g(0)$  is the average of the values of  $\log g$  on the circle  $\partial(rB)$ ; the real part of  $\log g(z)$  is  $\log |g(z)|$ , so that the above formula is simply Cauchy formula for the real part of  $\log g$ , at the origin 0.

(ii) Since 
$$|1 - e^{i\vartheta}| = \sqrt{(1 - \cos\vartheta)^2 + \sin^2\vartheta} = \sqrt{2 - 2\cos\vartheta} = 2|\sin(\vartheta/2)|$$
, we have  

$$\int_0^{2\pi} \log|1 - e^{i\vartheta}| \,d\vartheta = 2\int_0^\pi \log(2\sin(\vartheta/2)) \,d\vartheta = 2\pi\log 2 + 2\int_0^\pi \log\sin(\vartheta/2) \,d\vartheta;$$

Now:

$$I = \int_0^{\pi} \log \sin(\vartheta/2) \, d\vartheta = \int_0^{\pi} \log \sin((\pi - t)/2) \, dt = \int_0^{\pi} \log \cos(t/2) \, dt,$$

so that

$$2I = \int_0^\pi \log(\sin(t/2)\cos(t/2)) \, dt = \int_0^\pi \log\sin t \, dt - \pi \log 2,$$

and

 $\int_0^\pi \log \sin t \, dt = \int_0^{2\pi} \log \sin(\vartheta/2) \, \frac{d\vartheta}{2} = \int_0^\pi \log \sin(\vartheta/2) \, d\vartheta = I,$ r log 2.

so that  $I = -\pi \log 2$ .

(JENSEN'S FORMULA). Let f be holomorphic on an open set containing the closed disk rB; assume that  $f(0) \neq 0$ . Let  $d_1, \ldots, d_n$  be the zeros of f in rB, repeated according to multiplicity. Then we have

$$|f(0)| \prod_{j=1}^{n} \frac{r}{|d_j|} = \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \log|f(r e^{i\vartheta})| \, d\vartheta\right).$$

*Proof.* Let  $d_1, \ldots, d_m$  be the zeros of f in  $r\Delta$ , and  $c_1, \ldots, c_p$  be the zeros on  $\partial rB$ , all repeated according to multiplicity. Consider the function

$$h(z) = \prod_{j=1}^{m} r \frac{z - d_j}{\bar{d}_j z - r^2} \prod_{k=1}^{p} \frac{c_k - z}{c_k};$$

notice that each of the factors in the first product has absolute value 1 if  $z \in \partial rB$ . If we define

$$g(z) = \frac{f(z)}{h(z)} = f(z) \prod_{j=1}^{m} \frac{\bar{d}_j z - r^2}{r(z - d_j)} \prod_{k=1}^{p} \frac{c_k}{c_k - z},$$

the function g is holomorphic on an open set containing rB and never 0 on rB, so that by the lemma

$$\log |g(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |g(r \, e^{i\vartheta})| \, d\vartheta;$$

Remember now that  $\left|\prod_{j=1}^{m} \frac{\bar{d}_j z - r^2}{r(z - d_j)}\right| = 1$  if |z| = r; if  $c_k = r e^{i\alpha_k}$  for  $k = 1, \ldots, p$  we have

$$\frac{1}{2\pi} \int_0^{2\pi} \log|g(r\,e^{i\vartheta})|\,d\vartheta = \frac{1}{2\pi} \int_0^{2\pi} (\log|f(r\,e^{i\vartheta})| - \sum_{k=1}^p \log|1 - e^{i(\vartheta - \alpha_k)}|)\,d\vartheta = \frac{1}{2\pi} \int_0^{2\pi} \log|f(r\,e^{i\vartheta})|\,d\vartheta,$$

using also the other observation on the integral of the chords. The computation of |g(0)| gives

$$|g(0)| = |f(0)| \prod_{j=1}^{m} \frac{r}{|d_j|} \prod_{k=1}^{p} 1,$$

and the proof ends.

We can write also, if  $||f||_r = \max\{|f(z)|: |z| = r\}$ , observing that  $\log |f(re^{i\vartheta})| \le \log ||f||_r$ :

$$(\text{JENSEN'S INEQUALITY}) \qquad |f(0)| = \frac{1}{r^n} \prod_{j=1}^n |d_j| \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \log|f(r\,e^{i\vartheta})|\,d\vartheta\right) \le \prod_{j=1}^n |d_j| \frac{\|f\|_r}{r^n}$$

Let us derive the following corollary:

COROLLARY. Let  $f \in \mathcal{O}(\Delta)$  be bounded. Assume that  $f(0) \neq 0$ , and order the zeros of f in a sequence  $(d_n)_{n\geq 1}$  such that  $|d_n| \leq |d_{n+1}|$ . Then, for every  $m \in \mathbb{N}$  we have

$$|f(0)| \le \prod_{n=1}^{m} |d_n| \, \|f\|_{\Delta}$$

*Proof.* Given m, take r such that if  $m(r) = \{n \ge 1 : |d_n| \le r\}$  we have  $m(r) \ge m$ . We have

$$|f(0)| \le \prod_{n=1}^{m(r)} \frac{|d_n|}{r} \|f\|_r \le \prod_{n=1}^{m(r)} \frac{|d_n|}{r} \|f\|_{\Delta_r}$$

and since  $|d_n|/r \leq 1$  we have also

$$\prod_{n=1}^{n(r)} \frac{|d_n|}{r} \le \prod_{n=1}^m \frac{|d_n|}{r},$$

so that

$$|f(0)| \le \prod_{n=1}^{m} \frac{|d_n|}{r} ||f||_{\Delta},$$

for every r < 1 large enough. Passing to the limit as  $r \to 1^-$  we get

$$|f(0)| \le \prod_{n=1}^{m} |d_n| ||f||_{\Delta}$$

as required.

Then the infinite product  $\prod_{n=1}^{\infty} |d_n|$  converges to a real number not smaller than  $|f(0)|/||f||_{\Delta}$  (notice that the sequence of partial products is decreasing). This of course happens if and only if the series  $\sum_{n=1}^{\infty} \log |d_n|$  is convergent, and since this series has negative terms, its convergence is absolute, and equivalent to the convergence of the series  $\sum_{n=1}^{\infty} (1 - |d_n|)$ .

**3.6. Functions with prescribed singular parts.** Given  $d \in \mathbb{C}$ , a singular part at d is a function  $q \in \mathcal{O}(\mathbb{C} \setminus \{d\})$  with limit 0 at infinity; q has a unique representation as a Laurent series

$$q(z) = \sum_{n=1}^{\infty} \frac{c_{-n}(d)}{(z-d)^n},$$

where the series converges for every  $z \in \mathbb{C} \setminus \{d\}$ . We call *finite* a singular part when d is a pole for q, that is, all but a finite number of the  $c_{-n}(d)$  vanish; equivalently q is a rational function. If a function h is holomorphic on a region D except for isolated singularities then the set of these singularities is locally finite in D, and at every isolated singularity d the function h has a well defined singular part  $h_d^- \in \mathcal{O}(\mathbb{C} \setminus \{d\})$ , so that h has a singular part distribution, a function  $h_D$  defined on D, which associates to every  $d \in D$  the singular part  $h_d^-$  of h at d, whose support is then the set of non-removable singularities of h; it is of course a locally finite subset of D.

The problem we now pose is the following:

Given a singular part distribution on D with support a locally finite subset S of D, find a function h holomorphic on  $D \setminus S$  whose singular parts at every  $d \in S$  are the given ones.

Of course this is trivial if S is finite: simply take the sum of the singular parts.

3.6.1. Mittag-Leffler series. A Mittag-Leffler series for the singular part distribution  $(q_d)_{d\in S}$  is a series  $\sum_{d\in S}(q_d - g_d)$ , such that each  $g_d$  is holomorphic on D, and the series normally converges in  $D \setminus S$ . The terms  $g_d$  are often called *convergence summands* for the singular part distribution. In fact if these  $g_d$  exist then it is clear that:

PROPOSITION. If  $\sum_{d \in S} (q_d - g_d)$  is a Mittag-Leffler series for the singular part distribution  $(q_d)_{d \in S}$ , then the sum h of this series is holomorphic on  $D \setminus S$ , and at every  $d \in S$  has singular part  $q_d$ .

*Proof.* The only statement which needs proof is the last. Given  $d \in S$ , take r > 0 such that  $B(d,r] \subseteq D$ , and  $B(d,r] \cap S = \{d\}$ . If  $\gamma = \partial B(d,r]$  then the negative coefficients of the Laurent series of h at d are given by

$$c_{-m} = \frac{1}{2\pi i} \int_{\gamma} h(\zeta) \, (\zeta - d)^{m-1} \, d\zeta; \qquad m = 1, \, 2, \, 3 \, \dots$$

Since  $[\gamma]$  is a compact subset of  $D \smallsetminus S$  we have by uniform convergence:

$$\int_{\gamma} h(\zeta) \left(\zeta - d\right)^{m-1} d\zeta = \sum_{c \in S} \int_{\gamma} \left(q_c(\zeta) - g_c(\zeta)\right) \left(\zeta - d\right)^{m-1} d\zeta,$$

and since all functions, but for c = d, are holomorphic on the whole disc B(d, r] all these integrals vanish, but for c = d; the integral with  $g_d$  vanishes, too, so the only one left is

$$\int_{\gamma} q_d(\zeta) \, (\zeta - d)^{m-1} \, d\zeta,$$

obviously equal to  $c_{-m}(d) 2\pi i$ . The proof is concluded.

#### 3.6.2. Mittag-Leffler theorem.

THEOREM. Let S be a locally finite subset of the open subset D of  $\mathbb{C}$ , and let  $(q_d)_{d\in S}$  be a singular part distribution. There exists a family  $(g_d)_{d\in S}$  of rational functions holomorphic in D such that  $\sum_{d\in S} (q_d - g_d)$ is a Mittag-Leffler series for the given distribution. More precisely, we can choose the functions  $g_d$  as either polynomials, or rational functions with a single pole at the boundary of D and limit 0 at infinity.

Proof. As in the factorization theorem we partition the set S into two disjoint sets, A with no accumulation points in  $\mathbb{C}$ , and B such that given a bijective enumeration  $n \mapsto d_n$  of the points of B we have  $\lim_n \operatorname{dist}(d_n, S') = 0$  (either set may be empty). We treat 0 apart, removing it from A if it belongs to it. Given a bijective enumeration  $n \mapsto d_n$  of A, we prove that there is a sequence of polynomials  $g_n$  such that the series  $\sum_{n=1}^{\infty} (q_n - g_n)$  converges normally in  $\mathbb{C} \smallsetminus A$  (here  $q_n = q_{d_n}, g_n = g_{d_n}$ ). We simply take as  $g_n$  a polynomial in the Taylor development of  $q_n$  at 0, the polynomial of smallest degree  $k_n$  such that for  $z \in B(0, |d_n|/2]$  we have  $|q_n(z) - g_n(z)| \leq 1/2^n$ . In this way we get a Mittag–Leffler series for  $(q_d)_{d \in A}$ : given any compact subset K of  $\mathbb{C}$ , since  $\lim_{n\to\infty} |d_n| = +\infty$  there exists  $n_K \in \mathbb{N}$  such that  $K \subseteq B(0, |d_n|/2]$  for  $n \geq n_K$ ; then for  $n \geq n_K$  we get  $||q_n - g_n||_K \leq 1/2^n$ , and normal convergence follows. The sum  $f_A$  of the series is then holomorphic in  $\mathbb{C} \smallsetminus A$ , and as observed above (3.6.1) has  $q_n$  as singular part at every  $d_n \in A$ .

In the case of B we fix a bijection  $n \mapsto d_n$  of  $\{1, 2, 3, ...\}$  onto B, and for every n we pick  $c_n \in S' = B'$ such that  $|d_n - c_n| = \text{dist}(d_n, S')$ , exactly as in the proof of the product theorem 3.3.6. We then consider the Laurent development of  $q_n = q_{d_n}$  in the annulus  $\{|z - c_n| > |d_n - c_n|\}$ , outside of the disc  $B(c_n, |d_n - c_n|]$ ; it is a series of the form

$$q_n(z) = \frac{c_{-1}(c_n)}{z - c_n} + \frac{c_{-2}(c_n)}{(z - c_n)^2} + \dots + \frac{c_{-k}(c_n)}{(z - c_n)^k} + \dots,$$

(remember that  $\lim_{z\to\infty} q_n(z) = 0$ , so that there can be no terms with non negative exponents). The convergence to  $q_n(z)$  is uniform on every annulus like  $\{|z - c_n| \ge r\}$  for every  $r > |d_n - c_n|$ , so that we can pick  $k_n \ge 1$  to have

$$\left| q_n(z) - \sum_{k=1}^{k_n} \frac{c_{-k}(c_n)}{(z - c_n)^k} \right| \le \frac{1}{2^n} \quad \text{for every } z \text{ such that : } |z - c_n| \ge 2|d_n - c_n|.$$

Setting

$$g_n(z) = \sum_{k=1}^{k_n} \frac{c_{-k}(c_n)}{(z - c_n)^k},$$

we get a Mittag–Leffler series for  $(q_d)_{d\in B}$ , in  $\mathbb{C} \smallsetminus S'$ . In fact, if K is a compact subset of  $\mathbb{C}$  disjoint from S' we have that  $\mu = \min\{\operatorname{dist}(z, S') : z \in K\} > 0$ ; if  $n_K \in \mathbb{N}$  is such that  $\operatorname{dist}(d_n, S') < \mu/2$  for  $n \ge n_K$ , then for  $n \ge n_K$  we have  $K \subseteq \mathbb{C} \smallsetminus B(c_n, 2|d_n - c_n|[$  (remember that  $|d_n - c_n| = \operatorname{dist}(d_n, S'))$ ), hence  $||q_n - g_n||_K \le 1/2^n$  for  $n \ge n_K$ . The sum  $f_B$  of the series is then holomorphic in  $\mathbb{C} \smallsetminus (B \cup S')$  and has  $q_d$  as singular part at  $d \in B$  (3.6.1). Finally,  $f = f_A + f_B$  is holomorphic on  $D \smallsetminus S$  and has the required singular parts.

REMARK. For future use we note that in the case of B the rational function constructed has as residue exactly the coefficient  $c_{-1}(d)$  of the function  $q_d$ , for every  $d \in B$ . This follows immediately from the residue theorem applied to  $q_d$ , integrating  $q_d = q_{d_n}$  on a circle centered at  $c = c_n$  of radius larger than  $|d - c| = |d_n - c_n|$ , and from the characterization of the coefficients of the Laurent development.

### 3.6.3. A corollary.

COROLLARY. Let D be a region, let S be a locally finite subset of D, and let  $f : D \setminus S \to \mathbb{C}$  be holomorphic; assume that  $q_d$  is the singular part of f at  $d \in S$ . There is then a family  $(g_d)_{d \in S}$  of rational functions with pole set in  $\mathbb{C} \setminus D$ , such that the series  $\sum_{d \in S} (q_d - g_d)$  is normally convergent in  $D \setminus S$ , and a function  $\varphi$  holomorphic on D such that:

$$f(z) = \sum_{d \in S} (q_d(z) - g_d(z)) + \varphi(z).$$

*Proof.* By the preceding theorem the sequence  $g_d$  of rational functions with the stated properties exists; the singular part at  $d \in S$  of the sum  $h(z) = \sum_{d \in S} (q_d(z) - g_d(z))$  is exactly  $q_d$ , so that f - h has no singularity at any point of D; that is, the difference  $\varphi = f - \sum_{d \in S} (q_d(z) - g_d(z))$  is holomorphic on D.

3.6.4. An application to ideals of the ring  $\mathcal{O}(D)$ . In this number D is always a region of  $\mathbb{C}$ , a connected open set. As an application of the product theorem for regions we have seen that in the integral domain  $\mathcal{O}(D)$  every non empty subset S has a g.c.d., any element whose order function is  $\mathfrak{d}(z) = \min\{\operatorname{ord}(f, z) : f \in S\}$ . As an application of the Mittag–Leffler theorem we now prove that  $\mathcal{O}(D)$  is a *Bézout domain*: given  $f_1, \ldots, f_n \in \mathcal{O}(D)$  there exist  $u_1, \ldots, u_n \in \mathcal{O}(D)$  such that

$$gcd(f_1,\ldots,f_n) = \sum_{k=1}^n u_k f_k.$$

It is enough to prove the following particular case

. If 
$$f, g \in \mathcal{O}(D)$$
 are coprime, i.e.  $gcd(f, g) = 1$ , then there exist  $u, v \in \mathcal{O}(D)$  such that  
 $u f + v g = 1$ .

*Proof.* (WEDDERBURN) We assume f, g both nonzero and nonunits. Coprimality is then equivalent to asserting that the zero-sets Z(f) and Z(g), both non-empty, are disjoint. Consider the meromorphic

function 1/(fg); by the preceding corollary there exist rational functions  $(g_d)_{d \in Z(f) \cup Z(g)}$ , whose pole sets are contained in  $\mathbb{C} \setminus D$ , and  $\varphi \in \mathcal{O}(D)$ , such that

$$\frac{1}{fg} = \sum_{d \in Z(f) \cup Z(g)} (q_d - g_d) + \varphi,$$

where  $q_d$  is the singular part of 1/(fg). Write

$$\frac{1}{fg} = \left(\sum_{d \in Z(g)} (q_d - g_d) + \varphi\right) + \left(\sum_{d \in Z(f)} (q_d - g_d)\right),$$

and deduce that

$$1 = \left(\sum_{d \in Z(g)} (g q_d - g g_d) + \varphi g\right) f + \left(\sum_{d \in Z(f)} (f q_d - f g_d)\right) g$$

It is now easy to see that

$$u = \sum_{d \in Z(g)} (g q_d - g g_d) + \varphi g; \quad v = \sum_{d \in Z(f)} (f q_d - f g_d),$$

are holomorphic on D (the series are normally convergent in  $D \setminus (Z(f) \cup Z(g))$ , actually on all of D by inward spreading of compact convergence, see 2.7.4) and all singularities are removable, as is immediate to check).

The general case follows easily from the particular case, in every integral domain (if gcd(f,g) = h then f/h and g/h are coprime, so that u(f/h) + v(g/h) = 1 for some u, v in the domain, hence h = u f + v g, and the assert is true for two functions. An easy induction, using associativity of the gcd, yields the result. We have obtained the following:

PROPOSITION. If D is a region, in the ring  $\mathcal{O}(D)$  of functions holomorphic on D every finitely generated ideal is principal.

#### 3.6.5. Mittag-Leffler osculation theorem.

. Let D be open in  $\mathbb{C}$ , and let S be a locally finite subset of D. Assume that for every  $d \in S$  we are given a series  $v_d(z) = \sum_{k \leq n(d)} c_k (z - d)^k$  which converges normally in  $\mathbb{C} \setminus \{d\}$ ; n(d) is an integer. Then there exists a function h, holomorphic on  $D \setminus S$ , such that at every  $d \in S$  the Laurent development of h at d has  $\sum_{k \leq n(d)} c_k (z - d)^k$  as a "section", that is, for  $n \leq n(d)$  the coefficients of the Laurent development of h at d coincide with  $c_n(d)$ .

Proof. We pick for every  $d \in S$  an integer m(d) > 0 strictly larger than n(d); by the product theorem 3.3.4 we have a function f holomorphic on D such that  $\operatorname{ord}(f, d) = m(d)$  for  $d \in S$ . Let  $q_d$  be the singular part of  $v_d(z)/f(z)$ , for every  $d \in S$ , so that  $v_d/f = q_d + u_d$  with  $u_d$  holomorphic in a neighborhood of d. By Mittag–Leffler theorem 3.6.2 there is  $g \in \mathcal{O}(D \setminus S)$  whose singular part at every  $d \in S$  is  $q_d$ . We claim that h = fg has the desired properties. Clearly h is holomorphic on  $D \setminus S$ . We prove that  $\operatorname{ord}(h - v_d, d) \geq m(d)(> n(d))$ : this proves that the Laurent development of  $h - v_d$  at d has all the coefficients of index < m(d) equal to zero, as required. We have  $v_d = f q_d + f u_d$  in a nbhd of d, so that

$$h - v_d = f g - f q_d - f u_d = f (g - q_d - u_d);$$

since  $g - q_d$  and  $u_d$  are holomorphic around d, so is  $w_d = q - q_d - v_d$ ; hence  $\operatorname{ord}(w_d, d) \ge 0$ , so that

$$\operatorname{ord}(h - v_d, d) = \operatorname{ord}(f w_d, d) = \operatorname{ord}(f, d) + \operatorname{ord}(w_d, d) \ge \operatorname{ord}(f, d) = m(d),$$

as required.

The osculation theorem implies in particular that given a region D, a locally finite subset  $(d_n)_{n \in \mathbb{N}}$  of D, and a sequence  $(b_n)_{n \in \mathbb{N}}$  of complex numbers, there exists a function  $f \in \mathcal{O}(D)$  such that  $f(d_n) = b_n$ . The construction of such an f may be made more explicit in the case  $D = \mathbb{C}$ , as the following exercise (taken almost verbatim from [Remmert]) shows. EXERCISE 3.6.5.1. Let  $(d_n)_{n \in \mathbb{N}}$  be a sequence of distinct points in  $\mathbb{C}$ , with  $d_0 = 0$  and  $\lim_{n \to \infty} |d_n| = +\infty$ . Let  $f \in \mathcal{O}(\mathbb{C})$  be such that  $\operatorname{ord}(f, d_n) = 1$  for every  $n \in \mathbb{N}$ , and  $f(z) \neq 0$  if  $z \notin \{d_0, d_1, d_2, \ldots\}$ . Then for every sequence  $(b_n)_{n \in \mathbb{N}}$  there exists a sequence  $k_n \in \mathbb{N}$  such that the series

$$b_0 \frac{f(z)}{f'(0) z} + \sum_{n=1}^{\infty} b_n \frac{f(z)}{f'(d_n) (z - d_n)} \left(\frac{z}{d_n}\right)^{k_n}$$

converges normally in  $\mathbb{C}$  to an entire function g such that  $g(d_n) = b_n$ .

This is to be compared with Lagrange's interpolation theorem: if  $d_0, \ldots, d_m$  are m+1 distinct points of  $\mathbb{C}$ , and  $b_0, \ldots, b_m \in \mathbb{C}$  then

$$p(z) = \sum_{k=0}^{m} b_k \left( \prod_{0 \le j \le m, \ j \ne k} \frac{z - d_j}{d_k - d_j} \right)$$

is the polynomial of degree  $\leq m$  such that  $p(d_k) = b_k$ , for  $k = 0, \ldots, m$ .

3.6.6. Logarithmic integration. Assume that D is a region, and that  $f \in \mathcal{O}(D)$  is not identically zero. A necessary condition for f to be a logarithmic derivative of some  $h \in \mathcal{O}(D)$  is that for every loop  $\gamma$  in D

$$\frac{1}{2\pi i} \int_{\gamma} f(z) \, dz$$
 is an integer.

In fact if f = h'/h it is easy to see that the above integral is  $\operatorname{ind}_{h\circ\gamma}(0)$ , the winding number of  $h \circ \gamma$ around the origin. We show that the preceding condition is also sufficient. Fix  $c \in D$ . For every  $z \in D$ pick a path  $\alpha_z$  in D originating at c and ending at z. We prove that the function

$$g(z) = \exp\left(\int_{\alpha_z} f(\zeta) \, d\zeta\right),$$

is well-defined on D. In fact, if  $\beta_z$  is another path in D with the same endpoints, we get

$$\exp\left(\int_{\alpha_z} f(\zeta) \, d\zeta - \int_{\beta_z} f(\zeta) \, d\zeta\right) = \exp\left(\int_{\alpha_z \, . \, \tilde{\beta}_z} f(\zeta) \, d\zeta\right) = \exp(2\pi \, im) = 1,$$

 $(\tilde{\beta}_z \text{ is the path opposite to } \beta_z)$  so that

$$\exp\left(\int_{\alpha_z} f(\zeta) \, d\zeta\right) = \exp\left(\int_{\beta_z} f(\zeta) \, d\zeta\right)$$

It is now easy to prove that g'(z)/g(z) = f(z) for every  $z \in D$ . In fact, given any point  $a \in D$ , and a disc  $B(a, r \subseteq D)$ , for every  $z \in B(a, r \in C)$  we can write ( $\alpha$  is a path from c to a, fixed once and for all):

$$g(z) = \exp\left(\int_{\alpha} f(\zeta) \, d\zeta + \int_{[a,z]} f(\zeta) \, d\zeta\right) = \exp\left(\int_{\alpha} f(\zeta) \, d\zeta\right) \, \exp\left(\int_{[a,z]} f(\zeta) \, d\zeta\right) = g(a) \, e^{h(z)},$$

having set  $h(z) = \int_{[a,z]} f(\zeta) d\zeta$  for  $z \in B(a, r[;$  it is well known that h'(z) = f(z) for every  $z \in B(a, r[.$  It follows that g is holomorphic in B(a, r[ and that we have:

$$g'(z) = g(a) e^{h(z)} h'(z) = g(z) f(z) \quad \text{for every } z \in B(a, r[.$$

The conclusion is now immediate.

It is quite easy now to give a proof of the product theorem by way of Mittag-Leffler theorem and logarithmic integration, avoiding even infinite products. A function f has a zero of order  $m \ge 1$  at dif and only if its logarithmic derivative has singular part m/(z-d) at d: this is well-known and easy to prove. Assume that D is a region and S is a locally finite subset of D, and that we are given an integer  $m(d) \ge 1$  for every  $d \in S$ . Given the singular part distribution  $d \mapsto m(d)/(z-d)$  find for it a Mittag-Leffler series

$$\sum_{d\in S} \left( \frac{m(d)}{z-d} - g_d \right);$$

remember that  $g_d$  is a polynomial, or a rational function such as

$$\frac{m(d)}{z-c} + \sum_{k=2}^{n(d)} \frac{c_{-k}}{(z-c)^k},$$

with the same residue as the given singular part (see 3.6.2 and the remark at the end). We claim that if  $h \in \mathcal{O}(D \setminus S)$  is the sum of the above Mittag–Leffler series, then h is a logarithmic derivative. In fact let  $\gamma$  be a loop whose trace  $[\gamma]$  is contained in  $D \setminus S$ . By normal convergence on  $[\gamma]$  we have

$$\frac{1}{2\pi i} \int_{\gamma} h(z) dz = \sum_{d \in S} \left( \frac{1}{2\pi i} \int_{\gamma} \frac{m(d)}{z - d} dz - \frac{1}{2\pi i} \int_{\gamma} g_d(z) dz \right);$$

the first integral in parentheses is m(d) ind<sub> $\gamma$ </sub>(d); the second is zero if  $g_d$  is a polynomial, and otherwise is m(d) ind<sub> $\gamma$ </sub>(c(d)). Then the parentheses are integers, so that the series

$$\sum_{d \in S} \left( \frac{1}{2\pi i} \int_{\gamma} \frac{m(d)}{z - d} dz - \frac{1}{2\pi i} \int_{\gamma} g_d(z) dz \right)$$

is a converging series of integers, and then is actually a finite sum of integers.

Any logarithmic primitive of h is a function which has zeros exactly on the points of S, with the prescribed multiplicities.

EXERCISE 3.6.6.1. Let  $m \ge 2$  be an integer, D a region. Prove that  $f \in \mathcal{O}(D)$  has an m-th root in  $\mathcal{O}(D)$  if and only if

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} \, dz \in m\mathbb{Z},$$

(i.e., it is divisible by m) for every loop  $\gamma$  in  $D \smallsetminus Z(f)$ ; describe a construction of a root.

## 4. Runge theory

We want to address the question of approximating holomorphic functions by particularly simple holomorphic functions, e.g. polynomial or rational functions. For instance, given a region D of  $\mathbb{C}$  is it always possible to approximate every  $f \in \mathcal{O}(D)$  by polynomials, uniformly on every compact subset of D? The answer is in general no: the function 1/z, holomorphic on  $\mathbb{C}_* = \mathbb{C} \setminus \{0\}$  cannot be approximated uniformly by a sequence of polynomials on the circle  $\partial B$ : in fact  $\int_{\partial B} p(z) dz = 0$  for every polynomial, while  $\int_{\partial B} dz/z = 2\pi i$ . However, rational functions suffice, as we shall see in this section. The theory has interesting topological asides.

4.0.7. Connected components. We recall that every topological space  $(X, \tau)$  can be partitioned into connected components: given  $a \in X$  we consider the union  $C_a$  of all connected subsets of X containing a; this is by definition the connected component of X containing a; it is a connected subset of X, and is the largest connected subset of X containing a, since the union of two non disjoint connected subsets of X is still connected. Since the closure of a connected subset is still connected,  $C_a$  is also closed in X. If we consider a subset S of  $\mathbb{C}$ , and Z is a connected component of  $\mathbb{C} \setminus S$ , then  $\partial Z \subseteq \partial S$  (if  $c \in \partial Z$ , every disc B(c, r[ centered at c contains points of Z and points of  $\mathbb{C} \setminus Z$ ; if  $B(c, r[\cap S = \emptyset, \text{ then } B(c, r[\subseteq \mathbb{C} \setminus S, \text{ and hence } Z \cup B(c, r[$  is connected, disjoint from S, and properly containing Z, a contradiction if Z is a component of  $\mathbb{C} \setminus S$ ).

Recall that every open subset of  $\mathbb{C}$  has open connected components; if K is compact, K is closed, hence  $\mathbb{C} \smallsetminus K$  has open connected components. If r > 0 is such that  $rB \supseteq K$ , then since  $\mathbb{C} \smallsetminus rB$  is connected it is contained in a component of  $\mathbb{C} \smallsetminus K$ , which then has only one unbounded component. If D is open in  $\mathbb{C}$ , then  $\mathbb{C} \smallsetminus D$  is closed, hence its connected components, closed in the relative topology of  $\mathbb{C} \smallsetminus D$ , are also closed in  $\mathbb{C}$ ; the bounded components of  $\mathbb{C} \smallsetminus D$  are then compact, and are called *holes* of D.

Every open subset D of  $\mathbb{C}$  can also be considered an open subset of  $\mathbb{C}_{\infty} = \mathbb{C} \cup \{\infty\}$ , the Riemann sphere, one point compactification of  $\mathbb{C}$ . The connected component of  $\mathbb{C}_{\infty} \setminus D$  containing  $\infty$  is obtained in this way: let E be the union of all unbounded components of  $\mathbb{C} \setminus D$ ; then the component is  $\mathrm{cl}_{\mathbb{C}}(E) \cup \{\infty\}$ . In fact, if  $\mathcal{E}$  is the set of unbounded components of  $\mathbb{C} \setminus D$  in  $\mathbb{C}$ , every  $C \in \mathcal{E}$  has  $\infty$  in its closure; the set  $\bigcup_{C \in \mathcal{E}} C \cup \{\infty\}$  is then connected, and its closure in  $\mathbb{C}_{\infty}$ , clearly coinciding with  $\mathrm{cl}_{\mathbb{C}}\left(\bigcup_{C \in \mathcal{E}} C\right) \cup \{\infty\}$ , is the required component.

Notice that, in particular, if the set  $\mathcal{E}$  of unbounded components of  $\mathbb{C} \smallsetminus D$  is finite (with D open) then the component of  $\mathbb{C}_{\infty} \smallsetminus D$  containing  $\infty$  is  $\bigcup_{C \in \mathcal{E}} C \cup \{\infty\}$ .

But we shall see in 4.2 that in fact, for every open subset D of  $\mathbb{C}$  the union of unbounded components of  $\mathbb{C} \setminus D$  is actually closed in  $\mathbb{C}$ ; hence the component of  $\mathbb{C}_{\infty} \setminus D$  containing  $\infty$  is simply  $E \cup \{\infty\}$ , with E the union of all unbounded components of  $\mathbb{C} \setminus D$ . If K is a compact subset of  $\mathbb{C}$  then the components of  $\mathbb{C}_{\infty} \setminus K$  are exactly the bounded components of  $\mathbb{C} \setminus K$  and  $\{\infty\} \cup E$ , where E is the unbounded component of  $\mathbb{C} \setminus K$ .

4.0.8. A negative result. Let K be compact in  $\mathbb{C}$ ; assume that Z is a bounded component of  $\mathbb{C} \setminus K$ . Pick  $c \in Z$  and consider 1/(z-c); this function has then a non-zero minumum modulus in K, that is  $\mu = \min\{1/|z-c| : z \in K\} > 0$ . Then, if g is holomorphic on some open set containing  $Z \cup K$ , we have  $\|1/(z-c) - g(z)\|_K \ge \mu$ . For, assuming the contrary we get  $|1 - (z-c)g(z)| < |z-c|\mu \le 1$  for every  $z \in K$ , hence  $\|1 - (z-c)g(z)\|_K < 1$ . This is impossible; in fact by the maximum modulus theorem we get (by the preceding observations  $\partial Z \subseteq \partial K \subseteq K$ ):

$$\|1 - (z - c) g(z)\|_{Z \cup \partial Z} = \|1 - (z - c) g(z)\|_{\partial Z} \le \|1 - (z - c) g(z)\|_{K} < 1,$$

but  $c \in Z$ , and for z = c we have 1 - (c - c)g(c) = 1, which implies  $||1 - (z - c)g(z)||_{Z \cup \partial Z} \ge 1$ , a contradiction.

4.0.9. Approximation of Cauchy transforms by rational functions.

**PROPOSITION.** Let  $\alpha : [a, b] \to \mathbb{C}$  be a path, let  $u : [\alpha] \to \mathbb{C}$  be continuous, and let

$$f_u: \mathbb{C} \smallsetminus [\alpha] \to \mathbb{C}, \qquad f_u(z) = \frac{1}{2\pi i} \int_{\alpha} \frac{u(\zeta)}{\zeta - z} \, d\zeta,$$

be the Cauchy transform of u. If K is a compact subset of  $\mathbb{C}$  disjoint from  $[\alpha]$ , and  $\varepsilon > 0$ , there exists a rational function R, all of whose poles lie in  $[\alpha]$ , such that

$$\|f_u - R\|_K \le \varepsilon.$$

*Proof.* The function  $h: (z,\zeta) \mapsto (1/(2\pi i))u(\zeta)/(\zeta-z)$  is continuous, and hence uniformly continuous, on the compact set  $K \times [\alpha]$ . Given  $\varepsilon > 0$  we can find  $\delta > 0$  such that if  $\max\{|z'-z''|, |\zeta'-\zeta''|\} \le \delta$  then

 $|h(z',\zeta') - h(z'',\zeta'')| \leq \varepsilon$ ; by uniform continuity of  $\alpha$  on [a,b] we find  $\rho > 0$  such that if  $t',t'' \in [a,b]$ ,  $|t'-t''| \leq \rho$ , then  $|\alpha(t') - \alpha(t'')| \leq \delta$ . Take points  $a_0 = a < a_1 < \cdots < a_m = b$  such that  $a_k - a_{k-1} \leq \rho$ , and let  $\alpha_k = \alpha_{|[a_{k-1},a_k]}$  for  $k = 1, \ldots, m$ ; moreover let  $w_k = \alpha((a_{k-1} + a_k)/2)$  for  $k = 1, \ldots, m$ , and put  $\zeta_k = \alpha(a_k)$ , for  $k = 0, \ldots, m$ . We have, for  $z \in K$ :

$$f_u(z) = \int_{\alpha} h(z,\zeta) \, d\zeta = \sum_{k=1}^m \int_{\alpha_k} h(z,\zeta) \, d\zeta;$$

now the smallness of the arc  $\alpha_k$  makes  $\int_{\alpha_k} h(z,\zeta) d\zeta$  very close to  $h(z,w_k)(\zeta_k-\zeta_{k-1})$ ; and we can write:

$$\int_{\alpha_k} h(z,\zeta) \, d\zeta = \int_{\alpha_k} \left( h(z,\zeta) - h(z,w_k) + h(z,w_k) \right) \, d\zeta = \int_{\alpha_k} \left( h(z,\zeta) - h(z,w_k) \right) \, d\zeta + \frac{1}{2\pi i} \int_{\alpha_k} \frac{u(w_k)}{w_k - z} \, d\zeta = \int_{\alpha_k} \left( h(z,\zeta) - h(z,w_k) \right) \, d\zeta + \frac{u(w_k)(\zeta_k - \zeta_{k-1})}{2\pi i} \frac{1}{w_k - z},$$

with the first term small; thus

$$f_u(z) - \sum_{k=1}^m \frac{c_k}{z - w_k} = \sum_{k=1}^m \int_{\alpha_k} (h(z, \zeta) - h(z, w_k)) \, d\zeta, \qquad z \in K,$$

where  $c_k = (u(w_k)(\zeta_{k-1} - \zeta_k)/(2\pi i))$ ; we estimate the right hand side:

$$\left| \int_{\alpha_k} (h(z,\zeta) - h(z,w_k)) \, d\zeta \right| \le \int_{\alpha_k} |h(z,\zeta) - h(z,w_k)| \, |d\zeta| \le \int_{\alpha_k} \varepsilon \, |d\zeta| = \varepsilon V(\alpha_k),$$

where  $V(\alpha_k)$  is the length of the arc  $\alpha_k$ ; thus

$$\left|\sum_{k=1}^{m} \int_{\alpha_{k}} (h(z,\zeta) - h(z,w_{k})) \, d\zeta\right| \leq \sum_{k=1}^{m} \left| \int_{\alpha_{k}} (h(z,\zeta) - h(z,w_{k})) \, d\zeta\right| \leq \sum_{k=1}^{m} \varepsilon \, V(\alpha_{k}) = \varepsilon \, V(\alpha),$$

for every  $z \in K$ . The function  $R(z) = \sum_{k=1}^{m} c_k/(z - w_k)$  is then a rational function whose pole set is contained in  $\{w_1, \ldots, w_m\} \subseteq [\alpha]$  which approximates  $f_u$  on K to less than  $\varepsilon V(\alpha)$ .

4.0.10. Pole shifting. For every compact subset K of  $\mathbb{C}$  the set  $\mathcal{C}(K)$  of complex valued functions continuous on K, with the topology of uniform convergence, i.e. normed with  $\|\cdot\|_K$ , is a Banach algebra, hence the closure of every subalgebra of  $\mathcal{C}(K)$  is still a subalgebra of  $\mathcal{C}(K)$ : it is here intended that a subalgebra contains the constant functions.

. POLE SHIFTING THEOREM Let K be a compact subset of  $\mathbb{C}$ ; for every  $c \in \mathbb{C} \setminus K$  denote by  $A_c(K)$ the uniform closure in  $\mathcal{C}(K)$  of the algebra of polynomials in 1/(z-c). Then  $A_a(K) = A_b(K)$  if a and b belong to the same connected component of  $\mathbb{C} \setminus K$ ; and if a is in the unbounded component of  $\mathbb{C} \setminus K$ then  $A_a(K)$  coincides with the closure  $A_{\infty}(K)$  of the subalgebra of  $\mathcal{C}(K)$  consisting of the polynomial functions.

*Proof.* Since  $A_c(K)$  is a closed subalgebra of  $\mathcal{C}(K)$ , we have that  $A_b(K) \subseteq A_c(K)$  if and only if  $1/(z-b) \in A_c(K)$ . Pick a component Z of  $\mathbb{C} \setminus K$ , and take  $a \in Z$ . Let  $S = \{b \in Z : A_b(K) \subseteq A_a(K)\}$ . We prove that if  $c \in S$ , and  $B(c, r \subseteq Z)$ , then  $B(c, r \subseteq S)$ . In fact:

$$\frac{1}{z-b} = \frac{1}{(z-c) - (b-c)} = \frac{1}{z-c} \frac{1}{1-(b-c)/(z-c)};$$

if  $z \in K$  we have |z - c| > r > |b - c|; hence  $|(b - c)/(z - c)| \le |b - c|/r < 1$ ; this proves that the series

$$\frac{1}{z-b} = \frac{1}{z-c} \frac{1}{1-(b-c)/(z-c))} = \sum_{n=0}^{\infty} \frac{(b-c)^n}{(z-c)^{n+1}}$$

is normally convergent in K, hence 1/(z-b) belongs to  $A_c(K)$ . We leave it to the reader to show that this implies S = Z. We have proved that  $A_b(K) \subseteq A_a(K)$  for every  $a, b \in Z$ ; then  $A_a(K) = A_b(K)$  for every  $a, b \in Z$ . If Z is the unbounded component of  $\mathbb{C} \setminus K$ , pick  $a \in Z$  with  $|a| > \max\{|z| : z \in K\}$ ; then we have, with normal convergence on K:

$$\frac{1}{z-a} = \frac{1}{a} \frac{1}{z/a-1} = \frac{-1/a}{1-(z/a)} = \sum_{n=0}^{\infty} \frac{-1}{a^{n+1}} z^n,$$

which proves that  $A_a(K) \subseteq A_{\infty}(K)$ ; and if  $r > \max\{|z| : z \in K\}$  then by Cauchy formula  $z = \int_{\partial(rB)} \zeta/(\zeta - z) d\zeta/(2\pi i)$  for every z in the interior of rB; by the preceding proposition z is uniformly

approximable on K by rational functions with poles in  $\partial(rB)$ . This means that  $z_{|K}$  is in  $A_a(K)$  for a in the unbounded component of  $\mathbb{C} \setminus K$ ; hence  $A_{\infty}(K) \subseteq A_a(K)$ .

REMARK. Using 4.0.8, we actually see that if a and b belong to different components of  $\mathbb{C} \setminus K$ , then  $A_a(K) \neq A_b(K)$ .

4.0.11. A contouring cycle. By cycle in an open set E we mean, roughly speaking, a finite set of loops in E; the precise concept belongs to homology theory; its usage in the next theorem will be in any case clear from the proof.

THEOREM. Let D be open in  $\mathbb{C}$ , and let K be a compact subset of D. Then there is a cycle  $\gamma$  in  $D \setminus K$  such that for every f holomorphic in D and every  $z \in K$  we have

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} \, d\zeta.$$

Proof. Let  $\rho = \operatorname{dist}(K, \mathbb{C} \setminus D) = \min\{\operatorname{dist}(z, \mathbb{C} \setminus D) : z \in K\}$ . Let  $\delta > 0$  be such that  $\delta\sqrt{2} < \rho$ , and let  $p_1, p_2, q_1, q_2 \in \mathbb{R}$  be such that the compact rectangle  $R = \{p_1 \leq \operatorname{Re} z \leq p_2, q_1 \leq \operatorname{Im} z \leq q_2\}$ contains K in its interior. We pick  $n \in \mathbb{N}$  so large that  $(p_2 - p_1)/n < \delta$  and  $(q_2 - q_1)/n < \delta$ . The lines  $\operatorname{Re} z = a_k = p_1 + k(p_2 - p_1)/n$  and  $\operatorname{Im} z = b_k = q_1 + k(q_2 - q_1)/n$ ,  $k = 0, \ldots, n$  divide the rectangle Rinto a finite set of small rectangles Q, each with diameter smaller than  $\delta\sqrt{2} < \rho$ , with pairwise disjoint interiors. A typical rectangle Q will have a boundary  $\partial Q$  identified with the polygonal loop

$$[a_{j-1} + i b_{k-1}, a_j + i b_{k-1}, a_j + i b_k, a_{j-1} + i b_k, a_{j-1} + i b_{k-1}],$$

oriented in this way. It is not difficult to prove that for every function f holomorphic on an open set containing Q the following Cauchy formula holds:

$$f(z) = \frac{1}{2\pi i} \int_{\partial Q} \frac{f(\zeta)}{\zeta - z} d\zeta \qquad z \in \operatorname{int}_{\mathbb{C}}(Q),$$
$$0 = \frac{1}{2\pi i} \int_{\partial Q} \frac{f(\zeta)}{\zeta - z} d\zeta \qquad z \in \mathbb{C} \smallsetminus Q.$$

Consider now the set F of all rectangles Q which have non-empty intersection with K; each  $Q \in F$  is contained in D, since the diameter of Q is smaller than  $\operatorname{dist}(K, \mathbb{C} \smallsetminus D)$ . Let  $P = \bigcup_{Q \in F} Q$ ; plainly we have  $K \subseteq P \subseteq D$ . If  $A = \bigcup_{Q \in F} \operatorname{int}_{\mathbb{C}}(Q)$  and  $z \in A$  we evidently have

$$f(z) = \frac{1}{2\pi i} \sum_{Q \in F} \int_{\partial Q} \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

(z belongs to exactly one  $Q \in F$ , hence all the terms in the sum are zero, except the one corresponding to the Q containing z, whose value is f(z)). Of course A is in general strictly contained in  $\operatorname{int}_{\mathbb{C}}(P)$ , but it is dense in P. Consider now the segments which are sides of more than one rectangle  $Q \in F$ . If  $\sigma$  is one such segment, observe that  $\sigma$  is oriented in opposite ways as part of the boundary of the two rectangles; in the sum of the integrals over the boundaries the integrals over these segments then cancel out. In other words we have, for every u continuous on  $\bigcup_{Q \in F} \partial Q$ :

$$\sum_{Q \in F} \int_{\partial Q} u(\zeta) \, d\zeta = \sum_{j \in J} \int_{\sigma_j} u(\zeta) \, d\zeta,$$

where  $(\sigma_j)_{j \in J}$  is a bijective indexing of the oriented segments contained in only one rectangle  $Q \in F$ ; clearly the trace of every such segment is contained in  $D \setminus K$ , otherwise all the rectangles containing the segment, at least two, would be in F. We have our cycle  $\gamma$ : it is the family of oriented segments  $(\sigma_j)_{j \in J}$ , and we agree to write:

$$\int_{\gamma} u(\zeta) \, d\zeta := \sum_{j \in J} \int_{\sigma_j} u(\zeta) \, d\zeta,$$

for every function continuous on the trace  $[\gamma] = \bigcup_{j \in J} [\sigma_j]$  of  $\gamma$  (warning: in general  $\gamma$  is not a loop. However it can be proved that it is a finite family of polygonal loops, see 4.1.2). Notice now that the integral

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

defines a continuous function of z on  $\mathbb{C} \setminus [\gamma]$ , which coincides with f on A, and since A is dense in P, we have by continuity:

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta$$
 for every  $z \in P \smallsetminus [\gamma]$ .

Now  $K \subseteq P \setminus [\gamma]$ : we have observed that  $K \subseteq P$  and that K is disjoint from every segment  $\sigma_j$ . The construction is concluded.

4.0.12. A topological lemma and its corollary. We need one more lemma from topology:

LEMMA. Let D be open in  $\mathbb{C}$  and define  $L = \{z \in \mathbb{C} : \operatorname{dist}(z, \mathbb{C} \setminus D) \ge \rho\} \cap (rB)$ , for some  $r, \rho > 0$ . Then every bounded component of  $\mathbb{C} \setminus L$  contains a component of  $\mathbb{C} \setminus D$ . Moreover, in this situation every component of  $\mathbb{C}_{\infty} \setminus L$  contains a component of  $\mathbb{C}_{\infty} \setminus D$ .

*Proof.* If we put

$$M = \mathbb{C} \setminus \{z \in \mathbb{C} : \operatorname{dist}(z, \mathbb{C} \setminus D) \ge \rho\} = \{z \in \mathbb{C} : \operatorname{dist}(z, \mathbb{C} \setminus D) < \rho\} = \bigcup_{z \in \mathbb{C} \setminus D} B(z, \rho[; z \in \mathbb{C} \setminus D) < \rho\}$$

then we have  $\mathbb{C} \smallsetminus L = M \cup (\mathbb{C} \smallsetminus rB)$ ; since  $\mathbb{C} \smallsetminus rB$  is connected, it is contained in the unbounded component of  $\mathbb{C} \smallsetminus L$ ; every bounded component Z of  $\mathbb{C} \smallsetminus L$  is therefore disjoint from  $\mathbb{C} \smallsetminus rB$ , hence contained in M. Thus  $B(c, \rho[\cap Z \neq \emptyset \text{ for some } c \in \mathbb{C} \smallsetminus D$ ; but then  $B(c, \rho[\subseteq Z, \text{ and hence } c \in Z \text{ (every}$ disk  $B(c, \rho[$ , being disjoint from L and connected is contained in a connected component of  $\mathbb{C} \smallsetminus L$ ; if it intersects the bounded component Z of  $\mathbb{C} \smallsetminus L$  it is then contained in it). The connected component of  $\mathbb{C} \smallsetminus D$  containing c is then contained in Z (since  $\mathbb{C} \smallsetminus D \subseteq \mathbb{C} \smallsetminus L$ , every connected component of  $\mathbb{C} \smallsetminus D$ is contained in some connected component of  $\mathbb{C} \backsim L$ ).

For  $\mathbb{C}_{\infty}$  notice that the component of  $\mathbb{C}_{\infty} \setminus L$  containing  $\infty$  is simply  $\{\infty\} \cup E$ , where E is the unbounded component of  $\mathbb{C} \setminus L$ , and the other components of  $\mathbb{C}_{\infty} \setminus L$  are unaltered. All the unbounded components of  $\mathbb{C} \setminus D$  (if any exist) are contained in  $\{\infty\} \cup E$ , as is also  $\infty$ . Every bounded component of  $\mathbb{C} \setminus L$  contains a point of  $\mathbb{C}_{\infty} \setminus D$ , as we have just seen, and hence also a component of  $\mathbb{C}_{\infty} \setminus D$ .  $\Box$ 

COROLLARY. Let D be open in  $\mathbb{C}$ , and let K be a compact subset of D. Then there exists a compact subset L of D such that  $K \subseteq L$ , and every bounded component of  $\mathbb{C} \setminus L$  contains a component of  $\mathbb{C} \setminus D$ .

*Proof.* In the preceding lemma simply take  $\rho$  such that  $0 < \rho < \operatorname{dist}(K, \mathbb{C} \setminus D)$ , and r such that  $K \subseteq rB$ .

#### 4.1. Runge's approximation theorem.

. Runge theorem.

- (i) RUNGE THEOREM FOR COMPACT SETS Let K be a compact subset of  $\mathbb{C}$ , and let P be a set which intersects every bounded component of  $\mathbb{C} \setminus K$ . Then every function holomorphic on an open set containing K can be uniformly approximated on K by rational functions, all of whose poles lie in P.
- (ii) RUNGE THEOREM FOR OPEN SETS Let D be open in  $\mathbb{C}$ , and let P be a set whose closure intersects every hole of D. Then every  $f \in \mathcal{O}(D)$  can be compactly approximated on D by rational functions, all of whose poles lie in P.

*Proof.* (i) Given a compact subset K of  $\mathbb{C}$ , contained in some open subset D of  $\mathbb{C}$ , and  $f \in \mathcal{O}(D)$ , we take a cycle  $\gamma$  in  $D \smallsetminus K$  as in 4.0.11; by 4.0.9 the function f is approximable on K by rational functions with poles in  $[\gamma]$ ; that is, given  $\varepsilon > 0$  there is a rational function

$$R_{\varepsilon}(z) = p_{\varepsilon}(z) + \sum_{c \in F} p_c\left(\frac{1}{z-c}\right) \quad \text{such that} \quad \|f - R_{\varepsilon}\|_K \le \varepsilon/2;$$

here  $p_{\varepsilon}$  and  $p_c$  are polynomials, and F is a finite subset of  $[\gamma]$ . Since  $[\gamma]$  is contained in the complement of K, we can write

$$\sum_{c \in F} p_c \left( \frac{1}{z - c} \right) = \sum_{Z \in H} \left( \sum_{c \in F \cap Z} p_c \left( \frac{1}{z - c} \right) \right),$$
  
d components Z of  $\mathbb{C} > K$  such that Z  $\cap F$  is

where H is the set of bounded components Z of  $\mathbb{C}\smallsetminus K$  such that  $Z\cap F$  is non empty.

Now the rational function

$$A_Z(z) = \sum_{c \in F \cap Z} p_c\left(\frac{1}{z-c}\right),$$

1 \

can be approximated on K by rational functions with a single pole in  $Z \cap P$ ; pick one such function  $R_Z$  which differs from  $A_Z$  on K by less that  $\varepsilon/(2m)$ , where m is the cardinality of F. The conclusion is immediate: the rational function

$$p_{\varepsilon} + \sum_{Z \in H} R_Z$$

has its poles in P, and approximates f on K to less that  $\varepsilon$ . The proof of (i) is concluded.

(ii) Given a compact subset K of D, we first enlarge it to a compact subset L of D such that every bounded component Z of  $\mathbb{C} \setminus L$  contains a component of  $\mathbb{C} \setminus D$ , as in lemma 4.0.12. Since  $\overline{P}$  intersects every hole of D, for every bounded component Z of  $\mathbb{C} \setminus L$  the set  $P \cap Z$  is non empty. We now apply (i).

4.1.1. A corollary on nullhomologous loops. Runge's approximation theorem yields easily the following:

THEOREM. Let D be an open region, and let  $\gamma$  be a loop in D. Then the following are equivalent:

(i)  $\int_{\gamma} f(z) dz = 0$ , for every  $f \in \mathcal{O}(D)$ .

(ii)  $\gamma$  is nullhomologous in D, that is, for every  $z \in \mathbb{C} \setminus D$  we have  $\operatorname{ind}_{\gamma}(z) = 0$ .

*Proof.* (i) implies (ii) is trivial since  $\zeta \mapsto 1/(\zeta - z)$  is holomorphic in D if  $z \in \mathbb{C} \setminus D$ .

(ii) implies (i) Given  $f \in \mathcal{O}(D)$  pick a sequence  $R_n$  of rational functions, all of whose poles lie outside D, which uniformly approximates f on the trace  $[\gamma]$  of  $\gamma$ . Then  $R_n$  is of the form:

$$R_n(z) = p_n(z) + \sum_{a \in F_n} \left( \frac{\operatorname{Res}(R_n, a)}{z - a} + \sum_{k=2}^{m_a} \frac{c_{-k}(a)}{(z - a)^k} \right),$$

where  $p_n$  is a polynomial and  $F_n$  is the pole set of  $R_n$ , a finite subset of  $\mathbb{C} \setminus D$ . Hence the integral

$$\int_{\gamma} R_n(z) \, dz = 2\pi i \sum_{a \in F_n} \operatorname{ind}_{\gamma}(a) \operatorname{Res}(R_n, a)$$

is zero because  $\operatorname{ind}_{\gamma}(a) = 0$  for  $a \notin D$ , by the hypothesis made on  $\gamma$ . By uniform convergence we have

$$\int_{\gamma} f(z) dz = \lim_{n \to \infty} \int_{\gamma} R_n(z) dz = 0.$$

4.1.2. About cycles. The notion of cycle pertains to algebraic topology. Without entering too deeply into the subject, we say that a finite family  $\alpha = (\alpha_j)_{j \in J}$  of paths (all parametrized on the interval [0,1]) is a cycle if its homological boundary  $\sum_{j \in J} (\alpha_j(1) - \alpha_j(0))$  is 0: this sum is a formal sum, that is  $\alpha_j(1), \alpha_j(0)$  are not summed as complex numbers, but as elements of the free abelian group  $\mathbb{Z}^{(\mathbb{C})}$ , direct sum of  $\operatorname{Card}(\mathbb{C})$  copies of  $\mathbb{Z}$ ; thus for it to be 0 means that for every  $c \in \mathbb{C}$  we have

$$Card(\{j \in J : \alpha_j(1) = c\}) - Card(\{j \in J : \alpha_j(0) = c\}) = 0$$

(that is, for every  $c \in \mathbb{C}$  the number of paths ending at c is the same as the number of paths beginning at c). Let us prove:

. The finite family  $\alpha = (\alpha_j)_{j \in J}$  of paths is a cycle if and only if for every polynomial p we have

$$\int_{\alpha} p(z) dz \left( := \sum_{j \in J} \int_{\alpha_j} p(z) dz \right) = 0.$$

Moreover a cycle can be decomposed into a finite set of loops.

*Proof.* If P'(z) = p(z) we have, letting  $S = \{\alpha_j(1), \alpha_j(0) : j \in J\}$  be the set of extremities and origins of the paths  $\alpha_j$ 

$$\int_{\alpha} p(z) dz = \sum_{j \in J} \int_{\alpha_j} p(z) dz = \sum_{j \in J} (P(\alpha_j(1)) - P(\alpha_j(0))) = \sum_{c \in S} ((\operatorname{Card}(\{j \in J : \alpha_j(1) = c\}) - \operatorname{Card}(\{j \in J : \alpha_j(0) = c\}))P(c).$$

Clearly this expression is zero for every polynomial p if and only if  $\alpha$  is a cycle (for every given  $c \in S$  there exists a polynomial which is nonzero at c, and zero at every other  $d \in S$ ). Now we can observe

that  $\gamma$ , as constructed in the proof of 4.0.11 is such that  $\int_{\gamma} g(\zeta) d\zeta = 0$  for every function g holomorphic on D (given  $z \in K$ , apply the formula to  $f(\zeta) = (\zeta - z) g(\zeta)$ ). Thus we have  $\int_{\gamma} p(z) dz = 0$  for every polynomial, and  $\gamma$  is indeed a cycle.

By induction on the cardinality of J it is easy to prove that  $\alpha$  can be decomposed in a finite set of loops; we say that a finite set  $\{\alpha_1, \ldots, \alpha_m\}$  of paths is a loop if, up to a permutation, we have  $\alpha_j(1) = \alpha_{j+1}(0)$  for  $j = 1, \ldots, m-1$ , and  $\alpha_m(1) = \alpha_1(0)$ . Clearly a loop is a cycle; and if  $\{\alpha_j : j \in J\}$  is a cycle, and  $\{\alpha_1, \ldots, \alpha_m\}$  is a loop, with  $\{\alpha_1, \ldots, \alpha_m\} \subseteq \{\alpha_j : j \in J\}$ , then  $\{\alpha_j : j \in J\} \setminus \{\alpha_1, \ldots, \alpha_m\}$ is still a cycle. It should now be clear how to proceed: start with any  $c_0 \in S$  and let  $c_1$  be the extremity of a path  $\alpha_1$  beginning at  $c_0$ ; then get  $c_2$  extremity of a path  $\alpha_2$  beginning at  $c_1$ , etc.; by this process we select points in S, until we get  $c_k = c_j$  for some j < k. At the first such repetition we consider  $\alpha_{j+1}, \ldots, \alpha_k$  which clearly is a loop. We now remove the paths of this loop from the original cycle, and we get a cycle of strictly smaller cardinality, to which induction may be applied.

4.1.3. Characterizations of simple connectedness.

THEOREM. Let D be an open region of  $\mathbb{C}$ . Then the following are equivalent:

- (i) D is simply connected, that is, every loop of D is nullhomotopic in D (=the fundamental group  $\pi_1(D,c)$  is trivial, for any  $c \in D$ ).
- (ii) For every loop  $\gamma$  of D, and every f holomorphic on D we have

$$\int_{\gamma} f(z) \, dz = 0$$

- (iii)  $\mathcal{O}'(D) = \mathcal{O}(D)$ , that is, every holomorphic function on D has a primitive on D
- (iv) Every unit of D has a holomorphic logarithm.
- (v) D has the square root property: that is, every unit of D has a holomorphic square root.
- (vi) D is homeomorphic to the open unit disc.
- (vii) For every loop  $\gamma$  of D, and every  $z \in \mathbb{C} \setminus D$  we have

$$\operatorname{ind}_{\gamma}(z) := \frac{1}{2\pi i} \int_{\gamma} \frac{d\zeta}{\zeta - z} = 0$$

(this condition is sometimes expressed by saying that D is homologically simply connected; it is equivalent to asserting that the first homology group of D vanishes, i.e.  $H_1(D) = 0$ ).

- (viii)  $\mathbb{C}_{\infty} \setminus D$  is connected.
- (ix) Every function holomorphic on D can be compactly approximated on D by polynomials.
- (x\*) D has no holes, i.e.  $\mathbb{C} \setminus D$  has no bounded components.

*Proof.* (i) implies (ii): invariance of the integral under loop homotopy; that (ii) is equivalent to (iii) is well known; (iii) implies (iv): given a unit  $f \in \mathcal{O}(D)$  recall that a logarithm of f is any primitive g of f'/f such that for some  $c \in D$  we have that  $e^{g(c)} = f(c)$ ; (iv) implies (v): if g is a logarithm of the unit f, then  $e^{g/2}$  is a square root of f; (v) implies (vi): if  $D = \mathbb{C}$ , then  $z \mapsto z/\sqrt{1+|z|^2}$  is a homeomorphism of  $\mathbb{C}$  onto the open unit disc. If D is not  $\mathbb{C}$ , apply the Riemann mapping theorem. Finally, simple connectedness is a topological condition, hence preserved by homeomorphisms; then (vi) implies (i) and the first six conditions are equivalent.

Next we prove that (ii) implies (vii): if  $z \in \mathbb{C} \setminus D$ , then  $\zeta \mapsto 1/(\zeta - z)$  is holomorphic on D.

Proof that (vii) implies (viii), by contradiction: if  $\mathbb{C}_{\infty} \setminus D$  is not connected we have  $\mathbb{C}_{\infty} \setminus D = K \cup L$ , where K and L are closed disjoint non empty in  $\mathbb{C}_{\infty} \setminus D$ . If  $\infty \in L$ , then K is a compact subset of  $\mathbb{C}$ , and  $\mathbb{C}_{\infty} \setminus L = D \cup K$  is open in  $\mathbb{C}_{\infty}$  and hence  $E = D \cup K$  is an open subset of  $\mathbb{C}$ . By theorem 4.0.11 there is a cycle  $\gamma$  in  $E \setminus K = D$  around K such that

$$\operatorname{ind}_{\gamma}(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{d\zeta}{\zeta - z} = 1 \quad \text{for every} \quad z \in K.$$

The cycle can be decomposed into a finite set  $\gamma_1, \ldots, \gamma_p$  of loops; since

$$\operatorname{ind}_{\gamma}(z) = \sum_{j=1}^{p} \operatorname{ind}_{\gamma_j}(z), \text{ for every } z \in K,$$

we have  $\operatorname{ind}_{\gamma_i}(z) \neq 0$  for at least one j and  $z \in K$ , contradicting (vii).

(viii) implies (ix): take a compact subset K of D; by enlarging it if necessary we can assume that every bounded component Z of  $\mathbb{C} \setminus K$  contains a hole C of D (4.0.12). By hypothesis there is only one component of  $\mathbb{C}_{\infty} \setminus D$ , that containing  $\infty$ ; there cannot be bounded components of  $\mathbb{C} \setminus K$ , for any such component Z contains a bounded component C of  $\mathbb{C} \setminus D$ , which is also a component of  $\mathbb{C}_{\infty} \setminus D$ ; in fact Z is a neighborhood of C disjoint from all unbounded components of  $\mathbb{C} \setminus D$  (if  $Z \cap E \neq \emptyset$  for some unbounded component of  $\mathbb{C} \setminus D$  then  $E \subseteq Z$ , absurd since Z is bounded and E is unbounded). It follows immediately from Runge's theorem for compact sets that every function holomorphic on a neighborhood of K can be uniformly approximated on K by polynomials.

(ix) implies (ii): If f is holomorphic on D and  $\gamma$  is a loop in D we choose a sequence  $p_n$  of polynomials which converges to f uniformly on  $[\gamma]$ ; we then have

$$\int_{\gamma} f(z) \, dz = \lim_{n \to \infty} \int_{\gamma} p_n(z) \, dz = 0$$

since each integral  $\int_{\gamma} p_n(z) dz$  is zero.

REMARK.  $(x^*)$  has a special status to be discussed later (4.2). It trivially implies (viii), or (ix) by Runge's theorem, but the converse is not as easy as one might expect.

**4.2.** Holes of open subsets of  $\mathbb{C}$ . We complete here the analysis of the characterization of simple connectedness begun in 4.1.3. Recall that a locally compact (Hausdorff) space is a Hausdorff topological space in which every point has a compact neighborhood; it is well known that then the compact neighborhoods of a point are a base for the neighborhood system of the point, for every point of the space. We need a simple

LEMMA. In a compact space X, let  $(F_{\lambda})_{\lambda \in \Lambda}$  be a family of closed subsets; assume that the intersection  $\bigcap_{\lambda \in \Lambda} F_{\lambda}$  is contained in an open set W. Then there exists a finite subfamily  $(F_{\lambda})_{\lambda \in M}$  (M a finite subset of  $\Lambda$ ) such that  $\bigcap_{\lambda \in M} F_{\lambda}$  is also contained in W.

*Proof.* If  $F = X \setminus W$ , consider  $(F \cap F_{\lambda})_{\lambda \in \Lambda}$ ; this family of closed sets has empty intersection, so that some finite subfamily has also empty intersection.

We now prove the following theorem

. SURA-BURA'S THEOREM. Let X be a locally compact Hausdorff space. Let C be a connected component of X. If C is compact, then C is the intersection of all the open compact subsets of X containing it.

*Proof.* We first assume that X is compact. Take the family of all clopen (=open and closed) subsets of X containing C (this family contains at least X), and consider its intersection K; it is a closed subset of X containing C, and we claim that it coincides with C. If not, then K cannot be connected, hence  $K = A \cup B$ , with A and B closed disjoint and non empty; since C is connected we have either  $C \subseteq A$ or  $C \subseteq B$ ; we assume  $C \subseteq A$ . By normality of X there are open disjoint sets U and V with  $A \subseteq U$  and  $B \subseteq V$ . Now the family of all clopen subsets of X containing C has intersection  $K \subseteq U \cup V$ ; by the lemma, there is a finite family  $O_1, \ldots, O_m$  of clopen subsets of X containing C such that

$$K \subseteq O = \bigcap_{k=1}^{m} O_k \subseteq U \cup V.$$

Now O is clopen, hence  $O \cap U$  is open, but also closed, since  $O \cap U = O \cap (X \setminus V)$ . Thus  $O \cap U$  is a clopen subset containing C, but not containing K, a contradiction.

We now reduce the locally compact case to the previous one. For every  $x \in C$  we pick an open neighborhood U(x) of x in X with compact closure  $\overline{U}(x)$  in X; by compactness of C there are  $x_1, \ldots, x_m \in C$  such that  $C \subseteq U = U(x_1) \cup \cdots \cup U(x_m)$ ; let T be the closure of this open set U, that is  $T = \operatorname{cl}_X(U) = \bigcup_{k=1}^m \operatorname{cl}_X(U(x_k))$ . Clearly T is a compact subspace of X containing C. Then C is a connected component also of T, and by what we have just proved C is the intersection of the clopen subsets of T containing C. Since U is open in X, and hence in T, by the lemma there are clopen subsets of T containing C and contained in U; these are then open in U and hence in X, and are compact, being closed in the compact space T.

We now consider the case  $X = \mathbb{C} \setminus D$ , where D is an open subset of  $\mathbb{C}$ ; it is a closed subspace of  $\mathbb{C}$ , hence locally compact in the induced topology. Its compact components are exactly what we called holes of D (4.0.7). Trivially, a subset A of  $\mathbb{C} \setminus D$  is open in the relative topology of  $\mathbb{C} \setminus D$  if and only if  $D \cup A$  is open in  $\mathbb{C}$ . As a first consequence of Šura–Bura's theorem we have that unbounded components cannot accumulate on holes of  $\mathbb{C} \setminus D$ :

. For every open subset D of  $\mathbb C$  the union of the unbounded connected components of  $\mathbb C\smallsetminus D$  is closed in  $\mathbb C.$ 

*Proof.* We show that if  $p \in \mathbb{C}$  is not in this union E, then some neighborhood of p is disjoint from E. If  $p \in D$ , then D is the required neighborhood. Otherwise p is in some hole C of D; by the previous theorem, there is a compact subset A of  $\mathbb{C} \setminus D$  which is open in the relative topology of  $\mathbb{C} \setminus D$  and contains C. Since A is clopen in  $\mathbb{C} \setminus D$ , if it intersects a connected component of  $\mathbb{C} \setminus D$  it must contain it, but being compact it cannot contain unbounded sets. Then  $A \cup D$  is open in  $\mathbb{C}$ , has empty intersection with E, and contains C; since  $p \in C \subseteq A$ , the set  $A \cup D$  is a neighborhood of p disjoint from E.

### Hence:

PROPOSITION. If D is open in  $\mathbb{C}$ , then the components of  $\mathbb{C}_{\infty} \setminus D$  are the holes of D, and the union of all the unbounded components of  $\mathbb{C} \setminus D$ , with  $\infty$  added. Consequently, a connected open subset of  $\mathbb{C}$  is simply connected if and only if  $\mathbb{C}_{\infty} \setminus D$  is connected, and if and only if it has no holes.

# 5. The maximum modulus theorem

We recall the following version of the maximum modulus theorem for holomorphic functions, which is an immediate corollary of the Open Mapping Theorem:

. MAXIMUM MODULUS THEOREM, I Let D be a region of  $\mathbb{C}$ , and let  $f : D \to \mathbb{C}$  be holomorphic. If  $c \in D$  is a point of local maximum for |f|, then f is constant.

We shall present some results related to this theorem. In this section we use the following notation: if E is a set, and  $f: E \to \mathbb{C}$  is a function, then

 $||f||_E := \sup\{|f(z)| : z \in E\}, \text{ finite or } +\infty.$ 

We also use this terminology: if D is a region,  $\overline{D} = D \cup \partial D$  is the closure of D, and  $f : \overline{D} \to \mathbb{C}$  is continuous and holomorphic on D, then f is said to be c-holomorphic on D.

REMARK. If D is a region,  $f: D \to \mathbb{C}$  is holomorphic non constant, and  $M \ge 0$  is a real number such that  $|f(z)| \le M$  for every  $z \in D$ , then actually |f(z)| < M for every  $z \in D$ : otherwise, if |f(c)| = M for some  $c \in D$ , then c is a point of *absolute* maximum for |f| on D, so that f is constant on D.

5.0.1. Bounded regions. It is easy to see that:

. MAXIMUM MODULUS THEOREM, II Let D be a bounded region of  $\mathbb{C}$ , and let  $f: \overline{D} \to \mathbb{C}$  be continuous on  $\overline{D} = D \cup \partial D$  and holomorphic on D (i.e., f is c-holomorphic on D). Then  $||f||_{\overline{D}} = ||f||_{\partial D}$ , and if f is non constant then  $||f(z)| < ||f||_{\partial D}$  for every  $z \in D$ .

*Proof.* By Weierstrass theorem  $\max\{|f(z)|: z \in \overline{D}\} = ||f||_{\overline{D}}$  exists by continuity of f and compactness of  $\overline{D}$ ; then  $|f(z)| \leq ||f||_{\overline{D}}$  for every  $z \in D$ , and by the preceding remark we have  $|f(z)| < ||f||_{\overline{D}}$  unless f is constant on D. For f nonconstant the maximum modulus is then attained only on  $\partial D$ .  $\Box$ 

**5.1. Unbounded regions.** We have just seen that on bounded regions, the supremum of a c-holomorphic mapping is assumed on the boundary, and only on the boundary if the function is non-constant. If we remove the boundedness assumption for D this is no longer true; e.g. the exponential function has modulus constantly 1 on  $i \mathbb{R} = \partial T$ , where  $T = \{\operatorname{Re} z > 0\}$  is the right half-plane, but clearly  $\| \exp \|_T = \infty$ . As a second example, consider the function  $f(z) = \exp(e^{iz})$  in the strip  $D = \{z \in \mathbb{C} : |\operatorname{Re} z| < \pi/2\}$ , whose closure is  $\overline{D} = \{z \in \mathbb{C} : |\operatorname{Re} z| \le \pi/2\}$ ; we have, if z = x + iy:

$$|f(z)| = \exp(\operatorname{Re}(e^{i\,x-y})) = \exp(\operatorname{Re}(e^{-y}\,(\cos x + i\,\sin x))) = \exp(e^{-y}\,\cos x);$$

on  $\partial D = (\pi/2 + i \mathbb{R}) \cup (-\pi/2 + i \mathbb{R})$  we have |f(z)| = 1, constantly; but clearly  $f(iy) = \exp(e^{-y}) \to \infty$  if  $y \to -\infty$ . But if we assume f bounded on all of  $\overline{D}$ , then we still have:

. MAXIMUM MODULUS THEOREM, III Let D be an unbounded region of  $\mathbb{C}$ , and let  $f: \overline{D} \to \mathbb{C}$  be c-holomorphic on D. If  $||f||_{\overline{D}} < \infty$ , then  $||f||_{\overline{D}} = ||f||_{\partial D}$ , and if f is non constant then  $|f(z)| < ||f||_{\partial D}$  for every  $z \in D$ .

*Proof.* We assume f non constant. We first prove the theorem under the extra assumption that

$$\lim_{z\in\bar{D},\,z\to\infty}f(z)=0.$$

Let  $\mu$  be any real number,  $\mu > \|f\|_{\partial D}$ ; we prove that  $K = \{|f| \ge \mu\} := \{z \in \overline{D} : |f(z)| \ge \mu\}$  is empty. In fact, K is closed in  $\overline{D}$  by continuity of |f|, hence also closed in  $\mathbb{C}$ ; and since  $\lim_{z \in \overline{D}, z \to \infty} f(z) = 0$  we find R > 0 such that if  $z \in \overline{D}$  and |z| > R we have  $|f(z)| < \mu$ . Then  $|z| \le R$  if  $z \in K$ , so that K is also bounded, and hence compact. Clearly  $K \cap \partial D = \emptyset$  (on  $\partial D$  we have  $|f(z)| \le \|f\|_{\partial D} < \mu$ ). If K is non empty then |f| has a maximum on K, which clearly is also the absolute maximum of |f| on D (outside of K we have  $|f(w)| < \mu$ ); if this maximum is attained on  $c \in K \subseteq D$ , by the first theorem we get f constant on D; the proof with the extra assumption is concluded.

We now turn to the general case. Fix  $a \in D$ , and consider the auxiliary function  $g: \overline{D} \to \mathbb{C}$ , where g(z) = (f(z) - f(a))/(z - a) for  $z \in \overline{D} \setminus \{a\}$ , g(a) = f'(a); it is easy to see that g is bounded and c-holomorphic on D, with limit 0 at infinity, hence so also is  $\varphi(z) = f(z) g(z)$ . By what just proved we get  $\|\varphi\|_{\overline{D}} \leq \|\varphi\|_{\partial D}$ . Since  $\|\varphi\|_{\partial D} \leq \|f\|_{\partial D} \|g\|_{\overline{D}} \leq \|f\|_{\partial D} \|g\|_{\overline{D}}$  we have

$$|\varphi(z)| = |f(z)| |g(z)| \le ||f||_{\partial D} ||g||_{\bar{D}}$$
 for every  $z \in D$ 

The same can be repeated for  $\varphi_n = f^n g$ , for every  $n = 1, 2, 3, \ldots$ , obtaining

$$|\varphi_n(z)| = |f(z)|^n |g(z)| \le ||f||_{\partial D}^n ||g||_{\bar{D}} \quad \text{for every } z \in D,$$

so that, if  $z \in D$  with  $g(z) \neq 0$  we have:

$$|f(z)|^{n} \leq \frac{\|f\|_{\partial D}^{n} \|g\|_{\bar{D}}}{|g(z)|} \Rightarrow |f(z)| \leq \|f\|_{\partial D} \frac{\|g\|_{\bar{D}}^{1/n}}{|g(z)|^{1/n}},$$

for every  $n \ge 1$  and every  $z \in D \setminus Z(g)$ ; letting n tend to infinity we have

$$|f(z)| \le ||f||_{\partial D}$$
 for every  $z \in D \smallsetminus Z(g)$ ,

and since  $D \setminus Z(g)$  is dense in  $\overline{D}$  we conclude.

REMARK. Notice that if f is bounded on D and zero on the boundary then it is identically zero on all of D.

EXERCISE 5.1.0.1. Prove that boundedness of f on D can be replaced by the hypothesis that f is  $O(\log |z|)$  as  $z \to \infty$ , that is, there are A, r > 0 such that  $|f(z)| \le A \log |z|$  for  $z \in D$  and  $|z| \ge r$ .

5.1.1. A generalization. In all the preceding theorems, and in the ones to follow, the hypothesis that f be c-holomorphic, that is continuously extendable to the boundary of its domain, can be dropped if  $||f||_{\partial D}$  is replaced with

$$||f||_{\partial D} := \sup\{ \limsup_{w \in D, \ w \to z} |f(w)| : z \in \partial D \}.$$

We also consider the boundary  $\partial_{\infty}D$  of D in the extended plane  $\mathbb{C}_{\infty} = \mathbb{C} \cup \{\infty\}$ , so that for  $D \subseteq \mathbb{C}$  we have  $\partial_{\infty}D = \partial D$  if D is bounded, and otherwise  $\partial_{\infty}D = \partial D \cup \{\infty\}$ . Let's prove the generalization of the preceding proposition:

. Let D be a region of  $\mathbb{C}$ , and let  $f: D \to \mathbb{C}$  be holomorphic. Then, unless f is constant, we have

$$|f(z)| < ||f||_{\partial_{\infty}D}$$
 for every  $z \in D$ ;

and if  $||f||_{\partial_{\infty}D} < \infty$  then  $||f||_{\partial D} = ||f||_{\partial_{\infty}D}$ .

Proof. If  $||f||_{\partial_{\infty}D} = \infty$  there is nothing to prove. Otherwise, let  $\mu > ||f||_{\partial_{\infty}D}$  be any real number. We prove that  $K = \{w \in D : |f(w)| \ge \mu\}$  is empty. For any  $z \in \partial_{\infty}D$  there is a set U(z) containing z, open in  $\mathbb{C}_{\infty}$ , such that  $\sup\{|f(w)| : w \in U(z) \cap D\} < \mu$  (if  $z = \infty$  we can assume  $U(\infty) = \mathbb{C}_{\infty} \smallsetminus rB$ ). Since  $\partial_{\infty}D$  is compact there exist  $z_1, \ldots, z_m \in \partial D$  such that  $\partial_{\infty}D \subseteq U$ , where  $U = U(z_1) \cup \cdots \cup U(z_m)$ . Then  $D \smallsetminus U = (D \cup \partial_{\infty}D) \smallsetminus U$  is a compact subset of D (it is closed in  $\mathbb{C}_{\infty}$ , hence compact); since  $|f(w)| < \mu$  for  $w \in U$  we have that  $K = \{w \in D : |f(w)| \ge \mu\} = \{w \in D \smallsetminus U : |f(w)| \ge \mu\}$  is compact. If K is nonempty there is  $c \in K$  such that  $|f(c)| = \max\{|f(w)| : w \in K\}$  then also  $|f(c)| = \max\{|f(w)| : w \in D\}$ , and by the Maximum modulus theorem I we conclude. The last assertion,  $||f||_{\partial D} = ||f||_{\partial_{\infty}D}$ , is obtained as above, multiplying by a function g such that  $\lim_{z \in D, z \to \infty} f(z) g(z) = 0 \ldots$ 

5.1.2. The three lines lemma. We know that  $|\Gamma(\sigma + it)| \leq \Gamma(\sigma)$  for  $\sigma > 0$  (by the integral representation or directly from the Gauss limit); in other words we have  $\Gamma(\sigma) = \max\{|\Gamma(\sigma + it)| : t \in \mathbb{R}\} = \|\Gamma\|_{\operatorname{Re} s = \sigma}$ ; and we have seen that the real gamma function is logarithmically convex on  $]0, \infty[$ , equivalently that  $\Gamma(\alpha a + \beta b) \leq (\Gamma(a))^{\alpha} (\Gamma(b))^{\beta}$ , for  $a, b, \alpha, \beta > 0$  and  $\alpha + \beta = 1$ . In this sense the behavior of the gamma function is shared by all holomorphic functions bounded on strips, as we now see.

. THREE LINES LEMMA. Let  $f: \overline{S} \to \mathbb{C}$  be c-holomorphic, and bounded on the strip

$$S = \{ z \in \mathbb{C} : a < \operatorname{Re} z < b \}, \quad a, b \in \mathbb{R}, \ a < b.$$

Then, unless f is identically zero, the function  $\mu : [a,b] \to [0,\infty[$  defined by  $\mu(x) = \sup\{|f(x+iy)| : y \in \mathbb{R}\}$  is strictly positive and logarithmically convex in [a,b], that is

$$0 < \mu(\alpha x + \beta y) \le (\mu(x))^{\alpha} (\mu(y))^{\beta} \qquad x, y \in [a, b], \, \alpha, \beta \ge 0, \alpha + \beta = 1$$

*Proof.* It is clearly enough to prove the statement with x = a and y = b. We shall also prove that  $\mu(a)$ ,  $\mu(b) > 0$ ; in this proof A is any real number strictly larger than 0, if f(a + iy) = 0 for every  $y \in \mathbb{R}$ , otherwise  $A = \mu(a)$ ; same for B and  $\mu(b)$ . Given  $x \in [a, b]$ , let us find the weights  $\alpha, \beta \ge 0$  such that  $x = \alpha a + \beta b$  and  $\alpha + \beta = 1$ : we get  $\alpha = (b - x)/(b - a)$  and  $\beta = (x - a)/(b - a)$ . Let us consider the entire function

$$g(z) = A^{(b-z)/(b-a)} B^{(z-a)/(b-a)} = \exp\left(\frac{b-z}{b-a} \log A + \frac{z-a}{b-a} \log B\right);$$

we have, if z = x + i y:

$$|g(z)| = A^{\operatorname{Re}((b-z)/(b-a))} B^{\operatorname{Re}((z-a)/(b-a))} = A^{(b-x)/(b-a)} B^{(x-a)/(b-a)}$$

so that, in particular, |g(a+iy)| = A and |g(b+iy)| = B for every  $y \in \mathbb{R}$ ; moreover 1/|g(z)| is bounded in S (by max{1/A, 1/B}). Then f(z)/g(z) is bounded on S, and is bounded by 1 on the boundary of S, so that the preceding theorem implies  $|f(z)|/|g(z)| \leq 1$  for every  $z \in \overline{S}$ ; in particular

$$|f(x+iy)| \le |g(x+iy)| = A^{(b-x)/(b-a)} B^{(x-a)/(b-a)}, \quad x \in [a,b], \ y \in \mathbb{R},$$

so that

$$\mu(x) \le A^{(b-x)/(b-a)} B^{(x-a)/(b-a)} \quad \text{for every } x \in [a, b],$$

and letting  $A \to \mu(a)^+$  and  $B \to \mu(b)^+$  the proof is concluded; notice that since  $\mu(x) > 0$  for  $x \in ]a, b[$  (by the identity theorem) we must have  $\mu(a), \mu(b) > 0$ .

REMARK. With  $S = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$  we get, for every  $x \in [0, 1]$  and every  $y \in \mathbb{R}$ :

$$|f(x+iy)| \le (\mu(0))^{1-x} (\mu(1))^x$$

Hölder's inequality may be interpreted in this context: let  $(X, \mathcal{M}, \mu)$  be a measure space, and let  $u, v : X \to ]0, \infty[$  be strictly positive functions in  $L^1(\mu)$ . Then, for  $s \in \overline{S}$  the function

$$\varphi(s) = \int_X u^{1-s} \, v^s \, d\mu$$

is c-holomorphic (differentiation under the integral sign); moreover  $|\varphi(\sigma+it)| \leq \varphi(\sigma)$  for every  $\sigma \in [0, 1]$ , so that by the three lines lemma:

$$\varphi(\sigma) \le (\varphi(0))^{1-\sigma} (\varphi(1))^{\sigma} \quad \text{that is} \int_X (u(x))^{1-\sigma} (v(x))^{\sigma} d\mu(x) \le \left(\int_X u \, d\mu\right)^{1-\sigma} \left(\int_X v \, d\mu\right)^{\sigma},$$

and setting  $\sigma = 1/q$ ,  $1 - \sigma = 1/p$ ,  $u = |f|^p$ ,  $v = |g|^q$  we get the familiar Hölder's inequality.

5.1.3. As a corollary:

. THREE CIRCLES THEOREM (HADAMARD) Let  $f : \overline{A} \to \mathbb{C}$  be c-holomorphic, where  $A = \{z \in \mathbb{C} : u < |z| < v\}$  is an annulus,  $0 < u < v < \infty$ . For  $r \in [u, v]$  let  $M(r) = \max\{|f(z)| : |z| = r\}$ . Then, if  $\alpha, \beta \ge 0, \alpha + \beta = 1$  we have

$$M(u^{\alpha} v^{\beta}) \le (M(u))^{\alpha} (M(v))^{\beta}.$$

*Proof.* Consider  $g: \overline{S} \to \mathbb{C}$  defined by  $g(s) = f(e^s)$ , where  $\overline{S} = \{s \in \mathbb{C} : \log u \leq \operatorname{Re} s \leq \log v\}$ ; for  $u \leq r \leq v$  we have  $\mu(\log r) := \sup\{|g(\log r + it)| : t \in \mathbb{R}\} = M(r)$ . The three lines lemma says that

$$\mu(\alpha \log u + \beta \log v) \le (\mu(\log u))^{\alpha} (\mu(\log v))^{\beta} \quad \text{that is} \quad \mu(\log(u^{\alpha} v^{\beta})) \le (M(u))^{\alpha} (M(v))^{\beta},$$

or also

$$M(u^{\alpha} v^{\beta}) \le (M(u))^{\alpha} (M(v))^{\beta}$$

The three circles theorem may be expressed by saying that  $\log M(r)$  is a convex function of  $\log r$ .

5.2. The Phragmen–Lindelöf theorem for a strip. Even if it is bounded on the boundary of the unbounded region D, a c-holomorphic function is not necessarily bounded on the region, as seen at the beginning of 5.1. However, if the growth at infinity is "not too fast", where this notion of "fast" depends on the shape and size of D, then we can still deduce boundedness from boundedness on the boundary. We first state and prove the theorem for strips. Essentially the theorem says that if f is bounded on the strip, and at infinity is  $O(\exp(e^{b|z|}))$  for some b, with  $0 < b < \pi/(\text{width of the strip})$ , then f is bounded on the strip; there is also a more general result.

. PHRAGMEN-LINDELÖF THEOREM Let  $S = \{z \in \mathbb{C} : d - l < \text{Re } z < d + l\}$  be a vertical strip with bounded base  $(d \in \mathbb{R}, l > 0)$ . Let D be a region of  $\mathbb{C}$  contained in S, and assume that  $f : D \to \mathbb{C}$  is c-holomorphic and non constant on D, and that  $||f||_{\partial D}$  is finite. Put  $a = \pi/(2l)$ .

(a) If there is b, with 0 < b < a, and A, B > 0 such that

$$|f(z)| \le A \exp(B e^{b|z|})$$
 for every  $z \in D$ ,

then  $|f(z)| < ||f||_{\partial D}$  for every  $z \in S$ .

(b) If for every  $\varepsilon > 0$  there exists  $A_{\varepsilon} > 0$  such that

$$|f(z)| \le A_{\varepsilon} \exp(\varepsilon e^{a|z|}) \quad for \ every \ z \in D,$$

then  $|f(z)| < ||f||_{\partial D}$  for every  $z \in D$ .

*Proof.* Before going to the proof notice that since |z| is rotationally invariant the theorem holds for any strip with bounded base, horizontal, vertical or oblique, and that we may also translate the strip (this changes |z|, but it is easy to see that the hypotheses are still valid, since  $|z| - |c| \le |z - c| \le |z| + |c|$ ). So we consider the strip  $S = \{-l < \operatorname{Re} z < l\}$ .

(a) We pick c with b < c < a, and consider the function  $g_{\varepsilon}(z) = \exp(\varepsilon \cos(cz))$ , with  $\varepsilon > 0$  arbitrary. Notice that  $|g_{\varepsilon}(z)|^{-1} < 1$  on S: in fact

$$|g_{\varepsilon}(z)|^{-1} = |\exp(-\varepsilon \cos(cz))| = \exp(-\varepsilon \operatorname{Re}(\cos(cz)));$$

recalling that  $\operatorname{Re}(\cos(cz)) = \cos(cx) \cosh(cy)$  if  $x = \operatorname{Re} z$  and  $y = \operatorname{Im} z$ , since  $|cx| = c|x| \le cl < \pi l/(2l) = \pi/2$  we have, for  $x \in [-l, l]$ ,  $\cos(cx) \ge \cos(cl) > 0$  so that  $-\varepsilon \operatorname{Re}(\cos(cz)) < -\varepsilon \cos(cl) \cosh(cy) < 0$  for every  $z = x + iy \in \overline{S}$ , hence

$$|g_{\varepsilon}(z)|^{-1} \le \exp(-\varepsilon \cos(cl) \cosh(cy)) < 1$$
 for every  $z \in S$ .

If we define  $F_{\varepsilon}(z) = f(z) (g_{\varepsilon}(z))^{-1}$  for  $z \in D$ , we then have  $|F_{\varepsilon}(z)| \leq |f(z)|$  for every  $z \in \overline{D}$ , in particular also  $||F_{\varepsilon}||_{\partial D} \leq ||f||_{\partial D}$ . We now prove that  $\lim_{z\to\infty, z\in D} F_{\varepsilon}(z) = 0$ ; this implies that  $F_{\varepsilon}$  is bounded on D; by Maximum Modulus III we then get

$$|F_{\varepsilon}(z)| \leq ||F_{\varepsilon}||_{\partial D} \leq ||f||_{\partial D}$$
 for every  $z \in D$ ,

then

$$|f(z)| \le ||f||_{\partial D} \exp(\varepsilon \cos(cz))$$
 for every  $z \in D$ ,

which implies, letting  $\varepsilon \to 0^+$ , the assert.

Using the previous estimate for  $|g_{\varepsilon}(z)|^{-1}$  and the hypothesis we get

$$|F_{\varepsilon}(z)| \le A \exp(B e^{b|z|}) \exp(-\varepsilon \cos(cl) \cosh(cy)) \le A \exp(B e^{b(l+|y|} - \varepsilon \cos(cl) \cosh(c|y|));$$

let's prove that  $\lim_{y\to\pm\infty} (B e^{b(l+|y|)} - \varepsilon \cos(cl) \cosh(c|y|)) = -\infty$ ; in fact

$$B e^{b(l+|y|)} - \varepsilon \cos(cl) \cosh(c|y|) = B e^{b(l+|y|)} - \varepsilon \cos(cl) \frac{e^{c|y|} + e^{-c|y|}}{2} = e^{c|y|} \left( B e^{bl} e^{(b-c)|y|} - \varepsilon \cos(cl) \frac{1 + e^{-2c|y|}}{2} \right);$$

since b < c the expression in parentheses tends to  $-\varepsilon \cos(cl)/2 < 0$ , and the entire expression has limit  $-\infty$ , as  $y \to \pm \infty$ .

(b) Given  $\varepsilon > 0$  consider again  $g_{\varepsilon}(z) = \exp(\varepsilon \cos(az))$ ; again we have  $|g_{\varepsilon}(z)|^{-1} < 1$  in S (but this time  $|g_{\varepsilon}(z)|^{-1} = 1$  on  $\partial S$ ), so that if  $F_{\varepsilon}(z) = f(z) (g_{\varepsilon}(z))^{-1}$  we still have  $||F_{\varepsilon}||_{\partial D} \leq ||f||_{\partial D}$ . We want to prove that  $F_{\varepsilon}$  is bounded on the set  $D \cap i\mathbb{R}$ , intersection of D with the imaginary axis  $i\mathbb{R}$ ; this proves that  $F_{\varepsilon}$  is bounded on the boundary of the sets  $D_{-} = D \cap \{\operatorname{Re} z < 0\}$  and  $D_{+} = D \cap \{\operatorname{Re} z > 0\}$ ; since these sets are contained in the strips  $S_{-} = S \cap \{\operatorname{Re} z < 0\}$  and  $S_{+} = S \cap \{\operatorname{Re} z > 0\}$ , whose width is half that of S, we can apply part (a) of the theorem, replacing b with a, and a by 2a, to obtain boundedness of  $F_{\varepsilon}$  on  $D_{\pm}$ , and hence on D. By Maximum Modulus III then  $|F_{\varepsilon}|$  is bounded by  $||f||_{\partial D} \geq ||F_{\varepsilon}||_{\partial D}$ , so that, as above

$$|f(z)| \le ||f||_{\partial D} |\exp(\varepsilon \cos(az))| \quad \text{for every } z \in D,$$

and we conclude immediately, letting  $\varepsilon \to 0^+$ .

Take then  $\delta$  with  $0 < \delta < \varepsilon/2$ ; by hypothesis there is A > 0 such that

$$|f(z)| \le A \exp(\delta e^{a|z|}) \text{ for every } z \in D \cap (i\mathbb{R});$$

Then, for  $z = i y \in D \cap (i\mathbb{R})$ :

$$|F_{\varepsilon}(z)| \le A \, \exp(\delta \, e^{a|y|}) \, \exp(-\varepsilon \, \cosh(ay)) = A \, \exp(\delta \, e^{a|y|} - \varepsilon \, \cosh(ay)),$$

and consider the exponent

$$\delta e^{a|y|} - \varepsilon \cosh(ay) = e^{a|y|} (\delta - (\varepsilon/2)(1 + e^{-2a|y|}));$$

since  $\delta - \varepsilon/2 < 0$ , this exponent has limit  $-\infty$  as  $y \to \pm \infty$ , so that  $y \mapsto A \exp(\delta e^{a|y|} - \varepsilon \cosh(ay))$  is a continuous function zero at infinity, hence it is bounded. The proof ends.
Notice how delicate these estimates are: on the strip  $\{-\pi/2 < \text{Re } z < \pi/2\}$  the function  $f(z) = \exp(e^{iz})$  is unbounded but of modulus constantly 1 on the boundary (5.1).

5.3. The Phragmen–Lindelöf theorem for an angle. Here the critical value is  $\pi/\alpha$ , where  $0 < \alpha \leq 2\pi$  is the aperture of the angle.

. PHRAGMEN-LINDELÖF THEOREM Let  $A = \{r e^{i\vartheta} : r > 0, \varphi - \alpha/2 < \vartheta < \varphi + \alpha/2\}$ , with  $0 < \alpha \leq 2\pi$  be an angle, and let D be a region of  $\mathbb{C}$  contained in D. Assume that  $f : \overline{D} \to \mathbb{C}$  is c-holomorphic on D, with  $\|f\|_{\partial D} < \infty$ .

(i) If for some b, with  $0 < b < a = \pi/\alpha$  and A, B > 0 we have

 $|f(z)| \le A \exp(B|z|^b)$  for every  $z \in D$ ,

then  $|f(z)| \leq ||f||_{\partial D}$  for every  $z \in D$ .

(ii) If for every  $\varepsilon > 0$  there exists a constant  $A_{\varepsilon} > 0$  such that

 $|f(z)| \le A_{\varepsilon} \exp(\varepsilon |z|^a)$  for every  $z \in D$ 

(where, again,  $a = \pi/\alpha$ ), then  $|f(z)| \leq ||f||_{\partial D}$  for every  $z \in D$ .

Proof. Clearly the theorem is rotation invariant, and is is not restrictive to assume  $\varphi = 0$ ; in other words  $A = \{r e^{i\vartheta} \in \mathbb{C} : r > 0, |\vartheta| < \alpha/2\}$ . We can reduce this case to the case of a strip: observe that, if  $0 < \alpha \leq 2\pi$  the exponential function gives an holomorphic isomorphism of the horizontal strip  $S = \{s \in \mathbb{C} : -\alpha/2 < \text{Im } s < \alpha/2\}$  onto the angle A of the theorem; set  $g(s) = f(e^s)$ , for  $s \in E = \{s \in S : e^s \in D\}$ , and observe that the hypothesis  $f \in O(\exp(B |z|^b)$  for z tending to infinity in D becomes  $g \in O(\exp(B \exp(b \operatorname{Re} s)))$  for  $\operatorname{Re} s \to +\infty$  with  $s \in E$ , hence also  $g \in O(\exp(B \exp(b |s|)))$ , and similar arguments work for case (ii).

Note again that the balance in the preceding theorem is delicate: if we consider  $f(z) = \exp(z^a)$ , with  $a = \pi/\alpha$ , then we have |f(z)| = 1 on  $\partial D$ , where  $z = r e^{\pm i\alpha/2}$ , yet  $f(x) = \exp(x^a) \to \infty$  as  $x \to \infty$  on the positive reals, so that f is unbounded.

For  $\alpha = 2\pi$  we have a function bounded on a ray, and the critical value is 1/2. So:

COROLLARY. If an entire function  $f : \mathbb{C} \to \mathbb{C}$  is bounded on a ray, and for every  $\varepsilon > 0$  we have that f is  $O(e^{\varepsilon \sqrt{|z|}})$  as  $z \to \infty$ , then f is a constant.

*Proof.* By Phragmen–Lindelöf f is bounded, and by Liouville's theorem it is then constant.

The function  $z \mapsto \cos(\sqrt{z})$  is a non-constant entire function bounded on the positive ray  $\mathbb{R}_+$ , and is  $O(e^{\sqrt{|z|}})$ , but it is asymptotic to  $e^{\sqrt{|z|}}/2$  as  $|z| = r \to \infty$  on the ray  $\{-r : r \ge 0\}$ . But there are nonconstant entire functions that are  $O(\exp(|z|^{\varepsilon}))$ , for every  $\varepsilon > 0$ ; they are of course unbounded on every ray (see, e.g. Bak-Newman, Complex Analysis, 15.7).

5.3.1. Entire functions of finite order. The previous results suggest the following:

DEFINITION. An entire function is said to be of finite order if it is  $O(\exp(|z|^b))$ , for some b > 0. The order of an entire function is the infimum of the set of all b > 0 such that the function is  $O(\exp(|z|^b))$ .

EXERCISE 5.3.1.1. Prove that if an entire function of finite order has polynomial growth on the boundary  $\partial S$  of a strip, then it has polynomial growth of the same order on S (that is, if there is  $m \in \mathbb{N}$  such that f(z) is  $O(z^m)$  for  $z \to \infty$ ,  $z \in \partial S$ , then f(z) is also  $O(z^m)$  for  $z \to \infty$ ,  $z \in S$  (hint: consider  $f(z)/(z-c)^m$  with  $c \notin \overline{S} \ldots$ ).

Solution. Following the hint, let  $g(z) = f(z)/(z-c)^m$  with  $c \notin \overline{S}$  fixed. Then g is bounded on the boundary of S. Moreover, if  $f \in O(\exp(|z|^c)$  for some c > 0 as  $|z| \to \infty$ ,  $z \in S$ , then clearly  $g \in O(\exp(e^{b|z|}))$  for every b > 0: in fact

$$|g(z)| \exp(-e^{b|z|}) \le |f(z)| |z-c|^{-m} \exp(-e^{b|z|}) \le A \exp(|z|^c) |z-c|^{-m} \exp(-e^{b|z|});$$

Setting for simplicity r = |z| we easily see that the preceding expression has limit 0 as  $r \to +\infty$  (note also that  $|z - c|^{-m} \sim r^{-m}$  as  $r \to +\infty$ ):

$$A \exp(r^c - e^{br}) r^{-m} \to 0 \text{ for } r \to +\infty,$$

since, clearly,  $\lim_{r \to +\infty} (r^c - e^{br}) = -\infty$ . Taking  $b < \pi/a$ , with a the width of S, we get that g is bounded on S, by  $\|g\|_{\partial S}$ . Then

$$|f(z)| \le C|z-c|^m$$
 for every  $z \in S$ , with  $C = ||g||_{\partial S}$ .

EXERCISE 5.3.1.2. Suppose that f is a nonconstant entire function bounded on every ray. Prove that f is not of finite order (if not we can divide the plane into a finite number of small wedges...).

Solution. If f had finite order, then  $f \in O(\exp(e^{b|z|}))$  for some b > 0. Let us divide the plane into angles of aperture  $\alpha = 2\pi/m$  such that  $b < \pi/\alpha = m/2$ : that is, we pick m > 2b, and partition the plane into m equal angles. Since f is by hypothesis bounded on every ray it is bounded on the boundary of each of these angles, hence bounded on each of these angles by the Phragmen–Lindelöf theorem; but then f is bounded on  $\mathbb{C}$ , and by Liouville's theorem f is a constant.

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